Spectral Synthesis and Weak Amenability of Uniform Algebras

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1 Overview of the Research Project

The goal of the research project we pursued while at BIRS is to solve two distinct, but related, longstanding problems concerning the relationship between uniform algebras and two important Banach algebra properties, spectral synthesis and weak amenability, with connections to abstract harmonic analysis.

A uniform algebra on a compact Hausdorff space X is an algebra of continuous complex-valued functions that contains the constants, separates the points of X, and is uniformly closed in the algebra $C(X)$ of all continuous complex-valued functions on X. On every compact Hausdorff space there is the trivial example of a uniform algebra, namely $C(X)$ itself. A typical example of a nontrivial uniform algebra is the disc algebra which consists of the continuous complex-valued functions on the closed unit disc that are holomorphic on the open unit disc. The uniform algebras form a class of Banach algebras that is important both in the field of Banach algebras and in complex analysis. Uniform algebras also have applications to operator theory. The particular properties we are investigating, spectral synthesis and weak amenability, are of fundamental importance in abstract harmonic analysis.

As the example of the disc algebra suggests, there is a close connection between uniform algebras and analyticity, and there were once a number of conjectures that asserted the existence of various forms of analyticity associated with every nontrivial uniform algebra. These conjectures were disproved in the 1960s. However, in the current century, we and our coauthors have shown that under a variety of additional natural hypotheses, these conjectures are in fact true, and we have also constructed additional counterexamples further elucidating the extent to which the conjectures fail in general [1–6, 10–13, 15–22]. We believe that the time is ripe for developing a much better understanding of the connection between analyticity and uniform algebras and Banach algebras more generally.

One form of analyticity that was once conjectured to be present in every nontrivial uniform algebra was the existence of nonzero point derivations. For bounded point derivations, Wermer refuted this by constructing a Swiss cheese K (a type of compact planar set) such that $R(K)$, the algebra of complex-valued functions on K approximable by holomorphic rational functions with poles off K, is a counterexample [23]. Later a construction of Cole showed that even unbounded point derivations can fail to exist [9].

The problems we are aiming to solve concern two ways in which the condition that a bounded point derivation exists can be relaxed. One is to replace point derivations with derivations into an arbitrary commutative Banach module. The other way arises from reformulating the existence of nonzero bounded point derivations in terms of ideals. We then replace the existence of a nonzero bounded point derivation by the weaker condition that there exists a closed ideal that is not an intersection of maximal ideals. The first approach leads to the notion of weak amenability, the second approach to the notion of spectral synthesis.

A commutative Banach algebra A is *weakly amenable* if every bounded derivation from A into a commutative Banach A-module is zero. By a theorem of Bade, Curtis, and Dales [7], it is equivalent to require only that every bounded derivation into the dual A-module A^* is zero. A commutative Banach algebra A has *spectral synthesis* if for every closed set E in the maximal ideal space of A and every closed ideal I in A we have $\mathrm{hull}(\ker(E)) = E$ and $\ker(\mathrm{hull}(I)) = I$.

2 Objective

Our objective is to answer the following two questions which are over 30 years old:

1. Does there exist a nontrivial uniform algebra that is weakly amenable?
2. Does there exist a nontrivial uniform algebra with spectral synthesis?

3 Scientific Progress Made

We devoted most of our time at BIRS to investigating possible approaches to the construction of a nontrivial weakly amenable uniform algebra. Since the principal way of constructing nontrivial uniform algebras with no nonzero bounded point derivations is Cole's method of iteratively adjoining square roots, we began by investigating the effect that adjoining square roots to a uniform algebra A has on bounded derivations from A into the dual A-module A^*. We found that the effect on bounded derivations into the dual module is completely different from the effect on bounded point derivations.

If one starts with a uniform algebra A on which there is a nonzero bounded point derivation d, and one adjoins a square root to a suitable function in A, one obtains a new uniform algebra, containing A as a subalgebra, to which the point derivation d does not extend. Furthermore, if one adjoins square roots to a suitable family of functions in A, one obtains a uniform algebra containing A to which none of the bounded point derivations on A extend. There may be new nonzero bounded point derivations on the the new uniform algebra, but by iterating the process of adjoining square roots, and passing to a limit, one eventually obtains a uniform algebra on which there are no nonzero bounded (or even unbounded) point derivations.

Each bounded point derivation on a uniform algebra A induces in a natural way a corresponding bounded derivation into the dual A-module A^*. Surprisingly, we found that the operation of adjoining a square root that eliminates a bounded point derivation does not in general eliminate the corresponding derivation into the dual A^*. Specifically, starting with the disc algebra $A(D)$ and adjoining a square root to the identity function z, the bounded derivation $A(D) \to A(D)^*$ induced by differentiation at the origin *does* extend to a bounded derivation $A(D)[\sqrt{z}] \to A(D)[\sqrt{z}]^*$. (Of course the extended derivation is *not* induced by a point derivation on $A(D)[\sqrt{z}]$.) Furthermore, the norm of the extended derivation equals the norm of the original derivation. The uniform algebra $A(D)[\sqrt{z}]$ is isomorphic to the disc algebra $A(D)$, so we can iterate the process. Finally, we can pass to a limit to obtain a new uniform algebra that is *not* isomorphic to the original uniform algebra. However, because at each step of the extension process the norm of the derivation remains the same, the original derivation extends even to the final limit uniform algebra.

More generally, we found that every bounded derivation $A(D) \to A(D)^*$ extends to a bounded derivation $A(D)[\sqrt{z}] \to A(D)[\sqrt{z}]^*$. Here a curious situation arises. We do not know whether, in general, the norm of the extended derivation exactly matches the norm of the original derivation, but we showed that there is an equivalent norm, arising from an isomorphism of the space of derivations $A(D) \to A(D)^*$ with a function space given by work of Choi and Heath [8], under which the norm of the extension *does* coincide with the norm of the original derivation. Consequently, even after iterating the process and passing to a limit, *every* bounded derivation on the original uniform algebra extends to a bounded derivation on the final limit uniform algebra.

We also looked at the possibility of adapting Wermer's Swiss cheese construction to obtain a nontrivial weakly amenable uniform algebra. Here too the elimination of bounded derivations into the dual module appears to be very different from the elimination of bounded point derivations.

As a result of our investigations we obtained a much greater understanding of the weak amenability problem for uniform algebras. Since the only known methods of eliminating bounded point derivations seem to be incapable of eliminating all bounded derivations, we now think that it is likely that there are *no* nontrivial weakly amenable uniform algebras. Our next step, therefore, will be to dissect the proof that every *amenable* uniform algebra is trivial and attempt to push it further to cover the case of weakly amenable uniform algebras.

We also discussed the spectral synthesis problem. However, we spent only a little time on that since we focused on the weak amenability problem where we found we were making more progress.

We also made use of our time together at BIRS to revise a draft paper we wrote earlier on weak sequential completeness of uniform algebras. Being together at BIRS enabled us to find a more elementary proof that every weakly sequentially complete uniform algebra is finite-dimensional. The paper has now been accepted for publication in the Bulletin of the Irish Mathematical Society [14].

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