Interactions between Symplectic and Holomorphic Convexity in 4-dimensions 23w5123

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1 Overview of the Field

Convexity and convex function theory are classical topics in geometry and analysis, with variations of these notions appearing in a wide assortment of sub-fields. In complex geometry a vital notion of convexity is holomorphic convexity or, by the solution to the Levi problem, *pseudoconvexity*, while symplectic topology includes the key notion of *symplectic convexity*. It has been known for a long time (since work of Weinstein, Bishop, Eliashberg, Gompf and many others) that these ideas are closely linked, and the interplay has led to fascinating developments over the past decades.

In the last few years, however, there have been new indications of subtle interactions between symplectic and complex convexity, particularly in real dimension 4. Notably, new techniques from the theory of (pseudo-) holomorphic curves, Floer homology and gauge theory have led to fascinating and surprising examples and counterexamples in complex geometry, and ideas and results from complex geometry seem poised to provide new directions for study in symplectic topology. The main goal of the workshop was to introduce researchers on each side—symplectic and smooth low-dimensional topology on one hand, and complex geometry and function theory on the other—to the techniques, ideas, and questions on the other, in hopes of sparking collaborations and new insights as well as attracting new mathematicians to this fascinating field.

2 Recent Developments and Open Problems

The following is a sample of some of the problems and themes that were highlighted at the workshop.

Convexity and domains in $\mathbb{C}^2$

Compact domains in affine space, particularly domains of holomorphy, are a well-studied and very active area of research in complex geometry; it is understood that domains of holomorphy are the same as *pseudoconvex* domains. From the point of view of smooth topology, a fundamental open problem is the question of which compact 3-dimensional manifolds admit embeddings in $\mathbb{R}^4$, which amounts to asking whether they arise as boundaries of *smooth* compact domains. Recent research suggests that much progress can be made by combining the points of view, i.e. asking for embeddings of 3-manifolds that satisfy a convexity condition:

- Using techniques from symplectic geometry, Nemirovsky and Siegel [10] gave the first examples of domains in $\mathbb{C}^2$ that are isotopic to pseudoconvex domains, but not to rationally convex domains. Their work uses the close relationship between rational convexity and symplectic convexity: it was shown by Cieliebak-Eliashberg that [2, 3] *rational* convexity, naturally part of holomorphic geometry, is essentially equivalent to the simultaneous requirements of *pseudoconvexity* and *symplectic* convexity.
- Taking different approach, using tools from Floer homology and gauge theory combined with 3-dimensional contact topology, Mark and Tosun [9] recently showed that no 3-dimensional Brieskorn homology sphere can be the boundary of any symplectically convex domain in $\mathbb{C}^2$ (in particular, no rationally convex domain). A Brieskorn sphere is the link of a certain natural complex surface singularity; many of them admit smooth embeddings in $\mathbb{C}^2$ but the full answer to this problem in the smooth category is unknown.

This result concerning Brieskorn spheres mentioned above was proved using techniques from low-dimensional topology and gauge theory (Floer homology), and these methods also give various examples of 3-manifolds that bound pseudoconvex domains but not symplectically convex ones. On the other hand, while Nemirovsky-Siegel's examples of similar phenomena involve somewhat more direct symplectic- and complex-geometric methods, their examples are not accessible to the gauge-theoretic techniques: this already suggests some of the benefits that may be derived from the sort of interactions this workshop was designed to encourage.

The topology of contact type hypersurfaces

A "contact type hypersurface" in $\mathbb{R}^4$ (or $\mathbb{C}^2$) is essentially the same thing as the boundary of a symplectically convex domain. By the work mentioned above, if W is a rationally convex domain in $\mathbb{C}^2$, then its boundary is a contact type hypersurface. Conversely, if it is known for some 3-manifold $Y \subset \mathbb{R}^4$ that Y cannot be made into a contact type hypersurface, then no domain bounded by Y can be rationally (or even symplectically) convex. Contact type hypersurfaces were some of the first examples of contact manifolds for which Weinstein's conjecture concerning periodic orbits of the Reeb vector field was proved.

While many examples exist in higher dimensions, there are few explicit constructions of contact type hypersurfaces in $\mathbb{R}^4$. In fact, both Nemirovsky-Siegel's results and the result concerning Brieskorn spheres can be taken as evidence for the following, appearing in the work of W. Chen [1].
Conjecture: the only contact type hypersurfaces in $\mathbb{R}^4$ having vanishing first Betti number are diffeomorphic to the 3-sphere.

The topology of polynomially convex domains

There are a very limited number of examples of polynomially convex domains in $\mathbb{C}^2$. In fact, Cieliebak and Eliashberg [3] pose the following problem.
Conjecture: A simply connected polynomially convex domain in $\mathbb{C}^n$, $n > 2$, must be *subcritical*, meaning it admits a defining Morse function whose critical points all have index less than n. When $n = 2$, an analogous question would ask whether a polynomially convex domain must be a 1-handlebody.

While it concerns a strictly complex-geometric property (polynomial convexity), this conjecture seems closely tied to subtle issues in symplectic topology: notably the poorly-understood distinction between Liouville and Weinstein symplectic cobordisms. One expects that progress on this problem [4, 5], from either the complex-geometric or symplectic-topological side, will shed light on the other subject.

The topology of Stein boundaries

Many more examples are known, thanks in large part to work of Gompf, of 3-manifolds bounding pseudoconvex domains in $\mathbb{C}^2$, or equivalently Stein domains. In particular, Gompf [7] gave examples of 3-manifolds with vanishing first Betti number that bound Stein domains—even contractible Stein domains. Moreover, Gompf used his techniques to exhibit a Stein domain in $\mathbb{C}^2$ having the homotopy type of the 2-sphere, disproving a conjecture in complex geometry. However, it is still poorly understood which 3-manifolds can arise as the boundary of pseudoconvex domains in $\mathbb{C}^2$ (even among restricted classes such as Brieskorn spheres).

We hope that the sharing of expertise engendered by this workshop will spark progress on this question, particularly the following fascinating conjecture due to Gompf.
Conjecture: no Brieskorn sphere bounds a pseudoconvex domain in $\mathbb{C}^2$.

Convexity in other manifolds

Both symplectic convexity and pseudoconvexity are conditions that can be considered in more general spaces than $\mathbb{C}^n$. A natural next step is to consider domains in complex projective space, where several intriguing avenues are available.

Contact type hypersurfaces in $\mathbb{C}P^2$

Work of Evans and Smith shows that the family of lens spaces $L(p, q)$ that can be found as contact type hypersurfaces in the projective plane is quite constrained: for example, p must satisfy a certain Diophantine equation, and no more than three lens spaces may be embedded disjointly in this way. In fact, Evans-Smith [6] show that their criteria completely determine the lens spaces that arise as contact type hypersurfaces. The results, and particularly the techniques used (which involve pseudoholomorphic curves in orbifolds), are very suggestive of potential further progress. In one possible direction, smooth embeddings of lens spaces and other Seifert 3-manifolds in (orbifold) projective planes is closely related to the longstanding "Montgomery-Yang problem" of classifying smooth circle actions on the 5-sphere: this problem in smooth topology has resisted progress for decades.

Convexity and smooth topology

In some circumstances, a resolution of a problem involving complex or symplectic convexity may actually yield the resolution of a question in *smooth* topology. For example, Weimin Chen observed that obstructing a given 3-manifold from arising as a contact type hypersurface in $\mathbb{C}P^2$ is one step in a possible approach to constructing a smooth 4-manifold homeomorphic but not diffeomorphic to $\mathbb{C}P^2$. The existence, or not, of such an example is one of the major open questions in 4-dimensional topology, and it is natural to hope that Chen's techniques (and those of Evans-Smith above) may have adaptations or extensions that could lead to progress in this direction.

Perhaps the most important outstanding question in low-dimensional topology is the last remaining case of the Poincaré conjecture: that the 4-dimensional sphere has a unique differentiable structure. By the result of Eliashberg that the 3-sphere has a unique Stein filling, this has a reformulation in terms of pseudoconvexity: it is equivalent to the assertion that any compact contractible 4-manifold having boundary diffeomorphic to the 3-sphere admits the structure of a Stein domain [8].

On the other hand, this uniqueness vanishes if one considers open manifolds. Indeed Gompf proves that every open pseudoconvex convex domain contains an uncountable family of other such pseudoconvex domains, all of which are homeomorphic to the original but pairwise non-diffeomorphic. In particular, this yields uncountably many diffeomorphism types of domains of holomorphy in $\mathbb{C}^2$ homeomorphic to $\mathbb{R}^4$.

Working groups and open problems

As mentioned before the most essential purpose of the workshop was to bring together people from different areas (complex, symplectic, contact, and smooth topology/geometry) and initiate conversations to exchange knowledge, ideas, and questions related to different types of convexity and related problems. To contribute this, time was dedicated for discussion groups on Tuesday and Thursday of the conference week, each for at least 1.5 hours. On the first day of the conference, during an organizational meeting with all participants we made the following preliminary list of open problems and research topics.

1. **Gompf's Conjecture.** No Brieskorn sphere admits a pseudoconvex embedding in $\mathbb{C}^2$ with either orientation.

2. Is there a difference between Stein and Weinstein and Liouville cobordisms?

3. In high dimensions, there are infinitely many distinct Weinstein domains which are diffeomorphic to each other. Are there infinitely many diffeomorphic Weinstein structures which are not equivalent? (What if we control the Chern class?)
4. When is a (Lagrangian) cobordism of quasipositive knot (filling of a quasipositive knot) ribbon?
5. Given a Stein manifold, if you remove a holomorphic set, then the complement is Stein. If you remove a pseudoholomorphic set, do we know if the complement is Stein?
6. There are various types of convexity. Can embedded contact homology capacities be used to obstruct the different notions of convexity?
7. Can $\mathbb{R}P^2$ or S^2 be topologically embedded in $\mathbb{C}^2$ as a rationally convex subset?
8. Understand which singular Lagrangians in $\mathbb{C}^2$ have rationally convex neighborhoods, where we allow the singularities to be say cones over Legendrian torus knots.

3 Presentation Highlights

The first three days of the workshop included plenary lectures intended to introduce audience members to the specialties of the plenary speakers, roughly representing smooth, symplectic, and complex geometric aspects of the topic. The plenary lectures were preceded by introductory lectures by promising junior researchers. Additional lectures were presented by researchers in a variety of specialties, all roughly centered on the theme of convexity.

3.1 Introductory and Plenary Talks

- **Smooth topology and complex geometry**

  Kyle Hyden

  Title: The smooth topology of Stein manifolds

  This introductory lecture will explore the basic smooth topology of manifolds admitting Stein structures, with a focus on Stein surfaces (i.e., those of real dimension 4). Guided by the natural Morse functions carried by Stein manifolds, we will unpack Eliashberg's topological characterization of Stein manifolds, Gompf's handlebody construction of Stein surfaces, and the adjunction inequality. We will close with an application to the existence of exotic smooth structures on 4-space.

  Bob Gompf

  Title: Smooth and topological pseudoconvexity in complex surfaces

  Abstract: We will discuss several general tools for finding strictly pseudoconvex subsets of complex surfaces. An open subset U is smoothly isotopic to a Stein open subset if and only if its inherited complex structure is homotopic (through almost-complex structures) to a Stein structure on U. If we allow topological isotopy (homotopy through homeomorphic embeddings with no differentiability assumed), the condition on the complex structure can be dropped, and it is only necessary for U to admit a topological Morse function whose critical points have index at most 2. A deeper version of this shows that every finite 2-complex in a complex surface is topologically isotopic to a Stein compact, in fact, to a nested intersection of uncountably many homeomorphic Stein open subsets. This leads to a notion of pseudoconvexity for unsmoothably embedded 3-manifolds. We discuss examples and applications of such phenomena, with the hope of encouraging further exploration with these tools.

- **Complex geometry and convexity**

  Blake Boudreaux

  Title: Holomorphic convexity in several complex variables

  Abstract: In 1906, F. Hartogs discovered the existence of domains in $\mathbb{C}^n$ for which every holomorphic function can be extended to a larger domain. Domains that do not admit this extension phenomenon satisfy a complex type of convexity, known as pseudoconvexity. This type of convexity can be viewed as convexity "with respect to holomorphic functions", as opposed to geometric convexity, which is convexity "with respect to linear functions". In this talk, we will motivate and define pseudoconvexity. We will also compare and contrast its many equivalent formulations with that of classical convexity. We will also introduce a class of "pseudoconvex" manifolds and discuss their many properties. Notions of convexity with respect to other classes of functions will also be discussed.

  Resul Shafikov

  Title: Polynomial and Rational Convexity

  Abstract: In the first half of the talk I will give an overview of polynomial and rational convexity: I will give basic definitions, examples and outline some fundamental properties of polynomial and rationally convex compacts. In the second half of the talk I will discuss characterization of rational convexity of real submanifolds in complex Euclidean spaces and related problems.

- **Symplectic geometry**

  Joé Brendel

  Title': Toric reduction and applications

  Abstract: In this introductory lecture, we focus on a special case of symplectic reduction, in which the reduction is compatible with a toric group action. We recall the basic notions, discuss an example that will come up in Jonny's lecture and, if time permits, give further applications.

  Jonathan Evans

  Title Open problems around Lagrangian intersections

  Abstract: Let K be a Lagrangian submanifold and L_t be a family of Lagrangian submanifolds. Suppose you can displace K from each L_t. Can you displace K from all L_t simultaneously? If not, from how many L_t can you simultaneously displace K? We will discuss some specific problems which have this flavor and give some small results in this direction.

3.2 Additional Research Talks

- Luya Wang

  A connected sum formula of embedded contact homology

  Abstract: The contact connected sum is a well-understood operation for contact manifolds. I will focus on the 3-dimensional case and the Weinstein 1-handle model for the contact connected sum. I will discuss how pseudo-holomorphic curves in the symplectization behave under this operation. After reviewing embedded contact homology, we will see how this results in a chain-level description of the embedded contact homology of a connected sum.

- Joseph Breen

  Title: The Giroux correspondence in all dimensions

  Abstract: Twenty years ago, Giroux gave an influential result on the equivalence of contact structures in dimension 3 and open book decompositions up to stabilization. At the time, Giroux and Mohsen also partially extended the correspondence to all dimensions, albeit with different technology. From one point of view, the existence of open book decompositions can be viewed as a convexity statement for contact manifolds, and there are natural connections to symplectic convexity. In this talk, I will describe forthcoming joint work with Ko Honda and Yang Huang on establishing the Giroux correspondence in all dimensions using convex hypersurface theory.

- Purvi Gupta

  Title. Polynomially convex embeddings of compact real manifolds

  Abstract. A compact subset of $\mathbb{C}^n$ is said to be polynomially convex if it is cut out by a family of polynomial inequalities. Polynomial convexity grants certain approximation-theoretic properties to the underlying set. When the set is a real submanifold of $\mathbb{C}^n$, its convexity properties are partly influenced by its topology, and the local and global structure of its CR (complex-real) singularities. The minimum complex dimension into which all compact real manifolds of a fixed dimension admit smooth polynomially convex embeddings is not known (although some bounds can be deduced from the literature). In this talk, we will discuss some recent improvements on the previously known bounds. We will especially focus on the case where the h-principle has proved useful for producing the desired embeddings. This is joint work with R. Shafikov.

- Stefan Nemirovski

  Title: Complex Analysis 2.0

  Abstract: Peculiar features of low-dimensional differential, symplectic/contact topology affect the theory of holomorphic functions of two complex variables. The purpose of the talk will be to illustrate this principle with a few token examples and discuss open problems and possible research directions in this area.

- Marko Slapar

  Title: Representing homology classes of complex hypersurfaces in $\mathbb{C}P^3$

  Abstract: Thom conjecture, proven by Kronheimer and Mrowka in 1994, states that complex curves in $\mathbb{C}P^2$ are genus minimizers in their homology class. We will show that an analogous statement does not hold for complex hypersurfaces in $\mathbb{C}P^3$. This is joint work with Ruberman and Strle.

- Giancarlo Urzua

  Title: Exotic 4-manifolds and KSBA surfaces

  Abstract: Although exotic blow-ups of the complex projective plane at n points have been constructed for every $n > 1$, the only examples known by means of rational blowdowns satisfy $n > 4$. It has been an intriguing problem whether it is possible to decrease n. In this talk, I will show how to construct it for $n = 4$ from a configuration of 8 lines and 2 conics in a special position. This is part of a bigger picture to construct exotic $p\mathbb{C}P^2 \# q\mathbb{C}\bar{P}^2$ via the construction of particular Kollár–Shepherd-Barron–Alexeev (KSBA) singular surfaces. This is done by explicitly analyzing obstructions coming from

configurations of rational curves, and the use of computer searchers. This connection between the geography of configuration of rational curves and exotic 4-manifolds from KSBA surfaces leads to, I believe, a new view on this problem. There is a lot of data out of these searches, showing an intricate picture for KSBA surfaces. I hope to show that in this talk too. This is joint work with Javier Reyes.

- Angela Wu

  Title: On Lagrangian quasi-cobordisms

  Abstract: A Lagrangian cobordism between Legendrian knots is an important notion in symplectic geometry. Many questions, including basic structural questions about these surfaces are yet unanswered. For instance, while it is known that these cobordisms form a preorder, and that they are not symmetric, it is not known if they form a partial order on Legendrian knots. The idea of a Lagrangian quasi-cobordism was first defined by Sabloff, Vela-Vick, and Wong. Roughly, for two Legendrians of the same rotation number, it is the smooth composition of a sequence of alternatingly ascending and descending Lagrangian cobordisms which start at one knot and ends at the other. This forms a metric monoid on Legendrian knots, with distance given by the minimal genus between any two Legendrian knots. In this talk, I will discuss some new results about Lagrangian quasi-cobordisms, based on some work in progress with Sabloff, Vela-Vick, and Wong.

- Kyler Siegel

  Title: On rational curves with cusps and double points

  Abstract: A classic question in algebraic geometry asks what are the possible singularities for a plane curve of a given degree and genus. This is closely related to existence questions for singular Lagrangian surfaces in the affine or projective complex plane, which in turn connect with questions about the topology of rationally convex domains. In this talk I will describe a construction of various new families of rational plane curves with prescribed singularities, and I will wax poetic about how this ties in with the themes of this workshop.

- Morgan Weiler

  Title: ECH spectral invariants for toric contact forms

  Abstract: The embedded contact homology (ECH) chain complex has several natural filtrations, and applications of ECH to symplectic and contact geometry often rely on computing the associated spectral invariants. When the three-manifold is spherical (or more generally, toric), this means there is a precise function from ECH index to the minimal filtration value among cycles representing that index's homology class. ECH practitioners attempt to compute or estimate these functions. We will explain why those attempts are much more successful in the case of convex toric contact forms, including applications of the ECH knot filtration to surface dynamics and the ECH action filtration (aka the ECH spectrum) to symplectic embedding problems. The latter project is based on joint work with several coauthors, in arXiv:2010.08567, arXiv:2203.06453, and arXiv:2210.15069.

- Oliver Edtmair

  Title: Convexity, Hamitonian dynamics and symplectic embeddings

  Abstract: I will motivate several notions of convexity that play important roles in Hamiltonian dynamics and in the theory of symplectic embeddings. In particular, I will focus on the mysterious role convexity plays in Viterbo's conjecture on the systolic ratio and the symplectic capacities of convex domains in Euclidean space. I will end my talk by reviewing some recent progress towards this conjecture.

- Sümeyra Sakalli

  Title: Singular fibers in algebraic fibrations of genus two and their monodromy factorizations

  Abstract: Kodaira classified all singular fibers that can arise in algebraic elliptic fibrations. Later, Ogg, Iitaka and then Namikawa and Ueno gave a classification for genus two fibrations. In this work, we split these algebraic genus two fibrations into Lefschetz fibrations and determine the monodromies. More specifically, we look at four families of hypersurface singularities in $\mathbb{C}^3$. Each hypersurface comes equipped with a fibration by genus 2 algebraic curves which degenerate into a single singular fiber. We determine the resolution of each of the singularities in the family and find a flat deformation of the resolution into simpler pieces, resulting in a fibration of Lefschetz type. We then record the description of the Lefschetz fibration as a positive factorization in Dehn twists. This gives us a dictionary between configurations of curves and monodromy factorizations for some singularities of genus 2 fibrations. This is joint work with J. Van Horn-Morris.

4 Scientific Progress Made

For Tuesday discussion session session we had participants sorted into four groups, each of which had a group leader to facilitate/guide the discussions.

- **Group 1—Hutchings (Room 102)**

  Gupta, Lambert-Cole, Roy, Auyeung, Christian, Capovilla-Searle.

- **Group 2—Siegel (Room 106)**

  Boudreaux, Urzua, Gompf, Brendel, Knavel, Wan, Rodewald.

- **Group 3—Dimitroglou Rizell (Room 107)**

  Shafikov, Min, Ono, Choi, Magill, Nelson, Wang.

- **Group 4—Slapar (Room 202)**

  Wu, Sakalli, Park, Weiler, Lazarev, Breen, Edtmair.

For Thursday meeting we encouraged participants to choose and propose some promising focus topics. This was received well and quickly the following focus groups were formed.

- **Rational convexity in closed symplectic manifolds.**

  Shafikov, Dimitroglou-Rizell, Gupta, Mark, Lazarev, Tosun, Gompf, Boudareaux.

- **Symplectic and Algebro-Geometric approaches to Markov/Unicity Conjecture.**

  Urzua, Brendel, Park, Sakalli, Ono, Capovilla-Searle.

- **ECH contact class.**

  Nelson, Hutchings, Weiler, Magill, Lambert-Cole, Roy, Choi, Min.

Reports from the participants on their experiences and on the progress made were very positive. Particular items mentioned included:

- New examples of polynomially convex domains in $\mathbb{C}^2$ that are not subcritical, resolving a question mentioned in Section 2.

- Development of a strategy to extend results on rationally convex domains in $\mathbb{C}^2$ to other manifolds.
- Deeper understanding and study of the existence problem for rationally convex, topologically embedded 2-spheres in $\mathbb{C}^2$.
- Discussions and interactions among participants from different specialties on wide-ranging topics including Mori theory, exotic 4-manifolds, Floer theory of various sorts, and the subtle relations between Weinstein domains, rational convexity and the Stein condition.

5 Outcome of the Meeting

Feedback communicated to the organizers by participants was uniformly positive. Participants were particularly appreciative of the broad selection of topics and mathematically diverse group, which was described as unusual, refreshing, and mathematically beneficial.

Participants also praised the format, including the overview lectures and the organization of working groups.

Progress on particular mathematical topics was noted in the previous section; some participants have indicated that at least one preprint is in preparation on work initiated at this meeting.

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