Multitaper Spectral Analysis

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June 24-26, 2022

1 Overview of the Field

The multitaper method was developed by David Thomson of Bell Labs in the 1970's and 1980's culminating in a landmark 1982 paper "Spectral Estimation and Harmonic analysis" [30]. The method had stemmed from the author's own experimentation and data analysis in communications, especially from the careful inspection of millimeter waveguide measurements using the state-of-the-art in frequency domain statistical analysis of time series, the (windowed) periodogram. Connecting the estimation of spectra with the mathematical development of the prolate spheroidal wave functions of Slepian, Landau, and Pollak [26, 25, 27], as well as the computational methods that made them easy to compute, was the major catalyst for the development of the multitaper method, which went on to inspire forty years of innovation in applied problems from communications systems to seismology to medical applications. The astounding quantitative power of the multitaper quickly caught fire in the applied community when a junior geophysicist at Scripp's Institute of Oceanography, Alan Chave, invited Thomson to give a talk shortly after the 1982 paper was published. Thomson, together with Chave, expanded these methods to obtain simple error estimates using the newly developed theory of the bootstrap. Chave used these to study the electrical conductivity of the earth, among other things. Other early adopters included Frank Vernon, a seismologist who is now principal investigator on the USArray, a transportable seismic array placed in the continental US capable of such sensitive measurements as to detect not only terrestrial seismic activity, but through the use of spectral methods, can be used for nuclear treaty verification, and can detect small changes due to solar acoustic oscillations in the background seismic hum. (Linda Hinnov, Lou Lanzerotti)

Thomson went on to develop the theory for nonlinear and nonstationary time series and published a landmark 1995 climate paper that showed using a technique called complex demodulation, that the central england temperature series can be seen to contain signatures of anthropogenic climate forcings in the slope of the phase of the complex demodulate. Michael Mann, of the "hockey stick graph", and Jeffrey Park developed the multitaper singular value decomposition which allowed for the multivariate analysis of numerous climate time series in a similar technique to complex demodulation of a single series, to map similar shifts in sea surface temperatures, with the added ability to recognize spatial variations in ocean currents. Advanced methods, such as quadratic inverse theory [32], characterization of numbers of false detections of oscillations [34], and multitaper estimators of autocorrelation [21, 35], higher-order spectra [31, 3] and others, were to come. Additional significant scientific contributions were the applications to solar physics data, starting with [36], and climate change [33].

Meanwhile, the developments of Slepian, Landau, and Pollak became widely known in the mathematical community, with the extensions of the Cartesian and discrete versions to the sphere by F. Alberto Gr unbaum. Now the study of simultaneous "duration and bandwidth limiting" is a subject of a 2012 book [19]. Generalized Slepian sequences on the sphere are currently being applied in the geophysics community [24].

While these significant developments are all relevant to applied problems, this illustrates one particular feature of the multitaper method: one does not necessarily know that multitaper is necessary unless one analyzes real data. Parametric methods for time series, for example, generally revolve around simple statistical assumptions like stationarity or Gaussianity, and statisticians may rely too heavily on asymptotic unbiasedness of conventional estimators. But when the multitaper method is used on real data, it becomes clear that one never approaches the ground truth fast enough with a periodogram, all sample sizes are too small and no time series are collected with enough precision to rival the provable statistical advantages that one obtains with the multitaper method. This is why, forty years after its initial publication, multitaper is well-established as a gold standard for the analysis of data for which real scientific conclusions are desired.

2 Recent Developments

The most recent developments in the field as represented by those in attendance were in the form of mathematical, statistical, and applied contributions. Broadly grouping together the presentations, applied mathematical contributions advancing the study of discrete prolate spheroidal functions and their analogues in higher dimensions and dyadic spaces were the presentations of Grunbaum [13, 6, 16, 18, 17, 24] and the 2012 book of Hogan and Lakey entitled "Duration and Bandwidth Limiting" [19].

On the statistical front, new results were presented by Peter Craigmile and colleagues on the development of a novel semiparametric estimation method for the spectrum [29], and a fully parametric method of Adam Sykulski [28, 14] and others. New theoretical results concerning the optimality of the multitaper bias and variance were presented by Jose Luis Romero [1]. Chave and Haley also brought novel results which made use of the newly-developed missing-data Slepian sequences [5, 15].

Recent applied contributions not mentioned in the highlights to come were the developments of Proloy Das and collegues on the subject of nonstationary and nonlinear time series in the medical context [9, 8, 7] where state space models, modeling dynamics, and Bayesian techniques were drawn in to complement a multitaper paradigm for nonlinear dynamics in electroencephalogram data. Numerous novel results from the seismology laboratories of Frank Vernon, Jeff Park [12, 10, 11, 4] were presented as well as geophysical results from Linda Hinnov (to be discussed later) and Frederik Simons.

This is not an exhaustive list of the recent developments presented in the meeting and those not mentioned here are not particularly more important than the others, but an effort has been taken to give as broad an overview of the strong points of the meeting as is possible, while respecting space constraints. In the next section, we give three presentation highlights.

3 Presentation Highlights

3.1 Connection between prolate spheroidal wave functions and zeros of the zeta function

Published within the last three months of this meeting is the remarkable contribution of Fields Medalist Alain Connes and his long-time collaborator, Henri Moscovici. The paper [6] makes an exciting relation between the zeros of the Riemann zeta function, important in number theory, and the prolate spheroidal functions, important to multitaper spectrum analysis, but more generally relevant to the study of functions which are as time and band limited as possible given physical constraints. The paper outlines a self adjoint extension of the prolate second-order differential operator in the interval $[-\lambda, \lambda]$ to the real line. It is found that the spectrum of this operator has negative eigenvalues in Hilbert space related to the squares of the zeros of the zeta function. A historical overview of this remarkable result was given in [13] and presented during the meeting.

3.2 Nonstationarity

Textbook examples of time series ordinarily assume that the processes involved are wide sense or weakly stationary, or can be broken into blocks that are approximately stationary. Unfortunately, most geophysical and climate time series carry diurnal cycles, solar cycles, tides, and signatures of the Earth's rotation, solar rotation, or both, and characterization of these oscillations is central to the main reasons for their study. For many of these processes, it is appropriate, then, to convert both temporal indices of the autocorrelation function to frequency. A landmark work of Loéve [20] describes a theoretical decomposition for such spectra involving two frequency indices, however no nonparametric estimation technique was put forth. Recent work has focused on the modeling and description of cyclostationary and generalized almost cyclostationary processes [23], but only with the multitaper technique can one obtain a consistent estimation procedure for these spectra. While these estimations unfortunately produce large numbers of false detections, one can produce summaries of multitaper Loéve spectra by either summing along lines of constant frequency offset (obtained by rotating the Loéve spectrum by $-45°$), or by counting numbers of detections above some significance level along some offset frequency. In this applied talk, numerous motivational examples (i) from seismology (with additional cycles introduced by tidal frequencies), (ii) from astronomy in which a gravitationally-lensed black hole is observed having significant 1 cycle per day offset coherences, (iii) signatures of solar rotation producing nonstationary signatures in temperature data from Marseille going back as far as 1897, and (iv) barometric pressure data from Piñon flat observatory which shows short period signatures of solar pressure (or possibly gravity) modes with high significance, were presented along with numerous open theoretical questions. It is useful to apply multitaper Loéve spectra with or without summarizing the offset coherences for the purposes of sanity checking any sufficiently long time series with or without known nonstationarities, as these data may contain surprising features.

3.3 Point Process Spectrum

A point process generates discrete events at particular locations in space. The main questions regarding point processes center around characterizing the first order properties (number of events per unit area) and second order properties (the relationships between locations of pairs of events). The spectrum of a point process is a convenient representation of second order features, but has not been widely used beyond the development of a periodogram-like estimator due to Bartlett in 1963 [2]. Some interesting new developments in this area [22] return to the fundamental problem of an analogous spectral representation for a spatial point process, and estimators with good statistical properties. It turns out that (i) the formation of a direct spectral estimator using tapering is possible in a straightforward way, that (ii) there is a necessary first-order bias correction to remove a constant contribution at all wavenumbers which is not standard in the ordinary spectral analysis of time series, (iii) there are no particularly fast methods (such as the Fast Fourier transform) for computing spectrum estimates of realizations of point processes, and finally, (iv) when isotropy of the random process is assumed, one can determine the analogous mean and variance properties of tapered spectrum estimators. Results on simulated processes show significant improvements when tapers are introduced, and inspire the next generation of scientific investigation of point processes using these updated methods.

4 Scientific Progress Made

As outlined above in the research highlights, there was the identification of open problems in nonstationary time series, the very recent mathematical discoveries of Fields Medalist Alain Connes and the intruiging connections with pure mathematics, and finally the presentation of Sofia Olhede on the (very) underdeveloped and underrated aspects of point process models.

Connecting applied researchers with theoretical and other appiled researchers also presented some serendipitous findings. Linda Hinnov's presentation on stratigraphy and the strain in the Earth's mantle provided a very large and very complex dataset for discussion. It is a particularly difficult dataset to access and understand without the expertise of a domain expert. Several members of the audience went to download the publically available data and correspond with Linda about the analysis. Had the workshop schedule been more flexible, participants agreed that they would have liked to see a presentation from some on-the-fly analyses in the form of a case study.

While a google scholar search for papers on the subject of multitaper from the last 5 years returns 5,670 results, this group was responsible for approximately 50-60 of them. Of these, this organizer was able to identify 45 journal articles from the last five years authored by those in attendance. In addition to this, there were tens of works in progress, and several ArXiV manuscripts presented ahead of publication. The rapid dissemination of recent and in-progress results was made at this meeting, resulting in numerous opportunities for new contacts and collaborations.

5 Outcome of the Meeting

The participants would like to meet again in person in two years time to discuss the rapidly-evolving aspects of this field, both mathematical and statistical, as described above. We believe that the major success of this meeting is the confluence of applied researchers from diverse fields, applied mathematicians and statisticians bringing open problems together with half-baked solutions, and putting forth these challenges in an open forum.

Participants agreed that this was a very stimulating meeeting for one held entirely remotely. The meeting also had numerous talks from early career researchers and underrepresented minorities. Numerous early career people privately thanked the organizers for the opportunity to present their work with widely respected researchers whose books and papers they had studied very closely. Despite the fact that the schedule was held in the mountain time zone, participants from North America as well as Europe (and one participant returning from Korea!) attended the talks as late as midnight in their respective time zones, and all talks were well attended.

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