Graph Product Structure Theory (21w5235)

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1 Overview of the Field

Graphs are the standard mathematical model of many real-world entities: computer networks, road and highway networks, drainage systems, river networks, electrical networks, and so on. In some cases, these highly-complicated graphs are contained in the product of two or more much simpler graphs. In a recent breakthrough, it was shown that this is the case for planar or near-planar graphs like those that model river, road, and highway networks; any such graph is contained in the product of a simple tree-like graph and a path. This product structure gives deep insight into these graphs and their properties, allowing a host of mathematical and algorithmic tools to be applied to these graphs. Examples of graphs classes that can be described this way include planar graphs [11, 19], graphs of bounded Euler genus [11, 6], graphs excluding a fixed minor [11], and various non-minor-closed classes [12, 16]. These results have been the key to solving several open problems regarding queue layouts [11], nonrepetitive colouring [9], p-centered colouring [7], adjacency labelling [15, 8], and twin-width [2, 1]. The goal of this workshop is to continue the search for product structure in more general classes of graphs as well as to find new methods to exploit such product structure mathematically and algorithmically, when it is present.

2 Structure of the Workshop

This workshop featured four hubs of researchers in Banff, Melbourne, Korea and Poland. Each hub had 5–15 people meeting daily (in person), listening to talks, and collaborating on open problems. At the start of the workshop, participants were invited to pose open problems (in the workshop theme) for all participants to work on. The problems were posted to a coauthor website dedicated to the workshop, which enables discussions and for participants to record progress on the problems. On a few occasions during the workshops, two hubs had zoom conversations about specific open problems. Overall the hub model worked well. It enabled participants to have face-to-face conversations with people in their hub (which is essential for mathematical collaboration), while also allowing for interaction between people who could not travel to Banff.

There was one major problem with the running of the workshop: It took several days for recordings of talks to be put online. The goal of a hybrid workshop is to enable participation from around the world. For this to happen, participants need to be able to watch the talks at the first reasonable local time after the talk. So it is essential that videos be posted online within a few hours after the live talk. Unfortunately, most of the talks for this workshop took several days to be posted online. This defeated the purpose of the workshop. It appeared that the BIRS staff were more interested in fancy editing than doing what the participants actually needed. We suggest that unedited talks be placed online immediately after the talk in a place where only registered participants can access them. Later, edited videos can be posted in publicly.

3 Presentation Highlights

The workshop featured the following invited talks:

Torsten Ueckerdt "The planar graph product structure theorem"

Abstract. I will prove the planar graph product structure theorem, which states that every planar graph is a subgraph of the strong product of a graph with bounded treewidth and a path. Based on reference [11, 19].

Louis Esperet "Nonrepetitive colourings of planar graphs"

Abstract. A colouring of a graph is "nonrepetitive" if for every path of even order, the sequence of colours on the first half of the path is different from the sequence of colours on the second half. One of the first applications of the product structure theorem was to show that planar graphs have nonrepetitive colourings with a bounded number of colours, solving a problem raised by Alon, Grytczuk, Haluszczak and Riordan in 2002. I will explain the ideas of the proof and show how to extend the result to classes of graphs that are closed under taking topological minors. Based on reference [9].

David Wood "Queue layouts of planar graphs (and beyond)"

Abstract. We show that planar graphs have bounded queue-number, thus proving a conjecture of Heath et al. from 1992. The key to the proof is the Planar Graph Product Structure Theorem: Every planar graph is a subgraph of the strong product of some treewidth 8 graph and some path. We generalise the first result to show that every proper minor-closed class has bounded queue-number. Based on reference [11].

O-joung Kwon "Reduced bandwidth: a qualitative strengthening of twin-width in minor-closed classes (and beyond)"

Abstract. In a reduction sequence of a graph, vertices are successively identified until the graph has one vertex. At each step, when identifying u and v, each edge incident to exactly one of u and v is coloured red. Bonnet, Kim, Thomassé, and Watrigant [J. ACM 2021] defined the twin-width of a graph G to be the minimum integer k such that there is a reduction sequence of G in which every red graph has maximum degree at most k. For any graph parameter f, we define the reduced-f of a graph G to be the minimum integer k such that there is a reduction sequence of G in which every red graph has f at most k. Our focus is on graph classes with bounded reduced-bandwidth, which implies and is stronger than bounded twin-width (reduced-maximum-degree). We show that every proper minor-closed class has bounded reduced-bandwidth, which is qualitatively stronger than a result of Bonnet et al. for bounded twin-width. In many instances, we also make quantitative improvements using product structures. For example, all previous upper bounds on the twin-width of planar graphs were at least 2^{1000}. We show that planar graphs have reduced-bandwidth at most 466 and twin-width at most 583. Our bounds for graphs of Euler genus g graphs are $O(g)$. Lastly, we show that fixed powers of graphs in a proper minor-closed class have bounded reduced-bandwidth (irrespective of the degree of the vertices). Based on reference [2].

Gwenaël Joret "Sparse universal graphs for planarity"

Abstract. This talk focuses on the following two problems: (1) What is the minimum number of edges in a graph containing all n-vertex planar graphs as subgraphs? The best known bound is $O(n^{3/2})$, due to Babai, Chung, Erdös, Graham, and Spencer (1982). (2) What is the minimum number of *vertices* in a graph containing all n-vertex planar graphs as *induced* subgraphs? Here Bonamy, Gavoille, and Pilipczuk (2019) recently established a $O(n^{4/3})$ bound. We show that a bound of $n^{1+o(1)}$ can be achieved for these two problems. Based on reference [15, 8].

Rose McCarty "Representing graphs with sublinear separators"

Abstract. A class of graphs has "sublinear separators" if each of its n-vertex subgraphs has a balanced separator of size $O(n^{1-\epsilon})$, for a fixed $\epsilon > 0$. This property holds for classes with product structure, classes with a forbidden minor, and many types of geometric intersection graphs. But does every class with sublinear separators have a nice representation that displays all its separators? We find a new geometric representation which guarantees sublinear separators, generalizes most known constructions, and, unfortunately, still does not capture everything. Our approach is based on a connection with strong coloring numbers. Based on reference [13].

Pat Morin "Optimal vertex ranking of planar graphs (and beyond)"

Abstract. A (vertex) ℓ-ranking is a labelling $\phi : V(G) \to \mathbb{N}$ of the vertices of a graph G with integer colours so that for any path $u_0, \ldots, u_p$ of length at most ℓ, $\phi(u_0) \neq \phi(u_p)$ or $\phi(u0) < \max\{\phi(u_0), \ldots, \phi(u_p)\}$. We show that, for any fixed integer $\ell \geq 2$, every n-vertex planar graph has an ℓ-ranking using $O(\log n/ \log\log\log n)$ colours and this is tight even when $\ell = 2$; for infinitely many values of n, there are n-vertex planar graphs, for which any 2-ranking requires $\Omega(\log n/ \log\log\log n)$ colours. This result also extends to bounded genus graphs. These results rely on product structure theorems showing that every planar (or bounded genus) graph is contained in a graph product of the form $H \boxtimes P \boxtimes K$ where K is a fixed size clique, P is a path, and H is a planar graph of treewidth at most 3. In particular, it is critical that H is contained in a planar 3-tree (also known as a simple 3-tree). In developing this proof we obtain optimal bounds on the number of colours needed for ℓ-ranking graphs of treewidth t and graphs of simple treewidth t. These upper bounds are constructive and give $O(n \log n)$-time algorithms. Additional results that come from our techniques include new sublogarithmic upper bounds on the number of colours needed for ℓ-rankings of any graph with product structure, including apex minor-free graphs and k-planar graphs. Based on reference [3].

David Wood "Graph product structure theory for minor-closed classes"

Abstract. The Planar Graph Product Structure Theorem says that planar graphs are subgraphs of the strong product of a bounded treewidth graph and a path, which is to say they have bounded row treewidth. This talk explores similar theorems for minor-closed classes that are more general than planar graphs. First, I show that graphs of bounded Euler genus have bounded row treewidth. More generally, a minor-closed class has bounded row treewidth if and only if some apex graph is not in the class. I then show that graphs with bounded degree in any minor-closed class have bounded row treewidth. Finally, I present the Graph Minor Product Structure Theorem, which says that graphs in any minor-closed class have a tree-decomposition in which each torso is a subgraph of $(H \boxtimes P) + K_a$ where H has bounded treewidth, P is a path, and a is bounded. Based on reference [11, 10].

Pat Morin "Product structure for non-minor-closed classes"

Abstract. We will present a Product Structure Theorem for k-planar graphs, where a graph is k-planar if it has a drawing in the plane in which each edge is involved in at most k crossings. In particular, every k-planar graph is a subgraph of the strong product of a graph of treewidth O(k5) and a path. The proof works in a much more general setting based on so-called shortcut systems, which may be helpful for developing product structure for other non-minor closed graph classes. This talk will discuss the proof of this theorem for graphs constructed from shortcut systems, the tools that it uses, and its consequences for other non-minor closed graph classes including map graphs, string graphs, powers of bounded-degree graphs, and 2-dimensional k-nearest neighbour graphs. Based on reference [12].

Robert Hickingbotham "Shallow minors, graph products and beyond planar graphs"

Abstract. The planar graph product structure theorem of Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [J. ACM 2020] states that every planar graph is a subgraph of the strong product of a graph with bounded treewidth and a path. This result has been the key tool to resolve important open problems regarding queue layouts, nonrepetitive colourings, centered colourings, and adjacency labelling schemes. In this paper, we extend this line of research by utilizing shallow minors to prove analogous product structure theorems for several beyond planar graph classes. The key observation that drives our work is that many beyond planar graphs can be described as a shallow minor of the strong product of a planar graph with a small complete graph. In particular, we show that k-planar, (k, p)-cluster planar, fan-planar and k-fan-bundle planar graphs can be described in this manner. Using a combination of old and new results, we deduce that these classes have bounded queue-number, bounded nonrepetitive chromatic number, polynomial p-centred chromatic numbers, linear strong colouring numbers, and cubic weak colouring numbers. In addition, we show that k-gap planar graphs have super-linear local treewidth and, as a consequence, cannot be described as a subgraph of the strong product of a graph with bounded treewidth and a path. Based on reference [16].

Piotr Micek "Centered chromatic numbers and weak colorings numbers"

Abstract. A quick application of the product structure theorem gives the best known bound for p-centered chromatic number of planar graphs, which is $O(p^3 \log(p))$. The r-th weak coloring number of planar graphs is bounded by $O(r^3)$ but the proof applies another decomposition strategy known as chordal partitions. We are going to make an overview of the main proof ideas, recent developments, and complain about the most annoying open problems in the area. Based on reference [7].

Zdenek Dvorak "On graphs with polynomial growth"

Abstract. I will sketch the main ideas of the product structure theorem of Krauthgamer and Lee for graphs with polynomial growth, and explore connections to other topics (expansion of products, asymptotic dimension, ...). Based on reference [18].

Tony Huynh "Universal graphs for infinite planar graphs (and beyond)"

Abstract. Stanisław Ulam asked whether there exists a countable planar graph that contains every countable planar graph as a subgraph. János Pach (1981) answered this question in the negative. We strengthen this result by showing that every countable graph that contains all countable planar graphs must contain (i) an infinite complete graph as a minor, and (ii) a subdivision of the complete graph K_t, for every finite $t \in \mathbb{N}$. On the other hand, we construct a countable graph that contains all countable planar graphs and has several key properties such as linear colouring numbers, linear expansion, and every finite n-vertex subgraph has a balanced separator of size $O(\sqrt{n})$. The graph is the strong product of the universal treewidth-6 countable graph (which we define explicitly) and the 1-way infinite path. More generally, for every $\in \mathbb{N}$ we construct a countable graph that contains every countable K_t-minor-free graph and has the above key properties. Our final contribution is a construction of a countable graph that contains every countable K_t-minor-free graph as an induced subgraph, has linear colouring numbers and linear expansion, and contains no subdivision of the countably infinite complete graph (implying (ii) above is best possible). Based on reference [17].

Vida Dujmović "Clustered colouring via products"

Abstract. Using a recent result on product structure of apex minor free graphs, we give simple proofs on clustered colourings of such graphs, more specifically those that exclude $K_{s,t}$ as a subgraph..

4 Scientific Progress

Over 30 open problems were posed during the workshop. A number of these problems were completely solved. For others, significant progress was made, which may lead to their solution in the future. Several new collaborations were started, which we hope will continue to prosper. Here are three specific problems that were solved.

Product structure for graphs on surfaces: The following question was posed at the start of the workshop: Is there a function f such that every graph of Euler genus g is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_{f(g)}$ for some graph H of treewidth 3 and some path P? This question was answered in the affirmative by Distel, Hickingbotham, Huynh and Wood [6]. The paper is submitted.

Stack-number of 3-dimensional products: The following question was posed at the start of the workshop: Is there a constant c such that for every path P the strong product $P \boxtimes P \boxtimes P$ has a c-page book embedding? This question was answered using a surprising method based on Gromov's Overlap Theorem. This is the first known example of a graph family with bounded degree and unbounded stack number. The initial result lead to numerous other discoveries that are now compiled in reference [14], which is submitted to a top journal. This paper is authored by three senior researchers (Eppstein, Norin, Wood) and three students (Hickingbotham, Merker, Seweryn), most of whom had not worked together before.

A linear time algorithm for product structure in planar graphs: The following question was posed at the start of the workshop: Is there a $O(n)$ time algorithm that, given an n-vertex planar graph, computes an isomorphism from G to $H \boxtimes P \boxtimes K_3$, where H has treewidth 3 and P is a path. This question was answered by Bose, Morin and Odak [4].

The field of Graph Product Structure Theory received a major boost from the workshop, and continues to product new and exciting results [5, 1].

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