Geometry of Real Polynomials, Convexity and Optimization

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1 Overview of the Field

The workshop was focussed on recent interactions among the areas of real algebraic geometry, the geometry of polynomials, and convex and polynomial optimization. A common thread is the study of two important classes of real polynomials, namely hyperbolic and nonnegative polynomials. These two themes are interacting deeply with optimization as well as theoretical computer science. This interaction recently led to solutions of several important open problems, new breakthroughs and applications.

The study of hyperbolic polynomials originated in PDE theory, with researchers like Petrovksy laying the foundations in the 1930s. This was further developed by Gårding [1959], Atiyah, Bott and Gårding [1972, 1973], as well as Hörmander. Interest and active research in the area was renewed in the early 1990s in connection with optimization. Since then, there has been an amazing amount of activity allowing the subject to branch into and have a significant impact on a wide range of fields, including systems and control theory, convex analysis, interior-point methods, discrete optimization and combinatorics, semidefinite programming, matrix theory, operator algebras, and theoretical computer science.

The other main focus of the workshop was the closely related study of nonnegative polynomials and polynomial optimization. A systematic study of nonnegative polynomials already appeared in Minkowski's early work in the 19th century. Then, Hilbert, in his famous address at the International Congress of Mathematicians in 1900, asked for characterizations of nonnegative polynomials, including this question in his research agenda setting list of foundational problems.

Since the 1990s, this has become closely related to polynomial optimization, i.e., the optimization of polynomials subject to polynomial constraints. This very general class of optimization problems subsumes many others. For example, many combinatorial optimization problems (e.g. MAXCUT) can be formulated as optimization of a quadratic objective on the discrete hypercube $\{0,1\}^n$, which is defined by quadratic equations $x_i(x_i-1)=0$. By duality, understanding polynomials nonnegative on a given the set is equivalent to optimization over that set. A natural way to approximate nonnegative polynomials is using sums of squares. Such approximations are tractable because testing whether a polynomial is a sum-of-squares can be done with semidefinite programming. The resulting sum-of-squares relaxations (also known as Lasserre relaxations) have gained prominence both for practical use and for theoretical considerations. For instance, one of the consequence of the Unique Games Conjecture is that the simplest sum-of-squares relaxation for MAXCUT (the famous Goemans-Williamson relaxation) gives the optimal approximation factor among all the polynomial time algorithms. On the practical side, sum of squares relaxations have also been used for computing Lyapunov functions and for certified control in robotics. While sum-of-squares relaxations often perform well in practice, the reason for this behavior is not well-understood and is of prime interest both theoretically and practically.

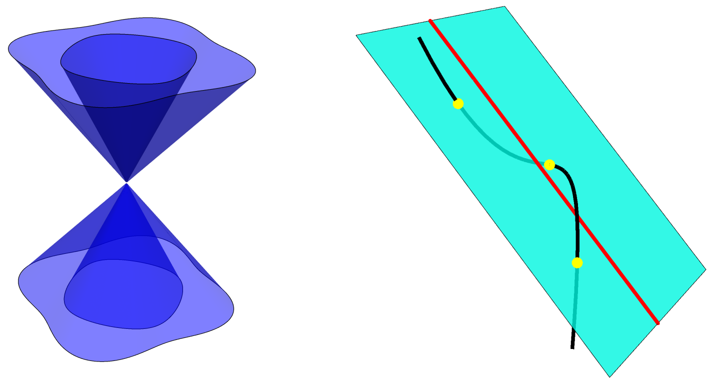

Figure 1: A hyperbolic hypersurface and hyperbolic space curve

2 Recent Developments and Open Problems

The areas of nonnegative polynomials, hyperbolic polynomials, mathematical optimization and theoretical computer science in collaboration and in interaction with each other have been producing many ground-breaking and surprisingly powerful results with far-reaching applications.

The theory of hyperbolic polynomials provides a very natural domain to formulate deep, far reaching yet elegant research problems. Because hyperbolicity appears in so many contexts, structural results about hyperbolic polynomials and their hyperbolicity cones can reverberate and have tremendous impact in several areas. For example, the works of Brändén, Gurvits, Helton-Vinnikov, Renegar together establish fundamental tools for understanding hyperbolicity cones, operations preserving hyperbolicity, determinantal polynomials, and inequalities appearing in coefficients. This increased facility with hyperbolic polynomials has helped enable a number of recent breakthroughs, including work of Gurvits on generalizations of Van der Waerden's conjecture and Bapat's conjecture as well as the recent proof of the Kadison-Singer conjecture by Marcus, Spielman and Srivastava. Recently these techniques have also been extended to a more general class of polynomials called Lorentzian or completely log-concave polynomials [1, 6]. Many properties of hyperbolic polynomials extend to this more general class, such as matroidal support and closure under derivatives, and have led to resolutions of open problems in matroid theory, including Mason's conjecture on the number of independent sets [2, 5] and the Mihail-Vazirani conjecture on the mixing time of certain Markov chains [1]. The analogues of many pieces of the well-developed theory of hyperbolic polynomials remain unresolved. Such generalizations would be desirable in their own right, as well as for applications to problems in combinatorial optimization and approximation theory.

Problem. *Extend the theory of hyperbolic polynomials (e.g. hyperbolicity cones, hyperbolic programming, characterization of operations preserving hyperbolicity) to the class of completely log-concave polynomials.*

Among the fundamental research problems in the area stands the "Generalized Lax Conjecture" relating hyperbolic and semidefinite programming. Several strict versions of this conjecture have been disproved, but the most general is still open.

Generalized Lax Conjecture. *Every hyperbolicity cone is a spectrahedron (i.e. linear section of the cone of positive semidefinite matrices).*

Work of Kummer on determinantal representations in [11] provides perhaps the strongest evidence for the conjecture, but there is a crucial positivity condition missing. Recent work by James Saunderson provides a necessary condition on the size of a spectrahedral representation based on the length of chains of faces of these cones [16]. A somewhat related question posed by Nemirovskii during the 2006 International Congress of Mathematicians, later conjectured by Helton and Nie, was disproved by Scheiderer in December 2016.

Theorem (Scheiderer [17])**.** *Not every convex semialgebraic set is the projection of a spectrahedron.*

Scheiderer's construction is closely related related to the complexity of sums-of-squares representations mentioned below, and the ramifications of this work are still being developed and understood.

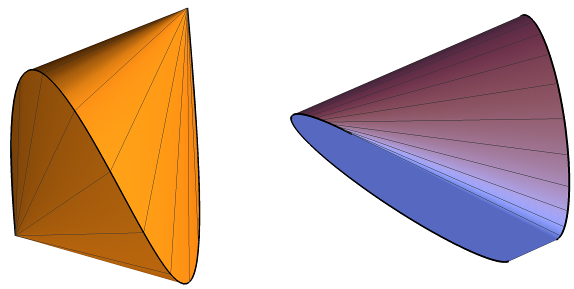

Figure 2: The spectrahedron of trigonometric moments given by the convex hull of the curve parametrized by $(\cos(\theta), \cos(3\theta), \cos(3\theta))$ and the dual cone of nonnegative trigonometric functions.

Part of the recent development in the theoretical understanding of sums-of-squares relaxations has come from connections with classical algebraic geometry. There seems to be a deep intrinsic connection between the quality of low-degree sum-of-squares relaxations and the classical properties of the algebraic set (e.g. Castelnuovo-Mumford regularity). Blekherman, Sinn, Smith, and Velasco have developed many of these connections [4], but several open questions remain. For example, one fundamental open question is the Pythagoras number of the d-th Veronese embedding of $\mathbb{P}^n$, which is related to is the number of squares needed to achieve all sums of squares of a fixed degree.

Question. *What is the lowest number $N(n, d)$ such that any sum of squares in $\mathbb{R}[x_0, \ldots, x_n]_{2d}$ is a sum of $N(n, d)$ squares?*

Even for $n = 2$, this is still open for $d \geq 4$, where it is known that $d + 1 \leq N(2, d) \leq d + 2$.

Another way that the sum of squares method is used is to certify graph density inequalities in extremal combinatorics (these are called Turán problems). Here, squaring takes place inside the *gluing algebra* of partially labelled graphs. Unlike in Hilbert's 17th problem, it was shown by Hatami and Norine in [8] that there exist graph density inequalities that cannot be certified using sums of squares of rational functions, but no explicit examples are known.

Question. *Find an explicit graph density inequality that cannot be written as a sum of squares of rational functions in the gluing algebra.*

This question may also be of interest in theoretical computer science, where sum of squares hierarchy is considered one of the most powerful methods of addressing combinatorial problems. Finding strong obstructions to sums of squares representability would shed more light on the power of the hierarchy. For recent progress on sums of squares for graph densities see Rekha Thomas' talk and [3].

On the one hand, each of these open problems and conjectures is interesting for the research areas of real algebraic geometry, operator theory, the geometry of polynomials, convex and polynomial optimization, as well as theory of computing and theoretical computer science. On the other hand, when we focus on a more detailed view from one of these research areas, many other variants of these open problems emerge.

For example, in the context of the Generalized Lax Conjecture above and Scheiderer's Theorem from December 2016, a convex and polynomial optimization viewpoint leads to a related open problem:

Question. *Is every hyperbolicity cone a spectrahedral shadow? (I.e., can we express every hyperbolicity cone as a projection of a spectrahedral cone?)*

If the Generalized Lax Conjecture is true, then the answer to the above question is trivially "yes". Interest in this version of the open problem partly stems from the fact that the set of linear optimization problems over extended formulations (or *lifted formulations*) essentially includes the set of linear optimization problems over the shadow (or the projection) as a special case.

When we consider Generalized Lax Conjecture and its variants from the viewpoint of computational complexity of solving the underlying convex optimization problems, further variants of these open problems emerge:

- Suppose K is a hyperbolicity cone that is spectrahedral. Based on the minimal defining hyperbolic polynomial of K, what is the smallest dimensional positive semidefinite cone which expresses K as a linear slice?
- Suppose K is a hyperbolicity cone that is spectrahedral shadow. Based on the minimal defining hyperbolic polynomial of K, what is the smallest dimensional positive semidefinite cone which expresses K as a projection (shadow) of a linear slice?

For recent progress in this direction, see [14] and Saunderson's talk.

3 Presentation Highlights

This workshop brought together established experts and young researchers from the areas of real algebraic geometry, the geometry of polynomials, and convex and polynomial optimization, to review the most recent significant discoveries and strive to forge new connections.

Each day of the workshop started with a one-hour talk that in addition to reporting on recent developments also gave an accessible overview of the background.

The workshop started with Victor Vinnikov's talk the first half of which gave a splendid introduction to the general area of stable polynomials and hyperbolic polynomials. Then in the second half Vinnikov focused on recent results on determinantal representations of hyperbolic polynomials. Vinnikov's talk made a connection to another BIRS workshop from 2010 when another participant, Petter Brándén had solved a major open problem in the field, leading to what is now called the Generalized Lax Conjecture.

The theme of determinantal representations of hyperbolic polynomials and stable polynomials continued throughout the workshop. Mario Kummer's talk addressed the representability question by focusing on convex cones constructed from algebraic curves by taking their convex hull.

Many talks served a bridging property between real algebraic geometry and convex optimization, convex geometry and convex analysis. Renegar presented a very general framework for subgradient methods. One of the most interesting cases being those optimization problems that are expressible as the intersection of a hyperbolicity cone and an affine space (hyperbolic programming problems). James Saunderson connected expressive power (via extended or lifted formulations) of classes of convex cones to the beautiful geometric measure of boundary structure of cones known as longest chain of faces. Jiawang Nie considered the classical saddle point problem in continuous optimization when the given functions are polynomials. Utilization of duality theory and Lasserre hierarchy and, exploiting the algebraic structure of the setup and that of Lagrange multipliers were highlighted. Simone Naldi considered the convex feasibility problem together with the theory of infeasibility certificates in the conic setup and brought the language and power of projective geometry to bear on the subject.

Didier Henrion attacked the classical problem of solving nonlinear PDEs using the Lasserre hierarchy. Jean-Bernard Lasserre proposed tractable semi-algebraic approximations employing the Christoffel-Darboux Kernel.

Rainer Sinn presented recent results on the joint numerical range of Hermitian matrices, which tie in with a general study of duality for hyperbolicity cones.

The theory of moment problems was treated in several talks, and some of these also dealt with the infinite dimensional generalizations. Maria Infusino considered extensions of tools for finite-dimensional moment problems to infinite dimensional settings in an hour-long overview. Salma Kuhlmann discussed the generalization of the theory of moments to the setting of symmetric algebras (algebra of symmetric tensors).

The theory and applications of completely log-concave polynomials or equivalently Lorentzian polynomials were featured in several of the talks. This class of polynomials strictly generalizes hyperbolic polynomials, yet they admit characterizations involving the Lorentzian signature (of the quadratic derivatives). Shayan Oveis Gharan gave an overview of how hyperbolic and completely log-concave polynomials can be used to approximate NP-hard counting problems using both deterministic and probabilistic algorithms. Nima Amari's talk followed up on this survey by focussing on log-concave polynomials coming from combinatorial structures called matroids. In particular, Amari talked about approximating the number of bases of a matroid by approximating the evaluation of a particular log-concave polynomial. Petter Brändén's hour-long survey completed the development of the theory of connections between hyperbolic polynomials, Lorentzian polynomials and discrete convexity.

Bachir El Khadir showed that all convex forms in 4 variables and of degree 4 are sums of squares. El Khadir also showed an attractive generalization of the Cauchy-Schwarz inequality. Rekha Thomas gave a survey talk on using sums of squares method to prove graph density inequalities and highlighted new results on the limitations of the sum of squares method, answering problems posed by Lovász. The number of squares used in a sum of squares decomposition is also an interesting quantity, useful in applications, such as Euclidean distance realization. Greg Smith's talk examined the number of squares needed to write any sum of squares on a projective variety (called Pythagoras number of the variety). He introduced a new algebraic invariant of the variety, called quadratic persistence, which is useful is giving lower bounds on the Pythagoras number.

Nonnegativity certificates given by sums of nonnegative circuit polynomials (Mareike Dressler) and sums of arithmetic-geometric mean exponentials (Riley Murray) were also studied.

The noncommutative theory of sums of squares and matrix inequalities, that ties in with functional analysis, was featured in two talks, by Igor Klep and by Jaka Cimpric.

The theme of complexity in real algebraic geometry was also central to a number of talks. Saugata Basu and Cordian Riener reported on joint work concerning the computation of the Betti numbers of semialgebraic sets that are invariant under the symmetric group.

Marie-Françoise Roy spoke about a new proof of the fundamental theorem of algebra from the intermediate value theorem for real polynomials. This forms part of a long-term project to find resp. improve complexity bounds for the real nullstellensatz. Mohab Safey El Din explained a novel approach for computing resp. approximating the volume of a semialgebraic set, significantly improving the previously known complexity bounds.

Eli Shamovich presented a modern extension of the classical Hermite method for counting the roots of a complex polynomial in the upper half plane to quadrature domains. Mihai Putinar tied in the geometry of polynomials with Hermitian sums of squares and demonstrated a more precise version of a classical Lemma due to Laguerre, developed in joint work with the late Serguei Shimorin.

3.1 List of speakers and the titles of their talks (in the order of appearance in the workshop)

1. Victor Vinnikov (Hyperbolicity, stability, and determinantal representations)
2. Mario Kummer (When is the conic hull of a curve a hyperbolicity cone?)
3. James Saunderson (Limitations on the expressive power of convex cones without long chains of faces)
4. Igor Klep (Noncommutative polynomials describing convex sets)
5. Saugata Basu (Vandermonde varieties, mirrored spaces, and cohomology of symmetric semi-algebraic sets)
6. Gregory Smith (Sums of squares and quadratic persistence)
7. Rainer Sinn (Kippenhahn's Theorem for the joint numerical range)
8. Shayan Oveis Gharan (From Counting to Optimization and Back using Geometry of Polynomials)
9. Nima Anari (Computing Log-Concave Polynomials)

10. James Renegar (A framework for applying subgradient methods)
11. Cordian Riener (Algorithms to compute topological invariants of symmetric semi algebraic sets)
12. Bachir El Khadir (On sum of squares representation of convex forms and generalized Cauchy-Schwarz inequalities)
13. Jiawang Nie (The Saddle Point Problem of Polynomials)
14. Simone Naldi (Conic programming: infeasibility certificates and projective geometry)
15. Rekha Thomas (Graph Density Inequalities and Sums of Squares)
16. Didier Henrion (Solving non-linear PDEs with the Lasserre hierarchy)
17. Jean-Bernard Lasserre (Tractable semi-algebraic approximation using Christoffel-Darboux kernel)
18. Maria Infusino (From finite to infinite dimensional moment problems)
19. Salma Kuhlmann (The moment problem for the algebra of symmetric tensors)
20. Mihai Putinar (Positive integral kernels for polar derivatives)
21. Marie-Francoise Roy (Quantative Fundamental Theorem of Algebra)
22. Mohab Safey El Din (On the computation of volumes of semi-algebraic sets)
23. Eli Shamovich (Counting the number of zeroes of polynomials in quadrature domains)
24. Jakob Cimpric (Some non-commutative nullstellensaetze)
25. Petter Brändén (Stable polynomials, Lorentzian polynomials and discrete convexity)
26. Mareike Dressler (Optimization over the Hypercube via Sums of Nonnegative Circuit Polynomials)
27. Riley Murray (SAGE certificates for signomial and polynomial nonnegativity)

3.2 Participants (in alphabetical order)

Anari, Nima (Stanford University)
Basu, Saugata (Purdue University)
Blekherman, Greg (Georgia Institute of Technology, Organizer)
Brändén, Petter (KTH Royal Institute of Technology)
Cimpric, Jaka (University of Ljubljana)
Dressler, Mareike (UC San Diego)
El Khadir, Bachir (Princeton University)
Helton, Bill (UC San Diego)
Henrion, Didier (LAAS-CNRS, University of Toulouse)
Infusino, Maria (University of Konstanz)
Klep, Igor (University of Auckland)
Kuhlmann, Salma (Universität Konstanz)
Kummer, Mario (TU Berlin)
Lasserre, Jean-Bernard (LAAS-CNRS 7, Toulouse)
Manevich, Dimitri (TU Dortmund University)
Murray, Riley (CalTech)
Naldi, Simone (Université de Limoges)
Nie, Jiawang (University of California San Diego)
Oveis Gharan, Shayan (University of Washington)
Piontek, Roland (TU Dortmund University)
Plaumann, Daniel (TU Dortmund University, Organizer)

Putinar, Mihai (UC Santa Barbara)
Renegar, James (Cornell University)
Riener, Cordian (University of Tromso)
Roshchina, Vera (University of New South Wales)
Roy, Marie-Francoise (Université de Rennes 1)
Safey El Din, Mohab (Sorbonne University)
Saunderson, James (Monash University)
Scheiderer, Claus (University of Konstanz)
Scholten, Georgy (North Carolina State University)
Shamovich, Eli (University of Waterloo)
Sinn, Rainer (Freie Universität Berlin)
Smith, Gregory G. (Queen's University)
Tetali, Prasad (Georgia Institute of Technology)
Thomas, Rekha (University of Washington)
Tunçel, Levent (University of Waterloo, Contact Organizer)
Vinnikov, Victor (Ben Gurion University of the Negev)
Vinzant, Cynthia (North Carolina State University, Organizer)

4 Scientific Progress Made

While it is too early to determine the long term impact of a week long conference, we are convinced that it successfully provided exposure of researchers from a diverse set of fields to each other's research and ideas.

One precedent for such success was the 2010 Banff meeting on "Convex Algebraic Geometry". This workshop had a significant overlap in participants with the current one. Over the intervening nine year, connections have developed into fully fledged research areas.

One such example of this fruition is the connection of matroids, stable polynomials, and determinantal representations that was made by Petter Brändén at the 2010 meeting, at which he disproved one version of the "generalized Lax conjecture" using tools from matroid theory. These connections between determinants, stable polynomials, and matroids led to papers by several of the participants of the current workshop and to the recent theory of Lorentzian and completely log-concave polynomials, which have resolved several long-standing open questions ranging in matroid theory (Mason's conjecture) and Markov-chains (Mihail-Vazirani conjecture). The three workshop talks on these recent developments introduced this material to several researchers in the field.

5 Outcome of the Meeting

For this workshop, we intentionally invited researchers from several different fields, ranging from real algebraic geometry, operator theory, convex optimization, and theoretical computer science. Both the morning introductory talks and the long lunch-breaks were designed to encourage participants to learn new material from each other. We received positive feedback from several participants about how many new people they were able to meet and that the conference had a greater exchange of ideas than usual. There are also several on going collaborations among the participants of the program coming from different countries and the opportunity to meet and exchange ideas in person is invaluable.

One of the other impact of the 2010 meeting at Banff was the development of a strong community of young researchers in the field. In fact, three of the the current organizers were junior participants at this conference. In order to continue this positive momentum, there were several (at least seven) excellent talks by young participants. This workshop introduced them to the experts in the field and we hope that they will develop to be leaders of the field.

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