Recent Advances in Banach lattices

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May 07 — May 12, 2023

This workshop focused on recent developments in the area of Banach lattices. The goal of the workshop was to bring together leading experts and active young researchers to discuss the current and future directions of these developments and to identify potential applications and the main open problems. We planned to understand the "big picture" of connections between these developments and other areas of Functional Analysis.

1 Overview of the Field

Various spaces of functions are ubiquitous in Analysis and Applications. Function spaces are the central objects of study of Functional Analysis. A *Banach space* is an abstraction of a function space, hence Banach space theory has been central to Functional Analysis. However, while Banach space theory captures the algebraic and the metric structures of function spaces, it "does not see" their order structures. Most classical function spaces have natural orders in the sense that for two functions f and g, one considers $f \leq g$ when $f(t) \leq g(t)$ for all t or for "almost all" t, depending on the context. Moreover, in most function spaces, this order is a lattice order. This naturally brings us to Banach lattices. Just as a *Banach space*, a *Banach lattice* is an abstraction of a function space, but it also takes its order into account. So a Banach lattice is a Banach space equipped with a lattice order, satisfying appropriate compatibility axioms. Most of the classical function spaces that appear in modern Functional Analysis are Banach lattices.

Banach lattice theory is a synthesis of Banach space theory and vector lattice theory. A *vector lattice* (Riesz spaces) is a vector space which is also a lattice, such that the two structures are compatible in a certain natural way. Vector lattices themselves form a rich and interesting category. Understanding of vector lattices is critical for Banach lattice theory.

The unofficial beginning of this area was the address of F. Riesz at the ICM in 1928, followed by works of H. Freudenthal and L. Kantorovich on vector lattices. In its early days (around the 1950s), most activities were devoted to vector lattices and were concentrated in several major schools: the Soviet school (L. Kantorovich and the Krein brothers), the Japanese school (H. Nakano and T. Ando), the German school (H.H. Schaefer), and the Dutch school (A.C. Zaanen and W. Luxemburg). Since then, the area has expanded and has developed connections with many other areas of mathematics, including Banach spaces, which led to the rise of Banach lattices. Several major advances in Banach lattices were made around the 1970s by H.H. Schaefer, P. Meyer-Nieberg, J. Lindenstrauss, L. Tsafriri, W. Johnson, and N. Ghoussoub. To mention a few other important connections (which were not addressed at the workshop): Spectral Theory of positive operators, Perron-Frobenius Theory of non-negative matrices, Ergodic Theory, Operator Semigroups, Convex Analysis, Inequalities, etc. There have been numerous applications of vector and Banach lattice to Math

Economics and Math Finance. These applications go back to works of L. Kantorovich and, more recently, of C. Aliprantis and I. Polyrakis. In these applications, vector and Banach lattices are used to model markets and the theory of positivity is used in the search for a market equilibrium.

The subject area of vector and Banach lattices (which is nowadays often referred to as "*Positivity*") is old and well established, and many parts of it have become classical. Members of the Positivity community have been distinguished in various ways. L. Kantorovich received the Nobel Prize in Economics for applications of Positivity in Economics. H. Schaefer was a member of the Mathematics and Natural Sciences Class of the Heidelberg Academy of Sciences and of the Academy of Sciences in Zaragoza. C. Aliprantis was a distinguished professor of Economics and Mathematics at Purdue University; he was also the founding editor of the journals Economic Theory and Annals of Finance. W. Luxemburg was a Fellow of the American Mathematical Society; he, A. Zaanen, and H. Freudenthal were members of the Royal Netherlands Academy of Arts and Sciences. In 2000, the International Commission on Mathematical Instruction of the International Mathematical Union instituted the Hans Freudenthal Medal. B. Johnson was awarded a Stefan Banach Medal in 2007. G. Curbera was a Curator at the International Mathematical Union over the period of 2011–2014.

There have been many regular and incidental conferences and workshops. In the 70s–80s, there was a series of workshops on Riesz spaces in Oberwolfach: June 24–30 1973, June 22–28 1975, June 19–25 1977, June 24–30 1979, June 27 – July 3 1982, July 1–5 1985, and April 23–29 Apr 1989. Currently, *Positivity* conferences are normally held every second year; they are the main events for the Positivity community. Below we list the locations of past Positivity conferences.

- Ankara, Turkey, 1999
- Nijmegen, Netherlands, 2001
- Rhodos, Greece, 2003
- Dresden, Germany, 2005
- Belfast, UK, 2007
- Madrid, Spain, 2009
- Leiden, Netherlands, 2013
- Chengdu, China, 2015
- Edmonton, Canada, 2017
- Pretoria, South Africa, 2019
- Ljubljana, Slovenia, 2023 (forthcoming)

Here we list a few recent "incidental" conferences and workshops:

- "Positivity in Functional Analysis and Applications" session at the 2006 Summer Meeting of the Canadian Mathematical Society, Calgary, Alberta.
- Symposium on Positivity and Its Applications in Science and Economics, Bolu, Turkey, September 2008.
- Workshop on Nonnegative Matrix Theory, American Institute of Mathematics, Palo Alto, California, December 2008.
- Ordered Spaces and Applications Conference, National Technical University of Athens, Greece, November 2011.
- Ordered Banach Algebras Workshop, Lorentz Center, Leiden, Netherlands, July 2014.
- Workshop on Operators and Banach lattices I and II, Universidad Complutense de Madrid, October 2012 and November 2016.

- Workshop on Recent Advances in Banach lattices, BIRS, Oaxaca, April 2018.
- Workshop on Banach spaces and Banach lattices I and II, ICMAT, Spain, September 2019 and May 2022.
- Conference on Ordered Structures and Applications, Tozeur, Tunisia, January/February 2023,
- Workshop on Ordered Vector Spaces and Positive Operators, Bergische Universität Wuppertal, Germany, March/April 2023

For further information about the field, we refer the reader to the following classical monographs: [1, 2, 23, 25, 26, 30].

2 Presentation Highlights — Recent developments

All talks at the workshop were devoted to important recent discoveries in the area of Banach lattices. Most of the talks were grouped by subject.

2.1 Free Banach lattices

A vector lattice X is a *free vector lattice* over a set A if $A \subseteq X$ and every map from A to an arbitrary vector lattice Y extends uniquely to a vector lattice homomorphism from X to Y. While free vector lattices have been known for a long time, see, e.g., [7], this area came to prominence thanks to a a recent paper [27], where B. de Pagter and A. Wickstead proved the existence of free Banach lattices over sets and characterized some of their properties. Then in [3], A. Avilés, J. Rodríguez, and P. Tradacete came up with the concept of a free Banach lattice over a Banach space. A Banach lattice X is a free Banach lattice over a Banach space E if E is a closed sublattice of X and every bounded operator T from E to an arbitrary Banach lattice Y extends uniquely to a lattice homomorphism from X to Y with the same norm. Not only did the authors of [3] prove that such an X exists, but they also found a representation of it as a function space, as a sublattice of the space of weak*-continuous functions on the unit ball of E^*. This was a major impetus to the area. This approach created a framework to study subspaces of Banach lattices as Banach spaces. From a categorical point of view, this can be seen as a functor from the category of Banach spaces and bounded linear operators into the subcategory of Banach lattices and lattice homomorphisms. It has become clear that understanding this functor is a key to properly understand the interplay between Banach space and Banach lattice properties, a goal that has been pursued ever since the first developments of these theories.

[3] led to numerous recent results, including [21], where free Banach lattices were constructed in several other categories, [29], where various properties of there spaces were established, [20], where complex free Banach lattices were constructed, and [12], where free Banach lattices were constructed in the category of dual spaces.

The workshop had three talks on free Banach lattices. The first talk, delivered by M. de Jeu, presented a historic overview of free objects in the framework of universal algebras, including free vector lattices and free vector lattice algebras. It followed by a talk by T. Oikhberg, outlining some of the results of [29], with a focus on connections between lattice homomorphisms between free Banach lattices and positively homogeneous weak*-continuous maps between the dual unit balls of the underlying spaces. Finally, D. Leung presented a very recent construction of free objects in the category of Banach lattices with upper p-estimates over Banach spaces.

2.2 Stable phase retrieval

Let X be a subspace of a Banach lattice E. Suppose that for every $f \in X$, one can recover f from $|f|$, up to a scalar multiple λ with $|\lambda| = 1$. We then say that X has a *phase retrieval* property. This has numerous applications as it means that one can recover a signal by only measuring its magnitude. Notable examples in physics and engineering which require phase retrieval include X-ray crystallography, electron microscopy, quantum state tomography, and cepstrum analysis in speech recognition.

In practice, we often have to recover f from $|Tf|$, where T is a linear transformation, e.g., the Fourier transform. As any application of phase retrieval would involve error, it is of fundamental importance that the recovery of f from $|Tf|$ not only be possible, but also be stable. If, moreover, we can do the retrieval in a Lipschitz-continuous way, we say that the subspace has *stable phase retrieval*.

The issue of phase retrieval has existed in applied mathematics (in particular, in frame theory) for a long time; see, e.g., [8]. Only recently it was recognized that the problem may often be solved using Banach lattice techniques. In the last few years, several papers and preprints have appeared, solving the stable phase retrieval problem for certain classes of subspaces of Banach lattices: [9, 10]. It has been discovered that stable phase retrieval problem is closely related to of asymptotically disjoint sequences explored by V. Kadec and O. Pełczyński.

At the workshop, M. Taylor and D. Freeman presented an overview of the subject and recent advances. They also proposed several open questions. This area of research clearly has a great potential.

2.3 Convergence structures

It has been known for a long time that several important convergences of sequences or nets in the theory of vector and Banach lattices are not topological. In particular, order convergence, unbounded order convergence, and relative uniform convergence are not induced by any topologies. These are not some exotic convergences: as a special case, almost everywhere convergence of sequences of measurable functions is not topological. These convergence came to the forefront a few years ago, when unbounded order convergence found applications in Math Economics, [13, 14, 15, 16, 17, 18].

On the other hand, there has long existed a theory of *convergence structures*, see, e.g., [6]. However, this theory was formulated in terms of filters, which made it less suitable for applications in analysis. It was only recently realized that the theory of convergence structures might equivalently be reformulated in terms of nets. This allows one to study convergences in vector and Banach lattices using the framework of convergence structures, see [28].

At the workshop, J.H. van der Walt presented an overview of the theory of convergence structures and showed how it could be applied to convergences in vector lattices. In particular, he introduced locally solid convergent structures, that naturally generalize locally solid topologies. E. Bilokopytov then presented recent applications of this theory to various problems in Banach lattices. In particular, he discussed relationship between adherences (or closures) of a sublattice with respect to various convergence structures.

2.4 Linear versus lattice embeddings

If a Banach lattice X has a closed subspace isomorphic to c_0, then X contains a closed sublattice that is lattice isomorphic to c_0. This result is one of the gems of Banach lattice theory; it goes back to P. Meyer-Nieberg in the 80s. It has been a long-standing open question whether the same is true for $C[0,1]$; some partial results were obtained in [24, 19]. The question was answered in the affirmative in [5]. Moreover, the authors characterized all Banach lattices with this property. At the workshop, G. Martinez-Cervantes presented a clear overview of these results and their proofs.

A motivation and a key ingredient in the proof is a recent result by D.H. Leung, L. Li, T. Oikhberg and M.A. Tursi that every separable Banach lattice embeds lattice isometrically into $C(\Delta, L_1)$, that was presented during the previous BIRS workshop in 2018. Another key ingredient is the recent concept of projectivity for Banach lattices introduced by B. de Pagter and A.W. Wickstead in their free Banach lattice paper [27]

2.5 Complemented subspace problem

It has been a long-standing open problem whether every complemented subspace of a Banach lattice is itself isomorphic to a Banach lattice. Recently, this problem was solved in the non-separable case. This result was presented at the workshop by D. de Hevia.

3 Building bridges between areas

An important feature of this meeting was building bridges between areas. This was reflected in the choice of participants and in the subjects of the presentations.

Since the theory of Banach lattices rests on two pillars, vector lattices and Banach spaces, special care was taken to bring together experts on these two topics. Several experts on Banach space theory were invited and connections between Banach lattices and Banach spaces were discussed at length. In particular, a considerable portion of the workshop was devoted to free Banach lattices over Banach spaces; in this subject the relationship between Banach spaces and Banach lattices is very important, so having experts in Banach spaces was invaluable.

There are also clear connections between free Banach lattices and Lipschitz-free metric spaces. To every metric space M, it is possible to assign a Banach space $F(M)$ generated by M in such a way that Lipschitz maps between metric spaces are converted into bounded linear operators between the corresponding Banach spaces. We call Banach space $F(M)$ the Lipschitz-free space over M (also known as the Arens-Eells space or Transportation Cost Space). The above universal property makes Lipschitz-free spaces an important tool in functional analysis because it allows the application of linear methods to nonlinear problems. While Lipschitz-free spaces are not Banach lattices, they are Banach spaces and vector lattices. The workshop was a natural place to explore connections between these structures. This connection was presented at the workshop by E. Pernecká.

Many problems in Banach lattice theory may be reduced to questions about spaces of continuous functions on topological spaces. This justified inviting several experts on such spaces. In particular, the solution to the Complemented Subspace problem, presented at the workshop, was based on a recent result about spaces of continuous functions [4].

P. Tradacete delivered a talk on *valuations* in Banach lattices. This connects Banach lattice theory with various applications, including measures, volumes, and finite-dimensional convex geometry.

Recent advances in convergence structures in vector lattices naturally suggest a link to convergence structures in general lattices and in even more general ordered sets. In the last few years, several results about order and unbounded order convergences in vector lattices have been extended to general lattices. It was therefore helpful to have several experts on this subject at the workshop.

4 Outcome of the Meeting

The meeting reached two main goals.

First, it exposed some very recent important advances (many of them not yet even published) to the community. So it contributed to dissemination of results, techniques, and ideas.

Second, it provided an environment for informal discussions. During the workshop, participants would often split into small groups exchanging ideas, making connections, and starting new collaborations and new projects. We expect that this workshop will result in several new projects.

In addition, it was helpful in bringing young mathematicians into the area. Several graduate students and postdocs participated in the workshop. The workshop provided them with opportunities to join the cutting edge research in this area, both by attending talks and by talking to other participants.

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