Women in Analysis (WoAN) - A Research Collaboration Conference for Women
Report on Workshop 25w5452
May 11 - 16, 2025

August 12, 2025

1 Short Overview

The research collaborative workshop 25w5452 is the second structural effort made by the Women in Analysis Research Network to engage women junior mathematicians in this broad area in carefully designed, cutting edge research projects. The main goal of this event was to further develop WoAN as a structured international network of women analysts as a mean of: establishing effective collaboration groups, developing new interdisciplinary research activities, training of junior mathematicians, increasing visibility and professional connections of participants, offering an avenue for informed career development pertinent to women in analysis, and creating a database of information and professional opportunities relevant to women in this mathematical field. Irina Mitrea (Temple University), Donatella Danielli (Arizona State University), and María Soria Carro (Rutgers University) served as organizers and the workshop hosted the following collaborative research teams:

1. **Harmonic Analysis**
2. **Linear and Nonlinear Boundary Problems**
3. **Nonlinear Dispersive Equations**
4. **Several Complex Variables**
5. **Scattering Theory**

Each team was led by internationally recognized women experts in these fields. Scientific activities at the workshop included introductory lectures and discussions, collaborative research time, poster and short talk sessions for junior participants, and wrap-up sessions in which teams reported on their progress. The workshop schedule also included a professional development session. Below we will elaborate on the scientific content of the workshop and the progress registered by the various collaboration teams.

2 Harmonic Analysis

The Harmonic Analysis Group consisted of: A. Čolović, G. Dafni, M. C. Pereyra, M. Ramirez, S. Sandberg, K. Taylor, and L. Ward.

The general area of Harmonic Analysis revolves around the study of classes of functions and operators, with emphasis on the quantitative properties of these objects, including boundedness, integrability, decay estimates, existence of gradients, and relations of curvature. Geometric measure theory provides a framework to describe structural properties of measures and sets, including density, rectifiability, and dimension. The group approached several questions at the overlap of harmonic analysis and geometric measure theory:

(1) non-linear two projection and Besicovitch theorems, and could Peter Jones β numbers help?,

(2) boundedness and compactness of commutators of the Cauchy integral on chord-arc curves,

(3) boundedness of the strong maximal function on strong BMO.

The challenge for the Harmonic Analysis group was to find areas of intersection in the participants' interests and to find problem(s) that would have a geometric flavor, involve some functional analysis, and have a possible dyadic optic.

What the Harmonic Analysis Group did in BIRS and afterwards

- The first group meeting at BIRS focused on problems that Krystal Taylor proposed (see Section 2.1) and it sparked our conversations for the rest of the week. Discussions on the notions of rectifiability and purely unrectifiable sets (see Section 2.1.1), lead the group to consider Peter Jones' β numbers, see Section 2.1.3, introduced in his solution to the analyst's Traveling Salesman Problem in $\mathbb{R}^2$ [19]. The β numbers can be used to characterize a class of chord-arc curves [6]. This sparked the group's curiosity, as it connected to Maricela Ramirez' work for the Cauchy integral on chord-arc curves, see Section 2.2.

- On the second group meeting, Maricela presented a dry run for her talk in the parallel sessions. Further questions arose, for example can we study compactness of commutators of the Cauchy integral on chord-arc or even Alhfors-regular curves and functions on VMO or some analogous space? This connected to work of L. Ward [20] and C. Pereyra [18], see Section 2.2.1, and to subjects close to Ana's and G. Dafni's areas of interest that we discussed in our fourth meeting, including endpoint results involving the Hardy space H^1 and the space of bounded mean oscillation BMO.

- In the third group meeting, Ana presented her thesis work on Hankel operators and composition of dyadic paraproducts, with connections to some of the problems proposed by Cristina Pereyra in the plenary lecture, and overlap with work done by Galia Dafni's student on multiparameter paraproducts. On our last group meeting, Lesley Ward explained in more detail one of the problems she had proposed in the plenary lecture, involving the strong maximal operator, and she showed us some interesting atomic measures [16, 15], see Section 2.2.2. Maricela and Ana prepared and delivered the wrap-up remarks for our group.

- The Harmonic Analysis group created a Dropbox to store papers/monographs that we think could be of interest to the group. Galia Dafni identified a 2002 monograph by Hervé Pajot [27] that touches on many of the themes we discussed and would like to explore in more depth: Peter Jones β numbers, Menger curvature, Cauchy integral, and analytic capacity. The Dropbox library is growing. Additionally the group has a joint overleaf, and met in zoom (a multicontinental meeting: North America, Australia and Europe) a month after the program ended, with plans to meet again early in July.

2.1 Rectifiability, Projection Theorems, and β Numbers

Rectifiability is a key concept that describes how much a set resembles a curve, see Section 2.1.1. Projections are used throughout mathematics, and are used in harmonic analysis to detect geometric and structural information about a set. There are many important projection theorems that are easy to state, including the two projection theorem [30]. A deeper problem related to projection theory is the Favard length problem, which concerns the rate of decay of the average projection length of a purely unrectifiable set, see Section 2.1.2. Tao proved both a quantitative two projection theorem and a quantitative Besicovitch theorem (which relates to the Favard problem) [29].

2.1.1 Four-corner Cantor Sets and Rectifiability

Rectifiability of a set refers to the extent to which one can fit countable union of Lipshitz curves onto the set. As an example of a purely unrectifiable set (one such that its intersection with any Lipschitz graph has zero length) we consider the four-corner Cantor set (also called Garnett's set), which is defined iteratively. Chord-arc curves are rectifiable, see 2.2.1.

Four-corner Cantor sets are examples of fractal sets, whose properties can often be measured using the Hausdorff measure. Hausdorff measure gives one a way to quantify fractal sets. Hausdorff measures do not, however, provide a way to determine whether a set is rectifiable or not. The four-corner Cantor set is purely unrectifiable (by the two-projection theorem below). A line set in a plane is rectifiable, but both of these sets have positive Hausdorff measure. To understand the way we can measure rectifiability of sets, the group looked towards projection theorems in geometric measure theory.

2.1.2 Projection Theorems

One can differentiate between rectifiable and unrectifiable sets by the way they behave under orthogonal projections. Let $\Pi_\theta : \mathbb{R}^2 \to \mathbb{R}$, for $0 \leq \theta < \pi$, be the orthogonal projection defined by:

$$\Pi_\theta(x, y) := (x, y) \cdot (\cos\theta, \sin\theta).$$

The two-projection theorem states ([29, Prop. 1.16]: *A compact set in the plane that has two distinct orthogonal projections of zero measure must be purely unrectifiable.* One common line of investigation identified was: **Can we prove an analogue of the two-projection theorem for nonlinear projection operators?**

Besicovitch's projection theorem states that for a purely unrectifiable compact set $E \subset \mathbb{R}^2$, with a positive Hausdorff measure, then $|\Pi_\theta(E)| = 0$ for almost every θ. In particular, the four-corner Cantor set has zero-length projections for almost every θ. The common line of investigation identified in this direction was: **Is there a Besicovitch's theorem for non-linear projections?** A version for a particular family of curve projections was shown in [14].

The Favard length is the average of the lengths of the orthogonal projections, namely

$$\mathrm{Fav}(E) := \frac{1}{\pi}\int_0^\pi |\Pi_\theta(E)|d\theta.$$

Purely unrectifiable compact sets must have zero Favard length. The Favard length of a set E in the unit square has a probabilistic interpretation: up to a constant factor, it is the probability that the "Buffon's needle," a long line segment dropped at random, hits E, [7]. For curved projections of sets of finite length and nearly purely unrectifiable, their Favard curved length is very small [14].

Can we say something about the probability that a curved Buffon's needle hits E?

2.1.3 Rectifiability and Beta Numbers

We now turn towards another way to quantify rectifiability of sets, namely β numbers. The traveling salesman problem asks whether given a set E, we can find a curve that contains the set. The analyst's traveling salesman problem asks whether we can find a rectifiable curve Γ that contains the set and if so, what the length of the curve would be. For sets in $\mathbb{R}^2$ the problem was solved by Peter Jones [**?**], who found that the length of the best curve can be estimated using a sum of an infinite series. He developed the concept of β numbers that measure how close the set is to being a line at different scales. To capture the notion of different scales, one can use dyadic cubes.

The β number of a set E with respect to the cube Q is given by

$$\beta(E, Q) := \frac{1}{\mathrm{diam}(Q)} \inf_L \sup\{\mathrm{dist}(z, L) : z \in 3Q \cap E\},$$

where the infimum is taken over all the lines L that intersect Q.

Peter Jones then proved that the curve of the smallest length containing $E \subset \mathbb{R}^2$ has length

$$l(\Gamma) \approx \mathrm{diam}(E) + \sum_Q \beta(E, Q)^2 \mathrm{diam}(Q).$$

If such a curve exists, the sum on the right will converge. One of the goals of our group was to understand how these two notions of rectifiability can be reconciled. To familiarize ourselves with these definitions, we worked out the β numbers for the four-corner Cantor set.

2.2 Boundedness of Specific Operators

Since the boundedness of operators is at the core of harmonic analysis, our group discussed possible directions for studying the boundedness of the commutator of the Cauchy integral and the strong maximal operator, and provided brief overviews of what an investigation into these problems could involve. Before diving into the specific problems discussed, we define the operators under consideration.

Let $\Gamma \subseteq \mathbb{C}$ be a rectifiable, orientable curve that separates the complex plane into two connected components. The boundary-to-domain Cauchy integral is defined for functions $f \in L^1\left(\Gamma, \frac{d\zeta}{1+|\zeta|}\right)$ by

$$C_\Gamma f(z) := \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{for all } z \in \mathbb{C} \setminus \Gamma.$$

One can then define the corresponding truncated, maximal, and principal value operators as usual. Given a singular integral operator T and a function b, the commutator of T with b is defined as follows:

$$[b, T](f) := bT(f) - T(bf).$$

The strong maximal function in the multi-parameter setting associated with a measure μ is given by

$$\mathcal{M}_{s,\mu} f(x) := \sup_{R \,:\, x \in R} \frac{1}{\mu(R)} \int_R |f(y)| \, d\mu(y),$$

where the supremum is taken over all axis-parallel rectangles R containing x.

2.2.1 The Commutator of the Cauchy Integral

Some of our group members have worked closely with the Cauchy integral. Namely, Cristina Pereyra and collaborators (see [18]) have studied related problems, while Maricela Ramirez worked on proving the uniform boundedness of the truncated Cauchy operator on chord-arc curves. Lesley Ward, along with collaborators, established the boundedness and compactness of the commutator of C_Γ, the Cauchy integral on Lipschitz curves. In particular, [20] shows that for real-valued locally integrable functions b, the commutator $[b, C_\Gamma]$ is bounded if and only if $b \in \mathrm{BMO}(\mathbb{R})$, the John-Nirenberg space of functions of bounded mean oscillation. Additionally, for $b \in \mathrm{BMO}(\mathbb{R})$, the operator $[b, C_\Gamma]$ is compact if and only if b is in $\mathrm{VMO}(\mathbb{R})$, the space of functions of vanishing mean oscillation (sometimes denoted $\mathrm{CMO}(\mathbb{R})$).

These results were established specifically for the case where the Cauchy integral is evaluated on a Lipschitz graph. With the goal of finding an interface between the group members' areas of focus, a natural question arose: **Can we extend the boundedness and compactness results discussed above to the setting where the Cauchy integral is defined on a chord-arc curve?**

2.2.2 The Strong Maximal function

In both [16] and [15], Lesley and her collaborator Guillermo Flores studied the boundedness of the strong maximal function. Of particular interest was a construction of an atomic measure μ, i.e., a measure supported on a countable set of points, called atoms, for which $\mathcal{M}_{s,\mu}$ is unbounded on $L^p(\mu)$. The group went over this construction in detail, leading to a potential future direction: **Characterize the measures μ for which $\mathcal{M}_{s,\mu}$ is bounded on $L^p(\mu)$ for all, none, or some $p \in (1, \infty)$.**

Since Galia Dafni specializes in different function spaces, the group discussed the multi-parameter analogue of the BMO space, the strong BMO space (also known as bmo or "little BMO"). This space is defined using averages over axis-parallel rectangles as opposed to dyadic cubes in the classical setting. This connection is relevant, as both the strong maximal function and BMO space naturally arise in the multi-parameter setting. This led to the following open question: **Is the strong maximal function $\mathcal{M}_{s,\mu}$ bounded on strong BMO?** These discussions highlighted promising directions at the intersection of operator theory, function spaces, and geometric measure theory.

3 Linear and Nonlinear Boundary Value Problems

Participants: Shalmali Bandyopadhyay (UT Martin, TN - virtual), Maya Chhetri (UNC Greensboro, NC), Briceyda B. Delgado (Infotec, Mexico), Melissa Glass (UNC Greensboro, NC), Nsoki Mavinga (Swarthmore College, PA), Maria Amarakristi Onyido (Northern Illinois University, IL), Diana Sanchez (National University of Colombia, Manizales, Colombia)

There has been a significant amount of interests in the study of nonlinear boundary value problems, especially nonlinear elliptic problems, since Bernstein's pioneering work in nonlinear elliptic equations in 1906 and Leray's work on hydrodynamical problems in 1933. Study of such problems is important for the understanding and modeling of nonlinear processes such as chemical, biological or ecological processes. The qualitative and quantitative study of these equations is essential to better understand and model such processes. The focus of the group was to introduce qualitative study of the questions of (a) existence, (b) nonexistence, (c) multiplicity, and (d) properties of solutions of some model problems which describe the long-term behavior of the corresponding time dependent parabolic problems.

3.1 Workshop Activities

This group focused on understanding one of the widely used tools called the method of sub- and supersolutions for second order elliptic problems with nonlinear boundary conditions of the form

$$\left.\begin{aligned} -\Delta u &= f(x,u) \quad \text{in } \Omega \\ \frac{\partial u}{\partial \eta} &= g(x,u) \quad \text{on } \partial\Omega \end{aligned}\right\} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary if $N \geq 2$ and a bounded open interval if $N = 1$, η is the outward normal derivative to the boundary $\partial\Omega$ and, Δ is the usual Laplacian operator. The nonlinearities $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

In a nutshell, it is **expected** that if the problem has a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of the problem above such that they are ordered, that is, $\underline{u} \leq \overline{u}$, then:

Existence there exists a (weak) solution u of the problem satisfying $\underline{u} \leq u \leq \overline{u}$.

Maximal/minimal solution there exist a minimal and a maximal solution u_* and u^*, respectively such that if u is any solution satisfying $\underline{u} \leq u \leq \overline{u}$ then $u_* \leq u \leq u^*$.

The literature generally deals with the question of **Existence** since in most cases, this is sufficient. However, the existence of both **Maximal** and **Minimal** solution, when they exist, provides more qualitative information about the solution, and is therefore an area of active investigation. In this group, we have focused on both the questions for various Elliptic problems with **nonlinear boundary conditions** when the subsolutions and supersolutions are not necessarily bounded.

3.2 Group work

Junior members had different level of exposure to the topic. Some of them had already co-authored articles with at least one or both co-leaders of the group, some were only the users of this tool in their own research but have not had opportunity to think about the method itself, and some had very brief and topical knowledge. To make sure everyone is on the same page, we started the discussion on the article [4] which deals with the case $f \equiv 0$, that is

$$\left.\begin{aligned} -\Delta u + u &= 0 \qquad\quad \text{in } \Omega \\ \frac{\partial u}{\partial \eta} &= g(x,u) \quad \text{on } \partial\Omega\,, \end{aligned}\right\} \tag{3.2}$$

we discussed the functional framework necessary to deal with the weak solution and the importance of the trace operator due to the presence of the nonlinearity g on the boundary $\partial\Omega$.

- if g is monotone, then a monotone iteration method yields the existence of a maximal and a minimal solution right away. Such monotone iteration technique is useful for numerical implementation.
- if g is **not** monotone, then a strategy has been to first prove the existence of a solution, then use the solution obtained to apply Zorn's lemma to find a maximal and a minimal solution.

Group activities:

- First part of the workshop at BIRS was spent on understanding the tools necessary to deal with the non-monotone case for the problem (3.2) and connecting with the problem (3.1).
- Then the group discussed several interesting applications, in particular, what are the sufficient conditions on the nonlinearity or nonlinearities that can guarantee the existence of an ordered pair of sub- and supersolution to guarantee a solution between them.
- Junior members were also introduced to the problems involving a bifurcation parameter multiplied nonlinearity and how to analyze the number of solutions with respect to the parameter depending on the behavior of the nonlinearities.
- One of the junior members also demonstrated the use of numerical experiments in validating the theory developed via *Matlab*.

3.3 Proposed problem and progress

- A subgroup has settled on extending the result to the coupled system of equations with coupled nonlinear boundary conditions of the form

$$\left.\begin{array}{rcll} -\Delta u_1 &=& f_1(x,u_1,u_2) & \text{in } \ \Omega\,, \\ -\Delta u_2 &=& f_2(x,u_1,u_2) & \text{in } \ \Omega\,, \\ \frac{\partial u_1}{\partial \eta} &=& g_1(x,u_1,u_2) & \text{on } \ \partial\Omega, \\ \frac{\partial u_2}{\partial \eta} &=& g_2(x,u_1,u_2) & \text{on } \ \partial\Omega\,. \end{array}\right\} \tag{3.3}$$

  These members have been meeting weekly (virtually) since returning from BIRS with the goal of wrapping up the manuscript over the summer. The plan is to have both the theoretical and numerical component in the work. The strategy is to combine the theory developed in [5] to the coupled system of equations case corresponding to the problem (3.2) with two preprints by N. Mavinga [22, 23] dealing with the problem (3.1). The challenge is to overcome the difficulty posed by the coupling in the differential equation through the nonlinearities in the domain Ω as well as on the boundary $\partial\Omega$.

- One of the doctoral students will be submitting her thesis this summer with the plan to defend in the fall semester and did not join the above project. However, she is already working with two separate projects with each of the co-leaders, as part of her thesis and one of them involve the nonlinear boundary condition. She plans to work on a BIRS related project once the thesis is completed.

- The most junior participant has also began working on her thesis projects involving problems of the form (3.1) from the mathematical ecology point of view. She will be joining some members of the group on a project as a result of this workshop during the fall semester.

3.3.1 Some Testimonials from junior participants

- *The combination of strengthened theoretical foundation, active collaboration, and clear deliverables makes this one of the most productive workshop experiences I've had. I'm particularly excited about contributing the computational component to our team's theoretical work, as this integration of numerical and analytical approaches often leads to the most robust and complete research outcomes.*

- *The Workshop on Analysis WoAN was an amazing experience; the opportunity to work in an inspiring environment was fantastic. In Mexico, there are very few women mathematicians working in analysis, so to participate in a full research week hearing about non-linear pde's, scattering theory, free boundary problems, harmonic and complex analysis, among others, was a very fruitful and motivating experience.*

  More than the results that can be obtained as a product of this workshop, I value and am very grateful for the opportunity provided by my team leaders (Maya and Nsoki), the knowledge shared, the constant support, and patience. Thanks also to the workshop organizers, Irina, Donatella, and Maria; without them, all this experience and learning would not have been possible.

- *Attending the Women in Analysis conference was a wonderful learning experience. I was in the Linear and Nonlinear Boundary Value Problems group. Our group worked together well and as a result we were very productive. There were varying levels of experience within our group so we spent a little over a day going over definitions, theorems, and the techniques of the proofs. The leaders were patient and made sure we were all grasping the concepts before we moved on.*

- *I consider that this workshop has been an excellent opportunity to establish connections with my peers and mentors. They taught me how valuable it is to work as a team to develop any skill and the importance of working individually to assimilate some concepts. Understanding something means being able to teach it to someone else and having them understand it. For this reason, I would like to share what I have learned with new members. I would like to organize this type of workshop for women in Colombia.*
  This experience has changed the way I do research and has also helped me develop social skills that I thought I did not have. Thanks to my mentors for inviting me to this workshop, I have learned many things about mathematics and about life. Thanks to this opportunity you gave me, I realized that I can be more independent and that I can do things I thought I could not. Thank you very much.

4 Nonlinear Dispersive Equations

Group Leaders: Nataša Pavlović, Gigliola Staffilani
Group Members: Iryna Petrenko, Jia Shi, Diana Son, Maja Tascović, Katja Vassilev, Luisa Velasco

This portion of the report summarizes the discussions and progress of various projects considered by the nonlinear dispersive equations group. The starting point of our discussions was three general families of problems suggested by the group leaders:

1. Fluid and water wave equations
2. Dispersive methods in the context of kinetic equations
3. Integrability in the context of kinetic equations

We include a complete list of these problems in Section 4.1. In addition, our discussions of these problems led us to propose several new problems, recorded in Section 4.2. For the workshop, we focused primarily on **P 1.3** (and related) and **P 2.1** from those lists, so that our primary focus became:

- Establishing blow up of solutions to dispersive equations using implosion of solutions to fluid equations.
- Using dispersive methods in the context of kinetic equations to establish well-posedness of initial value problems.

For the first point, we were able to make direct computations for several dispersive systems leading to the next steps, as well as show that certain dispersive systems are not suited to the current techniques. For the second point, we discussed the literature to see the advantages of the proposed technique.

4.1 Initial Problems

4.1.1 Fluid and water wave equations

In the context of fluid and water wave equations we propose the following problems, each of which with a connection to a related phenomenon identified in the context of some dispersive equations:

P 1.1 Understanding growth of Sobolev norms for the 2D SQG (Surface quasi-geostrophic waves) [21].

P 1.2 Establish Strichartz estimates for solutions linked to the water wave problem, based on the work [28].

P 1.3 Establish blow up for complex variable dispersive equations via self similar profiles in fluid equations , which has been done for NLS and complex wave equations [9, 10]. We can attempt the same technique for complex supercritical gKdV or ZK equation.

4.1.2 Dispersive methods in the context of kinetic equations

Motivated by the works of Chen-Denlinger-Pavlović [11, 12] on using ideas and methods from nonlinear dispersive equations in the context of Boltzmann equation, we propose the following problem:

P 2.1 Well-posedness analysis of a 4-wave kinetic equation on $\mathbb{R}^3$ based on transforming it to a dispersive equation via the inverse Wigner transform. For a formulation of the 4-wave kinetic equation see the recent work of Ampatzoglou [1].

4.1.3 Integrability in the context of kinetic equations

Motivated by the works of Mendelson, Nahmod, Pavlović, Rosenzweig and Staffilani [24, 25] on integrability of equations that appear in a context of derivation of NLS, we propose the following problem:

P 3.1 Investigating whether the equation obtained by the following process is integrable.

(a) Start with the GP (Gross-Pitaevskii) hierarchy corresponding to the 1D cubic NLS (see [24] for a formulation of the GP hierarchy). This hierarchy is integrable according to the works [24] and [25].

(b) Apply the Wigner transform to the equation under the previous step to obtain a kinetic-looking equation. For a basic info on using Wigner transform to connect kinetic and dispersive equations, see [11].

(c) Is this new equation integrable? Can we construct infinitely many conservation laws for this equation?

4.2 Additional Problems

Discussions on the problems in Section 4.1 led us to additionally propose the following questions:

P 4.1 Inspired by the blowup result for the NLS [10], in addition to **P 1.3** can one lift implosion from the NLS to the GP hierarchy? See the work [13] for blowup in GP hierarches.

- The GP hierarchy is an infinite, linear, coupled hierarchy given as follows

$$i\partial_t \gamma^{(k)} + (\Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k})\gamma^{(k)} = \sum_{j=1}^{k} B_{j,k+1}\gamma^{(k+1)},$$

  where the notation $\underline{x}_k = (x_1, x_2, \ldots, x_k)$ and $\gamma^{(k)} = \gamma^{(k)}(t, x_k, x'_k)$ has been used. Note that $\gamma^{(k)} = \prod_{j=1}^{k} u(t, x_j)\overline{u(t, x'_j)}$ is a solution of the GP hierarchy if u solves the NLS.
- To show blowup, we would begin by showing it for a factorized solution (this may be immediate)
- For other solutions, one can use the de Finetti approach.

P 4.2 Also inspired by **P 1.3**, can one provide an interpretation of NLS implosion at the level of the $N-$body hierarchy?

P 4.3 Also inspired by **P 1.3**, can one establish implosion of other dispersive equations? We considered:

label = () Modified Zakharov–Kuznetsov equation (This was looked at by Luisa and Katja)

$$\partial_t u + \partial_x \Delta u = |u|^2 \partial_x u, \qquad (t, x, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}. \tag{4.4}$$

This system is anisotropic (one dimension is favored). Therefore, in dimension $d = 2$ it is more useful to consider the symmetrized system:

$$\partial_t u + (\partial_x^3 + \partial_y^3)u = |u|^2(\partial_x + \partial_y)u, \qquad (t, x, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}. \tag{4.5}$$

lbbel = () Biharmonic NLS (This was looked at by Jia, Iryna, and Diana)

$$\partial_t u + \Delta^2 u = |u|^{p-1}u, \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^d. \tag{4.6}$$

lcbel = () Davey Stewartson system

$$\begin{cases} i\partial_t u + \sigma_1 \partial_{x_1}^2 u + \partial_{x_2}^2 u = \sigma_2 |u|^2 u + (\partial_{x_1}\varphi)u, \\ \alpha \partial_{x_1}^2 \varphi + \partial_{x_2}^2 \varphi = -\gamma \partial_{x_1}(|u|^2), \end{cases} \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \tag{4.7}$$

where $\sigma_1, \sigma_2 \in \{\pm 1\}$.

ldbel = () Hyperbolic NLS (This was looked at by Gigliola, Maja, and Natasa)

$$i\partial_t u + \Delta_x u - \Delta_y u = |u|^{p-1}u, \qquad (t, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m. \tag{4.8}$$

lebel = () Generalized KdV

$$\partial_t u + \partial_x^3 u = \partial_x(u^k). \tag{4.9}$$

Remark: We note that to perform the Madelung transformation we must have a complex valued solution u. Thus, in order to attempt this program for real-valued equations, we would need some kind of complexification of the solution.

4.4 Also inspired by **P 1.3**, can one extend the concept of implosion for 4-WKE from blowup of NLS? We noted that this is likely not possible since the derivation of the WKE uses random data and the blowup is very unstable.

4.3 Blow up of Dispersive Equations

Former work of Guderley [17], Merle-Raphaël-Rodnianski-Szeftel [26], and Buckmaster–Cao-Labora–Gomez-Serrano [8] established self-similar blowup for radial Euler, Navier-Stokes and some supercritical *defocusing* NLS equations. More recently, Cao-Labora–Gomez-Serrano–Shi-Staffilani extended these results to the non-radial case [9], [10]. This last work focuses on transforming the dispersive NLS system into a fluid equation. Together, the strategy of all of these works may be summarized as:

- **Step 1:** For complex-valued solution u to a dispersive equation, perform Madelung transformation

  $$u(t,x) = \sqrt{\rho(t,x)}e^{i\phi(t,x)},$$

  where ρ represents the density and $\nabla\phi = v$ represents the velocity.

- **Step 2:** Write equations for ρ_t, v_t, yielding "compressible Euler + small perturbation".

- **Step 3:** Write the solution in self-similar coordinates for α, γ constants:

  $$\rho(t,x) = \frac{1}{(T-t)^\alpha}\rho\left(\frac{x}{(t-T)^\gamma}\right),$$

  and similarly for v. In the radial case this yields a new system of ODEs.

- **Step 4:** Going from $(t,x) \to (s,y)$ for self-similar coordinates for Euler (note that $s \sim \log(T-t)$ and $y \sim e^s \cdot x$), blowup corresponds to $s \to \infty$. One may then separate and identify the "small terms", which should carry exponential decay in s. This will correspond to some restrictions for the chosen constants. For the ODE, one may then use a phase portrait to identify stationary *smooth* solutions $\overline{U}$ (corresponding to the velocity field) and $\overline{S}$ (corresponding to the density).

- **Step 5:** Once one has stationary solutions, consider the "small perturbation" terms and show that one still has a solution which blows up. If one considers the nonradial case, one additionally would then write $(\rho, v) = (\overline{S}, \overline{U}) + \eta(t,x)$, where η is nonradial. This yields an equation for η, which one must deal with.

Taking this general strategy, we looked at a variety of dispersive systems below, hoping to adapt this framework:

4.4 Modified ZK (4.5)

- Performed Madelung transform and self-similar change of variables; however, due to the odd number of derivatives, we were unable to rewrite the resulting equation in terms of $u = \nabla\phi$.
- After simplifying system by dropping all non-dominant terms, what remained was still a PDE rather than an autonomous ODE.

  Conclusion: Due to the third order derivatives in the ZK equation, we cannot apply this program to this equation to obtain blow-up. Different techniques must be applied.

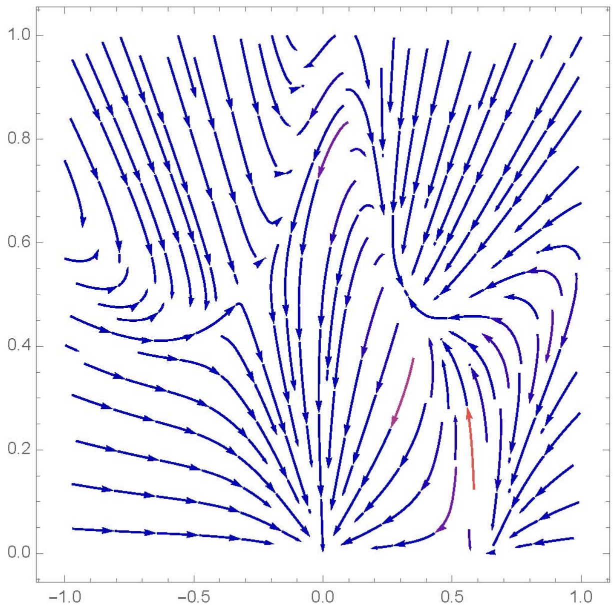
4.5 Biharmonic NLS (4.6)

- Performed Madelung transformation and self-similar change of variables to obtain the corresponding autonomous ODE system.

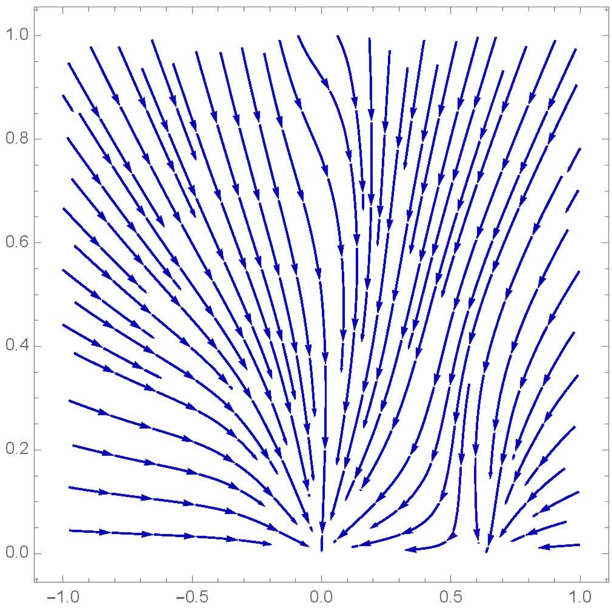
$$\begin{aligned} 0 &= \lambda_1\bar{W} - \frac{\lambda_2}{3}\bar{W} - \lambda_2\partial_\xi\bar{W} + 4\bar{W}^3(\partial_\xi\bar{W} + \frac{1}{3}\bar{W}) - 4\bar{V}^3(\partial_\xi\bar{V} + \frac{1}{3}\bar{V}) \\ 0 &= \lambda_1\bar{V} - \frac{\lambda_2}{3}\bar{V} - \lambda_2\partial_\xi\bar{V} + \frac{3}{2}(p-1)\bar{V}\ \bar{W}^2(\partial_\xi\bar{V} + \frac{1}{3}\bar{W}) + 4\bar{W}^3(\partial_\xi\bar{V} + \frac{1}{3}\bar{V}) \end{aligned} \tag{4.10}$$

- Studied this ODE via phase portraits which display the slope fields in the focusing and defocusing cases shown in the figures below.

Next step: Obtain smooth solutions, possibly with computer assisted proof.



Defocusing



Focusing

4.6 Hyperbolic NLS (4.8)

- No blow-up results are available for this equation and

$$H(u) = \int \left(|\nabla_x u|^2 - |\nabla_y u|^2 + \frac{2}{p+1}|u|^{p+1} \right) dxdy$$

  is conserved but not with definite sign.

- Since the Hyperbolic NLS bears such similarity to the NLS, some simplfiying assumptions are first considered: Assume $x \in \mathbb{R}^d$ and $y \in \mathbb{R}$. Then use the existing blow-up result for the NLS in $\mathbb{R}^d$, discussed above, by rescaling the solution u "faster" in x and "slower" in y to extend the result to the Hyperbolic NLS.

4.7 Dispersive methods in the context of kinetic equations

Inhomogeneous kinetic equations broadly take the form

$$\partial_t f + v \cdot \nabla_x f = \mathcal{Q}(f), \tag{KE}$$

where $\mathcal{Q}(f)$ is termed the *collision operator*. Such equations arise as the evolution of statistical behavior of a gas of particles or a wave system, where f represents, in some sense, the density of particles or waves. In the particle case, it's the well known Boltzmann equation for which local well-posedness was shown by Chen-Denlinger-Pavlović [11, 12] using the inverse Wigner Transform W^{-1}, so that one obtains

$$\begin{cases} W^{-1}(f)(x,x') := u(x,x'), \\ i\partial_t u + \Delta_x u - \Delta_{x'} u = B(u), \end{cases} \tag{4.11}$$

for some nonlinear term B. This allows dispersive techniques to be accessible.

More recently, local-well posedness has been established for wave kinetic equations, for instance [1, 2]. These results do not employ the Wigner Transform, which would hopefully allow for a well-posedness result in L^2 based spaces rather than the L^∞ and L^1 results of the previous papers.

4.8 Final Remarks

Three members of our group — Jia, Diana, and Iryna — participated fully remotely, yet they were highly engaged in the brainstorming sessions and even organized their own meetings to continue the discussions. The group as a whole collaborated exceptionally well, and we made significant progress.

As a group we would like to express our sincere gratitude to BIRS for the outstanding hospitality. In particular, having the opportunity to use the spectacular Kinnear Centre—with its stunning mountain views—was truly remarkable.

As it happens, all members of our group are based in the United States, where meetings of this kind are no longer funded. This made BIRS's support all the more meaningful, and we are deeply appreciative.

5 Scattering Theory

There were seven participants in the group on Scattering Theory: Emilia Blåsten, LUT University Fioralba Cakoni, Rutgers University, Anastasia Kisil, University of Manchester, Shari Moskow, Drexel University, Fatma Terzioglu, NC State University, Jingni Xiao, Drexel University, and Dana Zilberberg, Rutgers University. Most of us have a background in inverse problems, some were new to the topic, and nobody knew everything. So the group organized meetings in the following way: a) first, everyone gave an ex tempore presentation of their recent work in scattering theory, and then b) described some of the interesting open problems in that topic which we believe requires new ideas. We structure this summary in the same way, introducing the various topics very briefly.

5.1 Topics

5.1.1 Shape determination in scattering

The simplest problem that contains a core piece of scattering theory is the source scattering problem:

$$(\Delta + k^2)u^s = f \qquad \text{in } \mathbb{R}^n \tag{5.12}$$

where u^s must satisfy the Sommerfeld radiation condition and f is the source force term. The solution has the asymptotics

$$u^s(x) \sim \Phi(x)\varphi(\hat{x}), \quad x \to \infty, \quad \hat{x} = x/x \tag{5.13}$$

where φ is the far-field pattern of the scattered wave u^s and Φ is the outgoing fundamental solution to $(\Delta + k^2)$. In this simple case it turns out that

$$\varphi(\hat{x}) = c_n \int f(y)e^{-ik\hat{x}\cdot y}dy = \hat{f}(k\hat{x}) \tag{5.14}$$

The hope is that given k fixed, one could deduce something useful about the unknown scatterer f from its far-field pattern $\hat{f}(k\hat{x})$ on $\hat{x} = 1$.

Previous work has shown that if $f = \chi_P g$ where g is a smooth and positive function and P a *convex* polyhedron, then φ uniquely determines P and the values of g at the corner points of P. An open question is to which degree the same holds true for non-convex polyhedron, even in two dimensions.

5.1.2 Born series and new estimates

In potential scattering, given an incident wave u^i, the scattered wave for the Helmholtz equation $(\Delta + k^2 n(x))u = 0$ with refractive index $n(x)$ is $u^s = u - u^i$ which can be solved from the Lippmann–Schwinger equation

$$u(x) = u^i(x) + k^2 \int_{\mathbb{R}^n} \Phi(x-y)\big(n(y) - 1\big)u(y)dy \tag{5.15}$$

where Φ is the outgoing fundamental solution to $(\Delta + k^2)$. Iterating (5.15) gives the *Born series.*

As $x \to \infty$, we have asymptotically

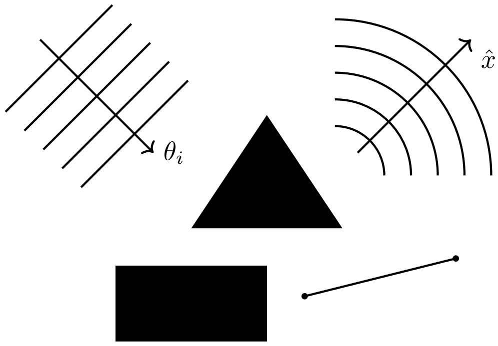
$$u^s(x) \sim \Phi(x)\varphi(\hat{x}), \qquad \hat{x} = x/x \tag{5.16}$$

and φ is called the far-field pattern (or scattering amplitude). Doing asymptotic analysis of the Born series gives a relation between the potential $m(x) = n(x) - 1$ and the far-field pattern φ:

$$\varphi = K_1 m + K_2 m \otimes m + \dots \tag{5.17}$$

Writing an ansatz $m = L_1\varphi + L_2\varphi\otimes\varphi + \dots$ we can study the operators L_i and investigate the convergence of $\varphi \mapsto m$. Classically Sobolev spaces have been used to make this formal. However it is interesting to look at whether recent Wasserstein-type estimates and methods could bring improvement. These methods capture inherent geometry better than Sobolev spaces.

5.1.3 Embedding formulas



An incident plane-wave with direction θ_i interacts with polyhedral obstacles producing a scattered wave whose far-field pattern φ we observe in direction $\hat{x}$. Given the far-field patterns for several incident directions $\theta_i \in \{\alpha_1, \dots, \alpha_m\}$, we can calculate an *embedding formula* for the far-field that would be produced with a new incident wave direction α

$$\varphi_\alpha(\hat{x}) = R\left(\hat{x}, \alpha, \left(\alpha_j, \varphi_{\alpha_j}\right)_j, \text{geometry}\right) \tag{5.18}$$

The rational function R is computed from the a-priori given incident directions and geometry by solving a linear system. The benefit is that this system does not need to be solved again for every new direction α, thus providing speedups in computing the fundamental object in scattering: the far-field pattern.

5.1.4 Interior transmission eigenvalue problem

An incident wave u^i of wave number k that does not cause scattering from an inhomogeneous refractive index n will be equal to the total field u outside of the scatterer $\Omega := x \in^n n(x) \neq 1$. This leads to a type of eigenvalue problem in Ω, written below for $w = u^i$ and $v = u$, that can be studied even without reference to an underlying scattering situation.

We say that the number k is an *interior transmission eigenvalue* (ITE) if there are non-trivial w, v satisfying

$$\begin{cases} (\Delta + k^2 n)v = 0, & \Omega, \\ (\Delta + k^2)w = 0, & \Omega, \\ v - w = 0, & \partial\Omega, \\ \partial_\nu v - \partial_\nu w = 0, & \partial\Omega. \end{cases} \tag{5.19}$$

This eigenvalue problem is extremely challenging, since it is not elliptic or even self-adjoint. Many of the typical research questions involve the existence, discreteness and accumulation points of ITE's, Weyl laws, completeness of eigenfunctions, and other questions from spectral theory.

A major open question related to this problem is what happens when $n - 1$ changes sign near the boundary. This is difficult to study because having $n > 1$ on part of the boundary and $n < 1$ on another prevents looking at the simplest case: radially symmetric solutions. However, if one sets

$$n(x) = 1 + \frac{1}{2}\sin\left(\frac{1}{1-x}\right), \qquad x < 1 \tag{5.20}$$

then one can hope to study the simpler radially symmetric case with varying sign at the boundary but with a more singular differential equation in the radial variable.

5.1.5 Homogenisation

Homogenisation studies what happens to periodic structures in partial differential equations as the fundamental cell size ε tends to zero.

If we apply homogenisation to the interior transmission problem in a domain Ω, we get the equation

$$(\Delta + k^2 n(x/\varepsilon))\frac{1}{n(x/\varepsilon) - 1}(\Delta + k^2)u_\varepsilon = 0 \quad \text{in } \Omega, \tag{5.21}$$

where n is the refractive index of waves of wave number $k \in_+$.

The point of homogenisation is that by letting $\varepsilon \to 0$, we get a simpler equation with solution u_0 which will give information about the "large scale" behaviour of the original solution u_ε, $\varepsilon \neq 0$. Also, the transmission eigenvalues of the two problems are related:

$$k_\varepsilon^2 = k_0^2 + \varepsilon\delta^{(1)} + o(\varepsilon), \quad \varepsilon \to 0. \tag{5.22}$$

To analyse the first order correction $\delta^{(1)}$ we need to do asymptotic analysis for u_ε, and get

$$u_\varepsilon \approx u_0 + \varepsilon^2\chi(x/\varepsilon)(\Delta + k^2)u_0 + \varepsilon\theta_\varepsilon \tag{5.23}$$

where χ solves a Poisson equation with periodic boundary conditions and source term being a transformation of the refractive index. Lastly, θ_ε is a term coming from the mismatch between the lattice ε and the boundary $\partial\Omega$.

The goal is to find the limit θ_0 . This shows in the first term asymptotic $\delta^{(1)}$ of the transmission eigenvalue. It involves the boundary effect on the eigenfunctions at the order ε and function on micro-properties of the highly oscillating media. We can determine a few transmission eigenvalues, assuming that the shape is known, the hope is to determine the homogenised n from k_0^2 and information on the microstructure from $\delta^{(1)}$.

5.1.6 Compton camera

Scattering can also lead to non-elliptic problems, or problems outside of partial differential equations. One topic we briefly discussed was the inverse problem for Compton cameras. The imaging target emits gamma rays which are detected in a sensor that can be moved around the target. Due to the physical processes involved in detecting the rays we do not have the exact incidence direction of the ray, but instead the location it hits the sensor and a (hollow) cone with apex there that contains the location of the ray's source. The measurements are additive. So the inverse problem is to determine an unknown function $f(x)$, $x \in^3$ from integrals over all cones parameterised by their apex $y \in \Gamma$, their axis direction β, and their opening angle θ. With suitable geometric conditions on the measurement locations, this cone-transformation of f can be reduced to a Radon transformation and thus inverted.

5.2 Conclusion

We were a large group with several senior members. The workshop provided a much needed environment for us to meet and share the latest details from each of our research fields and discuss future directions of investigation. Several collaborations emerged with the goal of tackling open problems presented in this short summary.

6 Several Complex Variables

The Complex Analysis group consisted of four members: Andreea Nicoara and Irina Markina served as the group leaders, while Caterina Stoppato and Elena Luca participated as members, with the latter joining remotely via an online platform. The research interests of A. Nicoara and I. Markina intersect in the field of geometry of manifolds in several complex variables. C. Stoppato has been increasingly engaging with complex analysis, transitioning from her background in quaternionic analysis. E. Luca, the most junior participant, continues to develop her growing interest in the area. During the introductory session, A. Nicoara presented three potential problems for group discussion:

Problem 1:
Construct a counterexample to the existence of sup norm estimates for $(p.q)$ forms on a smooth pseudoconvex domain if the q-type is infinite as the Sibony counterexample for $(0, 1)$ forms does not generalize.

Problem 2:
Prove a Diederich-Fornæss type theorem under a more general condition than pseudoconvexity such as there exist $k \in$ and $\epsilon > 0$ such that in a neighborhood of 0 the Levi form satisfies

$$\mathcal{L}(w, w) \geq -\epsilon|z|^k \quad \text{for all} \quad w = (w_1, \ldots, w_n) \in T^{1,0}\mathbb{C}b\Omega.$$

Problem 3:
Study Gromov hyperbolicity with respect to the Kobayashi distance of domains of finite D'Angelo 1-type with no assumption of convexity.

After careful discussing of the problems during the first session, we decided to concentrate on the study of the last problem, since it satisfied the interests of all the participants of the group.

Definition 6.1. *In complex geometry, the Kobayashi metric d_K in a space X is defined as the largest pseudometric on X such that*

$$d_k(f(x), f(y)) \leq \rho(x, y)$$

over all possible holomorphic imbeddings $f\colon \mathbb{D} \to X$ of the unit disc endowed with a hyperbolic distance $\rho(x, y)$ induced by the Poincare metric in $\mathbb{D}$.

Definition 6.2. *A (pseudo) metric space (X, d_K) is called Kobayashi hyperbolic if the pseudometric d_K is a metric on X.*

Consider an arbitrary metric space (M, d). For arbitrary points $y, z \in M$ we take a point $x \in M$ and write a triangle inequality

$$d(y, z) \leq d(y, x) + d(x, z).$$

Then the value

$$(y \cdot z)_x = \frac{1}{2}\Big(d(y, x) + d(x, z) - d(y, z)\Big)$$

is called the Gromov product and it measures how much we add to the distance $d(y, z)$ by going "around the corner" through the point $x \in M$.

Definition 6.3. *A metric space (M, d) is called Gromov hyperbolic if there is $\delta > 0$ such that*

$$(y \cdot z)_x \geq \min\{(y \cdot w)_x, (z \cdot w)_x\} \quad \textit{for all} \quad x, y, z, w \in M.$$

The classical hyperbolic space—whether in the ball model or the upper half-space model—is both Kobayashi and Gromov hyperbolic. In their work, Z. Balogh and M. Bonk [3] demonstrated that a bounded (and hence Kobayashi hyperbolic) strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with C^2 boundary is also Gromov hyperbolic with respect to the Kobayashi metric d_K. They posed an open problem: Can a similar result be established for domains of finite D'Angelo type, without assuming convexity?

A. Zimmer later addressed this question by proving that if $\Omega \subset \mathbb{C}^n$ is a bounded convex domain with C^∞ boundary, then the metric space (Ω, d_K) is Gromov hyperbolic if and only if Ω has finite D'Angelo 1-type. He also constructed an example of a Gromov hyperbolic domain that is neither convex nor smooth, and which contains an analytic disk in its boundary.

Recall that the D'Angelo 1-type at a boundary point of a domain in complex space is a numerical invariant that quantifies the maximal order of contact between complex analytic curves (i.e., 1-dimensional complex varieties) and the boundary at that point. This invariant plays a central role in the study of the $\bar{\partial}$-Neumann problem, subelliptic estimates, regularity theory, and the classification of the complex geometry of domain boundaries.

During our working sessions, we explored various approaches to defining the q-type property of the boundary of a domain $\Omega \subset \mathbb{C}^n$. In particular, we examined Catlin's approach, which defines the q-type via the generic value of the normalized order of contact, in contrast to D'Angelo's definition, which uses the infimum over such values. We also discussed several aspects of Gromov hyperbolicity in metric spaces.

We reviewed the main steps in the result of Balogh and Bonk, which established the Gromov hyperbolicity of bounded strictly pseudoconvex domains. Their approach involved constructing a Carnot–Carathéodory metric on the boundary using the strictly positive Levi form. In this context, we also familiarized ourselves with quasi-isometries and rough isometries, which serve as morphisms in the category of Gromov hyperbolic metric spaces.

Our current objective is to understand why convexity plays such a crucial role in establishing Gromov hyperbolicity. Additionally, we aim to investigate what can serve as a substitute for the Carnot–Carathéodory metric in the absence of a positive definite Levi form. We plan on holding Zoom meeting over the summer and then visiting each other during the academic year 2025-2026 in order to continue the work.

References

[1] I. Ampatzoglou, *Global well-posedness and stability of the inhomogeneous kinetic wave equation near vacuum*, Kinet. Relat. Models 17 (2024), no. 6, 838–854.

[2] I. Ampatzoglou and T. Léger, *Scattering theory for the Inhomogeneous Kinetic Wave Equation*, Preprint, arXiv:2408.05818.

[3] Z. Balog and M. Bonk, *Gromov hyperbolicity and the Kobayashi metric on strictly pseudoconvex domains*, Comment. Math. Helv. 75 (2000), no. 3, pp. 504–533.

[4] S. Bandyopadhyay, M. Chhetri, B. B. Delgado, N. Mavinga, and R. Pardo, *Maximal and minimal weak solutions for elliptic problems with nonlinearity on the boundary*, Electron. Res. Arch. 30 (2022), no. 6, 2121–2137. MR 4410308

[5] S. Bandyopadhyay, T. Lewis, and N. Mavinga, *Existence of maximal and minimal weak solutions and finite difference approximation for elliptic systems with nonlinear boundary conditions*, Electron. J. Differential Equations 2025 (2025), no. 43, 1–21.

[6] C.J. Bishop, *The traveling salesman theorem for Jordan curves*, Advances in Mathematics 404.5 (2022), pp. 1–27.

[7] M. Bond and A. Volberg, *Circular Favard Length of the Four-Corner Cantor Set*, Journal of Geometric Analysis, 21 (2011), pp. 40–55.

[8] T. Buckmaster, G. Cao-Labora, J. Gómez-Serrano, *Smooth imploding solutions for 3D compressible fluids*, Forum Math. Pi 13 (2025), Paper No. e6, 139 pp.

[9] G. Cao-Labora, J. Gómez-Serrano, J. Shi, G. Staffilani, *Non-radial implosion for compressible Euler and Navier-Stokes in* $\mathbb{T}^3$ *and* $\mathbb{R}^3$, Preprint, arXiv:2310.05325.

[10] G. Cao-Labora, J. Gómez-Serrano, J. Shi, G. Staffilani, *Non-radial implosion for the defocusing nonlinear Schrödinger equation in* $\mathbb{T}^d$ *and* $\mathbb{R}^d$, Preprint, arXiv:2410.04532.

[11] T. Chen, R. Denlinger and N. Pavlović, *Local well-posedness for Boltzmann's equation and the Boltzmann hierarchy via Wigner transform*, Comm. Math. Phys. 368 (2019), no. 1, 427–465.

[12] T. Chen, R. Denlinger and N. Pavlović, *Moments and Regularity for a Boltzmann Equation via Wigner Transform*, Discrete Contin. Dyn. Syst. 39 (2019), no. 9, 4979–5015.

[13] T. Chen and N. Pavlović and N. Tzirakis, *Energy conservation and blowup of solutions for focusing Gross–Pitaevskii hierarchies*, Ann. Inst. H. Poincaré C Anal. Non Linéaire 27 (2010), no. 5, 1271–1290.

[14] B. Davey and K. Taylor, *A Quantification of a Besicovitch Non-linear Projection Theorem via Multiscale Analysis*, The Journal of Geometric Analysis 32.138 (2022).

[15] G. Flores and L. Ward, *Zygmund type and flag type maximal functions, and sparse operators*, Proceedings of the American Mathematical Society, 151.2 (202), pp. 555–567.

[16] G. Flores and L. Ward, *Unboundedness for some strong maximal functions on weighted Lebesgue spaces*, Journal of Mathematical Analysis and Applications, 513.2 (2022).

[17] G. Guderley, *Starke kugelige und zylindrische Verdichtungsstösse in der Nähe des Kugelmittelpunktes bzw. der Zylinderachse* (German), Luftfahrtforschung 19 (1942), 302–311.

[18] Y. Han, J. Li, C. Pereyra, and B.D. Wick, *A note on weak factorization of Meyer-type Hardy space via Cauchy integral operator*, Studia Math 253 (2020), pp. 307–327.

[19] P.W. Jones, *Rectifiable sets and the traveling salesman problem*, Invent. Math., 102.1 (1990), pp. 1–15.

[20] J. Li, T.T.T. Nguyen, L. Ward, and B.D. Wick, *The Cauchy Integral, Bounded and Compact Commutators*, Studia Mathematica 250.2 (2020), pp. 193–216.

[21] A. Kiselev and F. Nazarov, *A simple energy pump for the quas-igeostrophic equation*, Nonlinear partial differential equations, 175–179, Abel Symp., 7, Springer, Heidelberg, 2012.

[22] N. Mavinga, T. Myers, and M.N. Nkashama, *Weak solutions of nonlinear elliptic problems with growth up to critical exponents*, Submitted.

[23] N. Mavinga, T. Myers, and M.N. Nkashama, *Weak solutions of nonlinear elliptic problems with monotone nonlinearities*, Submitted.

[24] D. Mendelson, A. Nahmod, N. Pavlović and G. Staffilani, *An infinite sequence of conserved quantities for the cubic Gross-Pitaevskii hierarchy on* $\mathbb{R}$, Trans. Amer. Math. Soc. 371 (2019), no. 7, 5179–5202.

[25] D. Mendelson, A. Nahmod, Nataša Pavlović, M. Rosenzweig and G. Staffilani, *Poisson commuting energies for a system of infinitely many bosons*, Adv. Math. 406 (2022), Paper No. 108525, 148 pp.

[26] F. Merle, P. Raphaël, I. Rodnianski, J. Szeftel, *On the implosion of a compressible fluid II: Singularity formation*, Ann. of Math. (2) 196 (2022), no. 2, 779–889.

[27] H. Pajot. Analytic Capacity, Rectifiability, Menger Curvature and the Cauchy Integral. Vol. 1799. Lecture Notes in Mathematics. Berlin: Springer-Verlag, 2002.

[28] Notes on a simplified model of the 3D capillary gravity water wave equation on tori. Note.

[29] T. Tao, *A quantitative version of the Besikovitch projection theorem via multiscale analysis*, Proceedings of the London Mathematical Society, 98.3 (2009), pp. 559–584.

[30] K. Taylor, *Connections Between Fractal Geometry and Projections*, Notices of the American Mathematical Society, 71.11 (2024), pp. 1458–1465.

[31] A. M. Zimmer, *Gromov hyperbolicity and the Kobayashi metric on convex domains of finite type*, Mathematische Annalen, (2016) p. 1425–1498.

[32] A. M. Zimmer. *Gromov hyperbolicity, the Kobayashi metric and* $\mathbb{C}$*-convex sets*, Trans. Amer. Math. Society, (2017) p. 8437–8456.