Interactions between Brauer Groups, Derived Categories and Birational Geometry of Projective Varieties

Nathan Grieve (Michigan State University),
Collin Ingalls (Carleton University)

November 10–15, 2019

The workshop *Interactions between Brauer Groups, Derived Categories and Birational Geometry of Projective Varieties* (19w5164) took place at the Banff International Research Station (BIRS) during the time period November 10–15, 2019. It was an international event and was attend by 35 participants. A total of 19 individuals gave 50-60 minute lectures on topics which were of interest to the proposed subject areas.

1 Overview of the Field

The workshop 19w5164 focused on three important areas of scientific research within algebraic and arithmetic geometry. These topics include:

(1) Brauer groups;

(2) Derived Categories; and

(3) Birational geometry.

Some of the initial sources of motivation for these proposed topics included the work of Clemens and Griffiths, [15], that of Artin and Mumford, [4], and work of Mukai [30] and [31]. A main goal of the works [15] and [4] was to construct algebraic varieties which are unirational but not rational. Recall, also, that the techniques of [15] are Hodge theoretic in nature whereas those of [4] are algebraic; they are largely based on using the theory of maximal orders to construct conic bundles over rational surfaces.

Two key results from [15] and [4], respectively, are summarized in the following way.

Theorem ([15]). The intermediate Jacobian $(J(X), \Theta)$, for X a complex rational threefold, is a product of Jacobians of curves

$$(J(X), \Theta) \simeq \prod_i (J(C_i), \Theta_i).$$

Theorem ([4]). If X is a projective non-singular complex variety, then the torsion subgroup of the third integral cohomology group $H^3(X, \mathbb{Z})$ is a birational invariant. In particular, $H^3(X, \mathbb{Z})$ is torsion free when X is rational.

In terms of bounded derived categories of coherent sheaves on projective varieties, an important question is the extent to which they are classification invariants. For instance, for the case of Abelian varieties, the question of isomorphism invariance was addressed in work of Mukai [30] and [31]. In those works, in addition to establishing many other important results, the concept of *Fourier-Mukai transform* was systematically developed and applied.

Theorem ([31]). Let $\mathcal{P}$ be the Poincaré line bundle on $A \times \hat{A}$, for A an Abelian variety with dual Abelian variety $\hat{A}$. Then, the Fourier-Mukai transform

$$\Phi_{\mathcal{P}}\colon \mathrm{D}^b_{\mathrm{coh}}(A) \to \mathrm{D}^b_{\mathrm{coh}}(\hat{A})$$

is an equivalence of categories.

The above theorem of Mukai motivates the question as to the extent to which the bounded derived category of sheaves on a given non-singular projective variety determines the variety up to isomorphism. For non-singular projective varieties with ample canonical or anticanonical class, this question has been addressed by Bondal and Orlov [9]. Their main result can be stated as:

Theorem ([9]). Let X and Y be non-singular projective varieties with ample canonical or anticanonical class. If $\mathrm{D}^b_{\mathrm{coh}}(X)$ and $\mathrm{D}^b_{\mathrm{coh}}(Y)$ are equivalent, then X and Y are isomorphic.

The standard expository introductory treatment of derived categories and Fourier-Mukai transforms is the text by Huybrechts [25]. The theory of Fourier-Mukai transforms within the context of orbifold Deligne-Mumford stacks was pursued by Kawamata [26].

Another significant application of Fourier-Mukai transform techniques, building on [30] and [31], is through the theory of *generic vanishing theory* which was pioneered by Green and Lazarsfeld [19]. That theory has been a tool for studying irregular varieties. Within the context of Abelian varieties, the use of derived categories is a means for further developing the generic vanishing theory of Green and Lazarsfeld. In particular, the Generic Vanishing Theorem for Abelian varieties was emphasized by Hacon [21].

Theorem ([21]). Let $\mathcal{F}$ be a coherent sheaf on a complex Abelian variety A and let $\hat{A}$ be the dual abelian variety. Then every irreducible component of the *cohomological support locus*

$$V^i(A, \mathcal{F} \otimes P) := \left\{ P \in \hat{A} : h^i(A, \mathcal{F} \otimes P) \neq 0 \right\}$$

has codimension at least i.

This generic vanishing theory for Abelian varieties was refined and significantly developed further in a series of articles by Pareschi and Poppa [34], [35], [36]. In more recent times, there has been significant further developments which pertain to these topics. As one such example, we mention the recent work of Lombardi and Popa, [29], which builds on earlier work of Popa [37] and Popa-Schnell [38].

Continuing with questions in Hodge theory and their relevance to the topics of the workshop, note that, for instance, similar to the case of cubic threefolds which was considered in [15], a natural question is the extent to which cubic fourfolds in $\mathbb{P}^5$ are rational. A Hodge theoretic approach for producing examples of rational cubic fourfolds was given by Hassett [22].

The work [1] is motivated by rationality questions for cubic fourfolds and builds on work of Kuznetsov [27]. It proves a number of results in the direction of derived categories of cubic fourfolds. Another feature of [1] is that the authors use results from [10] to express some of their Fourier-Mukai calculations in terms of Hodge theory.

In a compatible direction, the works of Ballard-Favero-Katzarkov, including [7], and Ballard-et-al, [6], have a number of implications for the bounded derived categories of coherent sheaves on projective hypersurfaces, and cubic fourfolds in particular. For example, in [7], the authors combine their [7, Theorem 1.2] with results of Orlov [33] and Hochschild-Kostant-Rosenberg, [23], to reprove Griffiths' description of the primitive cohomology of a projective hypersurface [20]. Indeed, the following theorem is established in [7].

Theorem ([7]). Let Z be a non-singular, complete projective hypersurface defined by a homogeneous degree d polynomial

$$F \in \mathbf{C}[x_1, \dots, x_n].$$

Then for all p, with $0 \leq p \leq n/2 - 1$, Orlov's theorem, [33], combined with the Hochschild-Kostant-Rosenberg isomorphism, induces an isomorphism

$$H^{p,n-2-p}_{\mathrm{prim}}(Z) \simeq \mathrm{Jac}(F)_{d(n-1-p)-n}.$$

In general, the theory of Fourier-Mukai functors and the concept of semiorthogonal decompositions for derived categories continue to be topics of significant interest. They have important connections to rationality questions and Brauer groups. Some recent works related to the scope of the workshop include [5] and [3]. In what follows, we describe in some detail these interconnections.

In [5], the authors use Clifford algebras and semiorthogonal decompositions for the derived category of quartic Del Pezzo fibrations to characterize the condition that a given generic degree four Del Pezzo fibration $X \to \mathbb{P}^1$ is rational. On the other hand, the theory of Fourier-Mukai functors has recently been used to study Jacobian elliptic fibrations [3]. In [3, Theorem 1.5], for example, certain twisted derived equivalences, which arise from such elliptic fibrations, are related to cyclic subgroups of quotients of Brauer groups.

Returning to extensions of the work of Artin-Mumford, [4], it is of continued interest to use tools from birational algebraic geometry to study ramfication in Brauer groups of function fields of algebraic varieties. The more recent developments should be seen as extensions to earlier work of Chan-Ingalls, [12], as well as Chan-Kulkarni [14]. As one more specific example, we mention the following form of the classical Castelnuovo's contraction theorem which was obtained in [12].

Theorem ([12]). Let S be a non-singular complex projective surface with function field $\mathbf{K}$. Let $\alpha \in \mathrm{Br}(\mathbf{K})$ be a Brauer class with ramification divisor Δ_α. Suppose that the ramification of α along irreducible curves in S is terminal. Then, with these assumptions, if E is an irreducible curve in S which has the two properties that: (i) $E^2 < 0$; and (ii) $(\mathrm{K}_S + \Delta_\alpha) \cdot E < 0$, then E is a (-1)-curve.

Finally, fixing an algebraically closed characteristic zero base field, by [8], [16], and [11], for instance, our basic understanding of the main theorems from higher dimensional birational geometry of projective varieties, especially those results which are the foundations of the minimal model program, has been clarified a good deal. Indeed, it is now understood that the main theorems of the minimal model program can be established as consequences of finite generation of adjoint rings.

Theorem ([16]). Let X be a non-singular complex projective variety and A an ample $\mathbb{Q}$-divisor on X. Let Δ_i be a $\mathbb{Q}$-divisor on X such that $\lfloor \Delta_i \rfloor = 0$, for $i = 1, \ldots, r$, and such that the divisor $\Delta_1 + \cdots + \Delta_r$ has simple normal crossings support. Then the adjoint ring

$$R(X; K_X + \Delta_1 + A, \ldots, K_X + \Delta_r + A)$$

is finitely generated.

By contrast, the abundance conjecture remains an extremely important open question.

The numerical abundance conjecture ([32]). If (X, Δ) is a complex projective klt $\mathbb{Q}$-Gorenstein pair, then the numerical and the Iitaka dimension of the log canonical divisor $K_X + \Delta$ coincide.

These concepts, tools and more recent developments, within the area of birational geometry, are foundational to the main topics and themes of the workshop. Indeed, as is apparent in all of the sections that follow, each of the workshop's lectures intersected on these themes in some form or another.

2 Recent Developments and Open Problems

The workshop identified the continued importance of three main research directions (see Section 1). While many of these topics have close ties to the specific subject areas of the workshop, some have important more tangential overlap. We refer to the workshop's lectures themselves for motivation, background and statements of many of these recent developments and open problems. We summarize these lectures in Section 3.

In this section, we content ourselves with motivating and stating some more specific problems from three more broad areas which we feel are especially important to the specific aims of the workshop. In more precise terms, we feel that the workshop identifies the following three topics as especially important to follow-up in more specific further details. We then include a discussion and justification for future research on these topics. These three themes include:

(1) Topological, arithmetic and Hodge theoretic aspects of derived equivalences for projective varieties;

(2) Variations of geometric invariant theory and higher dimensional birational geometry; and

(3) Effective and explicit methods for Brauer groups, ramification and related objects.

Indeed, the above three interrelated topics closely reflect the three main subject areas of the workshop. In terms of point (1), about a third of the workshop's lectures directly dealt with the subject of derived categories.

Specifically, they focused on topics such as: (i) orthogonal decompositions for twists of varieties over non-algebraically closed fields; (ii) the extent to which rational points may be considered as derived category invariants; and (iii) Hodge theoretic topological aspects of derived equivalence.

One reoccurring theme throughout the workshop was the classical problem about derived invariance of Hodge numbers. While several of the workshop's lectures reported upon the recent progress which has been made towards a complete understanding of this problem, they also indicated that continued research on that topic remains to be of interest. This is evidenced, for instance, by the lectures which were given by Lieblich, Bragg, Frei, McFaddin, Duncan, Gulbrandsen, Lombardi and Doran.

We state the problem of derived invariance of Hodge numbers in more precise terms below. For the case of (non-singular complex projective) threefolds, for example, an affirmative answer to the problem about derived invariance of Hodge numbers is known by work of Popa and Schnell [38].

Problem (Derived invariance of Hodge numbers). Suppose that X and Y are non-singular complex projective varieties with equivalent bounded derived categories of coherent sheaves

$$\mathrm{D}^b_{\mathrm{coh}}(X) \simeq \mathrm{D}^b_{\mathrm{coh}}(Y).$$

Then, within this context, does it follow that X and Y have equal Hodge numbers

$$h^{p,q}(X) = h^{p,q}(Y)$$

for all p and q?

The above mentioned problem about derived invariance of Hodge numbers, motivates the more general problem as to the extent to which derived equivalences between non-singular projective varieties can be seen as classification invariants. Again, this topic was a recurring theme throughout the workshop. The following problem was stated during the lecture by Lieblich. It has since been discussed, in more detail, in the recent arXiv preprint [28].

Problem (M. Lieblich and M. Olsson). Let X and Y be non-singular complex projective varieties. If there exists a filtered equivalence

$$\Phi \colon \mathrm{D}(X) \xrightarrow{\sim} \mathrm{D}(Y),$$

between derived categories of perfect complexes, then are X and Y birationally equivalent?

Turing to arithmetic questions, another topic of interest was the extent to which fields of definition enter into the picture in terms of questions about derived equivalences between pairs of projective varieties. As one example, the following problem of Addington-et-al arose during the lecture of Frei.

Problem (Addington-et-al). Suppose that X and Y are derived equivalent non-singular projective varieties over a number field $\mathbf{k}$. Then under what additional hypothesis (if any) does it hold true that X and Y simultaneously contain $\mathbf{k}$-rational points?

In terms of fields of definitions and existence of full exceptional collections over a given base field, the following conjecture of Orlov was emphasized during the lecture of McFaddin.

Conjecture (Orlov). Let X be a non-singular projective variety with field of definition a number field $\mathbf{k}$. If X admits a full exceptional collection which is defined over the base number field $\mathbf{k}$, then X is in fact rational.

From the view point of item (2), it is now understood that the logical link between the three topics of the workshop is achieved through the question of birational classification for higher dimensional algebraic varieties. An important role is also played through concepts which have origins in Mumford's Geometric Invariant Theory. As some entry points to these extremely important topics, we mention work of Dolgachev and Hu [17], Hu and Keel [24] and Reid [39].

Returning to the subject of bounded derived categories of coherent sheaves on projective varieties, the manner in which they relate to the concept of Variation of Geometric Invariant Theory Quotients (VGIT) has been studied in detail by Ballard-Favero-Katzarkov, [7], and their school. On the other hand, much can still be done to clarify and pin-down, in more precise terms, the exact manner in which VGIT is reflected in the overall context of birational classification.

The workshop's lecture by Lesieutre, about numerical dimension, emphasized the continued importance for the study of measures of positivity, growth and singularities within the overall context of algebraic geometry. The workshop touched on topics that have motivation from concepts in string theory and mathematical physics. For instance, the subject of *homological mirror symmetry* and its relation to the concept *Landau-Ginzberg models* was emphasized in talks of Katzarkov, Doran, Favero and Ballard.

As one more recent development, here we state a conjecture of Doran-Harder-Thompson which was emphasized in the lecture of Doran. It is discussed in more detail in the arXiv preprint [18].

Conjecture (Doran-Harder-Thompson). Let X be a Calabi-Yau variety which arises as a non-singular fiber of some *Tyurin degeneration* $\mathcal{V} \to \Delta$. Let X_1 and X_2 be quasi-Fano varieties which meet normally with common intersection a non-singular Calabi-Yau divisor X_0. Then the corresponding Landau-Ginzberg models $W_i \colon X_i^\vee \to \mathbb{C}$ of the pairs (X_i, X_0), for $i = 1, 2$, glue together to give a fibration $W \colon X^\vee \to \mathbb{P}^1$. Finally, the compact fibers of the Landau-Ginzburg models consist of Calabi-Yau manifolds which are mirror to the common anti-canonical divisor X_0.

Finally, in the direction of point (3), we mention that several of the workshop's lectures highlighted the need for effective and explicit methods for the study of Brauer groups, ramification and related objects. This is evidenced, for example, in the lectures which were given by Diaz, Dhillon, Gille, Grieve, Lewis, Parimala and Scully.

As one classical representative example, which was highlighted in the lecture by Parimala, the classical Milnor Conjecture, which is now a theorem of Voevodsky, continues to provide impetus for more recent developments which pertain to ramification and other invariants which may be associated to quadratic forms and Brauer groups. For the sake of completeness, we state this conjecture (theorem of Voevodsky) below.

The Milnor Conjecture (Theorem of Voevodsky). Let $\mathbf{k}$ be a field with characteristic not equal to 2. Let $W(\mathbf{k})$ be the Witt ring of equivalence classes of quadratic forms and $I(\mathbf{k}) \subset W(\mathbf{k})$ the ideal of even dimensional forms. Put $I^n(\mathbf{k}) = I(\mathbf{k})^n$. Recall, that the ideal $I^n(\mathbf{k})$ is additively generated by the n-fold Pfister forms $\langle\langle a_1, \ldots, a_n \rangle\rangle := \langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$, for $a_i \in \mathbf{k}^\times$. Then, with these notations and conventions, the map

$$e_n(\langle\langle a_1, \ldots, a_n \rangle\rangle) := (a_1) \cdot \ldots \cdot (a_n)$$

extends to give a homomorphism

$$e_n \colon I^n(\mathbf{k}) \to H^n(\mathbf{k}, \mathbb{Z}/2\mathbb{Z}).$$

This morphism is onto and has kernel equal to $I^{n+1}(\mathbf{k})$.

As a more recent conjecture, which pertains to ramification and Brauer groups, we state a form of the Period-index conjecture (following [2]).

The Perod-index conjecture. Suppose that $\mathbf{k}$ is either an algebraically closed, a C_1, or a p-adic field and set $e = 0, 1, 2$ accordingly. Let $\mathbf{K}$ be a field of transcendence degree n over $\mathbf{k}$. Then, for all $\alpha \in \operatorname{Br}(\mathbf{K})$, it holds true that

$$\operatorname{ind}(\alpha) | \operatorname{per}(\alpha)^{n-1+e}.$$

As another more specific representative problem, we mention a question which has origins in the work of Artin and Mumford [4]. It complements earlier related work of Chan and Kulkarni [13]. Its specific relation to the topics of the workshop was emphasized in the lecture of Grieve. Briefly, it asks to characterize, and distinguish between, the concepts of Brauer and orbifold birational logarithmic pairs.

Problem (Compare with [13]). Let S be a non-singular projective surface and (S, Δ) an *orbifold pair*

$$\Delta = \sum_i (1 - 1/m_i) C_i,$$

for m_i positive integers and C_i irreducible nonsingular curves in S. Suppose that Δ has simple normal crossings support. Let $\mathbf{K} = \mathbf{k}(S)$ be the function field of S, for $\mathbf{k}$ an algebraically closed characteristic zero field. Give necessary and sufficient conditions for the boundary orbifold divisor Δ to have shape

$$\Delta = \mathbb{D}(\alpha)_S = \Delta_\alpha$$

for $\mathbb{D}(\alpha)_S$ the *trace* of the *ramification birational divisor* $\mathbb{D}(\alpha)$ which is determined by α an element of $\operatorname{Br}(\mathbf{K})$, the Brauer group of $\mathbf{K}$.

One other topic which pertains to item (3) concerns the manner in which unramified cohomology groups can be used to verify or disprove the conclusion of the *integral Hodge conjecture*. For more precise statements, let X be a non-singular complex projective variety. The conclusion of the integral Hodge conjecture asserts that every class $\alpha \in H^{p,p}(X,\mathbb{Z})$ is algebraic. Recall, that when $p = 0$ or $p = \dim(X)$, the conclusion is indeed trivially true, while the case that $p = 1$ follows from the celebrated Lefschetz $(1, 1)$-theorem.

On the other hand, for other values of p, the conclusion of the integral Hodge conjecture is known not to hold true in general. Of particular interest, is the case that $n = \dim(X) = 3$ and $p = 2$. Within that context, various results (both positive and negative) are known. In general, the following problem remains open and of interest.

Problem (The integral Hodge conjecture). Let X be a non-singular complex projective threefold. Then, under what hypothesis is it true that every class $\alpha \in H^{2,2}(X,\mathbb{Z})$ is algebraic?

3 Presentation Highlights

In this section, we describe some aspects of the workshop's lectures themselves and place some emphasis on their especially important aspects. First of all, about 3-4 weeks prior to the workshop, the organizers sent invitations to selected participants asking if they would be interested in giving a talk at the workshop. This method of solicitation and selection of talks was effective. Indeed, as is evidenced in what follows, all of the talks had either direct relevance to the main themes of the workshop or important interconnections.

The workshop's lectures made for an attractive, exciting and fresh viewpoint to the topics of the proposed workshop. In the selection of talks, the organizers were also able to achieve a good balance between the overall demographics both in terms of the subject of the lectures as well as in terms of the speakers themselves. Indeed, the speaker list itself includes a good blend of junior and senior participants.

The full list of invited speakers and the title of their talks is summarized below. The abstracts and video recordings of the lectures are available on the workshop's webpage `http://www.birs.ca/events/2019/5-day-workshops/19w5164`.

(1) Ludmil Katzarkov: D modules and Rationality

L. Katzarkov's talk was a great start to the workshop. In only fifty minutes he was able to give context and motivation for much of the rest of the workshop. The topics which he discussed were wide ranging and included many examples. For instance, he motivated the subject of homological mirror symmetry and its relation to the Landsberg-Ginzberg model through an introductory question about the birational geometry of the projective plane. In doing so, he touched on the subject of toric varieties, the Monge Ampère Equation and the subject of Gromov-Witten Invariants, within the context of Hodge theory, rationality questions and orthogonal decompositions. Finally, he was able to achieve the main aim of his talk which was to propose a new approach to nonrationality questions.

(2) Ajneet Dhillon: Essential dimension of stacks of bundles

A. Dhillon gave a nice survey of the concept of essential dimension for gerbes. He motivated the discussion with examples and explained some of the most important numerical invariants that one can associate to a central simple algebra—the period and the index. Before explaining his recent joint work with I. Biswas and N. Hoffmann, he described a related conjecture of J.-L. Colliot-Thélène, N. Karpenko and A. Merkurjev which relates the essential dimension of a gerbe to its period. He also explained related work of P. Brosnan, Z. Reichstein and A. Vistoli.

(3) Max Lieblich: Torelli theorems for derived categories

M. Lieblich reported on joint work with M. Olsson. A main question was to detect birationally at the level of derived categories through the concept of a filtered derived equivalence. The idea is that these tools will allow for a suitable formulation, within the context of derived categories of projective varieties, of analogues to the more traditional Torelli-type theorems which arise via Hodge theory. Before explaining some of the more technical details to this project, including the concept of filtered derived equivalence and the use of Hochshild cohomology, he first surveyed some more classical results which were of direct interest to the subject of the workshop. These topics included the classical statement of the Torelli theorem combined with the works of Mukai, Bondal-Orlov and Kawamata.

(4) Daniel Bragg: Derived invariants of varieties in positive characteristic

Through the partnership between MSRI and BIRS, we were able to obtain travel funding to host D. Bragg who is currently a postdoctoral fellow at the University of California Berkeley. He is a former Ph.D. student of M. Lieblich. He reported on joint work with N. Addington and B. Antieau which pertained to the question of the extent to which Hodge numbers of projective varieties over fields of positive characteristic can be detected at the level of the derived category. In particular, they construct an example of derived equivalent threefolds, which are defined over a characteristic three field, which have different Hodge numbers. In disseminating these results, he explained some important techniques for working with differential forms in positive characteristic; for example the use of de Rham and crystalline cohomology theories, Dieudonné modules, and the theory of the de Rham Witt complexes.

(5) Sarah Frei: Rational points and derived equivalence

S. Frei is currently a postdoctoral fellow at Rice University and is a former Ph.D. student of N. Addington. We were able to provide her with travel funding through the BIRS-MSRI travel funding program. Frei's talk centred around arithmetic questions for derived categories. A main goal was to describe her joint work with N. Addington, B. Antieau and K. Horing which, among other things, gives examples which show that the concept of rational point for varieties over non-algebraically closed fields, cannot, in general, be considered as derived category invariants. This principal is illustrated by two kinds of examples. One is given by a certain torsor over a certain given Abelian variety; the other is given by a pair that consists of certain hyperkähler fourfolds. She aimed her talk at a wide audience. In doing so, she surveyed a number of topics and examples about the nature of derived invariants for varieties over non-algebraically closed fields.

(6) Charles Doran: Gluing Periods for DHT Mirrors

C. Doran reported on very recent joint work with F. You and J. Kostink. At the same time he explained the *DHT mirror symmetry conjecture*, which he made jointly with A. Harder and A. Thompson. As some of the background and motivation for these topics had already been given earlier in the week, for example in the talk by Katzarkov, Doran was able to proceed immediately with an explanation of these more recent developments. In doing so, he gave a detailed explanation of the Tyurin degenerations for Calbai-Yau manifolds and related topics. The idea is that if a Calabi-Yau manifold X admits a Tyurin degeneration to the union of two quasi-Fano varieties X_1 and X_2 intersecting along a smooth anticanonical divisor D, then the Landau-Ginzburg mirrors of (X_1, D) and (X_2, D) can be glued together to obtain the mirror variety of X. The main focus of the talk was to discuss the manner in which the periods of (X_1, D) and (X_2, D) related to the mirror of X. Moreover, it was explained how these techniques allow for a classification and construction of certain mirror polarized K3-surfaces and fibered Calabi-Yau threefolds.

(7) Humberto Diaz: Unramified cohomology and the integral Hodge conjecture

H. Diaz described his recent result which establishes existence of a new type of example which violates the integral Hodge conjecture. In more precise terms, after surveying some earlier related work of J. L. Colliot-Thélène and C. Voisin, he showed how one can construct complex Kummer threefolds, defined over number fields, which possess non-algebraic non-torsion $(2, 2)$-cohomology classes. These results were of interest to many of the workshop's more senior participants.

(8) Stephen Scully: On an extension of the separation theorem for quadratic forms over fields

S. Scully's talk dealt with the concept of *Witt index* for p, q-antistropic nondegenerate quadratic forms. First he gave some general motivation from the theory of quadratic forms. This included, for example, the question of the extent to which an anisotropic quadratic form remains isotropic after extension to the function field of a quadric. He then discussed how one is able to determine constraints on the Witt index, as a function of discrete invariants which were determined by p and q, which give insight to this interesting problem. Finally, he stated and proved some of his own more recent results on this topic after having first provided additional context via earlier related results by D. Hoffman.

(9) Patrick McFaddin: Twisted forms of toric varieties, their derived categories, and their rationality

P. McFaddin discussed his joint work with M. Ballard, A. Duncan and A. Lamarche which deals with the concept of twisted forms for toric varieties and the question of the extent to which existence of full-exceptional collections, in the derived category, can be used to deduce rationality properties. As some motivation for his joint work, he surveyed topics that surround a conjecture of Orlov. The idea is that if a variety admits a full exceptional collection over the base ground field, then it should be rational.

(10) Matthew Ballard: From flips to functors

M. Ballard reported on joint work, with a number of individuals including some of his graduate students. To begin with, he gave context and motivation by explaining a conjecture of Bondal and Orlov. The conjecture asserts that flops of smooth projective varieties should induce an equivalence of derived categories. The he sketched a construction as to how one may construct an integral Fourier-Mukai kernel, essentially starting from a flip of normal projective varieties. In doing so, he explained aspects of his joint work with Diemer and Favero. Later in the week, during his talk, D. Favero developed these themes further.

(11) Raman Parimala: Quadratic forms and Brauer groups

R. Parimala's talk encompassed many topics that surround the question of obtaining uniform bounds for splittings of elements in Brauer groups with some emphasis on quadratic forms. One theme was the manner in which quadratic forms and Clifford algebras may be used as a tool for studying the Brauer group. Of particular interest was the question of period index bounds for the Brauer group, of a given field, and the u-invariant. She also discussed the period index problem within the context of the recent work of B. Antieau-et-al which contains period index results for surfaces over p-adic fields. Finally, she reported on her more recent joint results which are in the direction of local global principles for existence of points for certain hyperelliptic curves with genus at least equal to two.

(12) John Lesieutre: Numerical dimension revisited

J. Lesieutre's lecture had subject the concept of numerical dimension for $\mathbb{R}$-Cartier divisors on non-singular projective varieties. After giving a self-contained introduction to this subject, he explained his recent example which shows that, in general, the numerical and Iitaka dimensions for $\mathbb{R}$-Cartier divisors need not hold in general. Finally, this example was illustrated via a computer animation which was worked out in conjunction with his undergraduate research students.

(13) Alexander Duncan: Indistinguishability and simple algebras

A. Duncan reported on his joint work with M. Ballard, A., Lamarche and P. McFaddin, which has subject étale forms for projective varieties over non-algebraically closed fields. There are analogous questions for separable algebras. Some more technical aspects for establishing these results include several constructions that surround certain kinds of coflasque resolutions for linear algebraic groups. Explaining these more technical details was of interest and occupied a good deal of the talk.

(14) James Lewis: Indecomposable K_1 classes on a Surface and Membrane Integrals

J. Lewis surveyed many topics that surrounded algebraic cycles and mixed Hodge structures. For instance, he first motivated and outlined the construction of the transcendental regulator map before stating the conjecture about this map which is due to Bloch and Griffiths. He then discussed topics that deal with his more recent joint results which aim to shed light on the Milnor-Beilinson-Hodge conjecture. Finally, he explained why the concept of membrane integrals should allow for a tool for detecting indecomposable classes in the first Milnor K-group of a given projective algebraic surface.

(15) Martin Gulbrandsen: Donaldson-Thomas theory for abelian threefolds

M. Gulbrandsen lectured about Donaldson Thomas invariants for Calabi-Yau and abelian threefolds. He first discussed the key concepts and related results, for example illustrating how these invariants provide virtual counts for the number of stable sheaves with given fixed numerical invariants. Then he focused on the case of Abelian threefolds with some emphasis on the interpretation in terms of derived categories. One idea was to illustrate the difference between the Calabi-Yau and the threefold case in general. Finally, he reported on results which are contained in his joints work with R. Moschetti including related results by Oberdieck-Shen and Oberdieck-Piyaratue-Toda.

(16) Nathan Grieve: Birational divisors and consequences for noncommutative algebra and arithmetic

The concept of birational divisor arises in different contexts; for example in work of V. V. Shokurov and, independently, P. Vojta. This lecture by N. Grieve reported on related recent developments. The main emphasis was to motivate and explain his joint work (with C. Ingalls) which deals with log pairs which are determined by ramification of classes in Brauer groups of function fields of algebraic varieties. A key point was to explain how to define and study a birationally invariant concept of Iitaka-Kodaira dimension. The other idea was to determine the nature of this invariant for suitable embeddings of division algebras. He was also able to briefly mention how the concept of birational divisor arises in his recent works which deal with complexity and distribution of rational points in projective varieties. Those results build on earlier work of M. Ru and P. Vojta (among others).

(17) David Favero: VGIT for CDGAs

D. Favero reported on his joint more recent works including those with M. Ballard and L. Katzarkov. Particular attention was given to the subject of Variation of Geometric Invariant Theory quotients within the context of differential graded algebras. He also motivated the topic of variation of geometric invariant theory quotients from the point of view of M. Reid as well as subsequent work by Dolgachev-Hu which built on earlier work of M. Thaddeus. Some other topics of interest were the many more recent results about orthogonal decompositions for Derived Categories of Moduli Spaces and Deligne Mumford stacks. More specifically, Favero explained related results of Kawamata, Orlov, Baytrev-Borisov including his joint work with Kelly and Doran. He also reported on a recent joint project with N. K. Chidambaram, whose is currently a graduate student at the University of Alberta . Finally, Favero followed-up, in some more detail, topics which were discussed earlier in the week by Ballard.

(18) Luigi Lombardi: Fibrations of algebraic varieties and derived equivalence

The purpose of L. Lombardi's lecture was to report on recent results that deal with special classes of fibrations, onto normal projective varieties, which admit a finite morphism to an abelian variety. Of particular interest is their behaviour under derived equivalence. One of his more specific aims was to explain applications of his earlier joint work with M. Popa. That work had subject the topic of derived invariance for the non-vanishing loci (attached to the canonical bundle) and the generic vanishing theory. For example, he explained how that work can be used to understand the manner in which derived equivalences for such fibrations can be used to understand isomorphism classes of fibrations onto smooth projective curves with genus at least equal to two. Finally, he explained how the conjectural derived invariance of Hodge numbers relates to the question of derived invariance for fibrations onto higher dimensional bases.

(19) Stefan Gille: A splitting principle for cohomological invariants of reflection groups

S. Gille's talk was an inspiring conclusion to the workshop. He spoke about various cohomological invariants that can be associated to Milnor K-groups and questions about ramification. The topics which he discussed complemented a number of lectures which took place earlier in the week including the lecture by Lewis. To begin with, he surveyed various background results with some emphasis on the treatment which was given in J.P. Serre's UCLA lectures. This allowed him to report on his joint work with C. Hirsch in some detail. Briefly, those results deal with generalizations of more classical splitting principals, for cohomological invariants of Weyl groups, so as to apply to the case of orthogonal reflection groups.

4 Scientific Progress Made

At the conclusion of the workshop, the organizers reached out to the workshop's participants to inquire about scientific research related activities and progress that may been seen as outputs to the workshop itself. Several participants mentioned to us that the blend of the three diverse yet overlapping topics of the workshop allowed for solid progress in terms of developing their interactions further. We received several enthusiastic responses some of which we describe here.

As one representative example, M. Ballard made mention of the fact that the subject mater of the workshop was diverse yet cohesive. He also said he enjoyed several discussions about these varying intersecting topics, which include, for instance, derived categories, birational geometry, noncommutative algebra and arithmetic. He also made mention of the fact that the workshop contained a sufficient amount of common core interests that enabled these interactions to be meaningful and productive.

While not all invited participants were able to attend, we have received evidence that the workshop has still been at least a tangential benefit to those individuals. For example, shortly after the conclusion of the workshop, we have been informed from such individuals that they have benefited from the reading of the workshop's abstracts and watching of the recorded lectures themselves.

As some further evidence of the workshop's immediate impact, A. Dhillon made explicit mention that the workshop lectures and discussions have inspired a collection possible future research projects. He was also able to meet new researchers and made some offers of invitations for seminar lecturers for seminars at his home institution.

The concept of numerical dimension was the main focus of the lecture by J. Lesieutre; it also arose in passing during the lecture of N. Grieve. At the conclusion of the workshop, we learned from T. Eckl that these lectures inspired a possible joint project between himself and Lesieutre.

Finally, at the conclusion of the workshop, J. Lewis wrote to say that he was very excited about having had participated and that the interdisciplinary aspect of this workshop provided stimulus for interaction among colleagues within the fields of algebraic geometry, algebra and K-theory. He also mentioned that the workshop assisted in his recruitment of two more junior participants for the thematic research semester which he is organizing at the Newton Institute during the 2020 Spring Semester.

5 Outcome of the Meeting

The workshop 19w5164 facilitated research collaborations amongst researchers with expertise in Brauer groups, derived categories, birational geometry and nearby areas. It was a success and will have a lasting impact on the participants and the mathematical community at large. In short, the overall broader impacts of this event will be felt by researchers near to its theme for several years to come.

The organizers made efforts to ensure participation by qualified Canadian and internationally based individuals, woman, junior researchers and other visible minorities. This is evidenced directly in the workshop's participant list. This success in attracting qualified personnel was made possible by a handful of factors. For example, the workshop dates were compatible with the fall reading break at the University of Alberta. This facilitated attendance by more junior participants (including graduate students).

The fact that the workshop was held in Banff was a contributing factor for the attraction of participants. Indeed, the workshop hosted a good collection of international participants (including researchers who are based in the United States, Norway, the United Kingdom, Italy and Germany). Many of these individuals indicated directly to us that the workshops of this kind, held in Banff, are attractive for them to participate in.

It remains challenging to ensure adequate travel funding for more junior participants. To facilitate this cause, prior to the workshop, the organizers reached out to selected junior participants at MSRI member US based institutions. In doing so, the workshop was able to take advantage of the MSRI-BIRS travel funding program. Via this program, two US based postdoctoral fellows received travel funding (S. Frei and D. Bragg).

When making the workshop schedule, the organizers made an effort to allow for informal interaction time amongst participants. Similarly, when formulating the participant and speaker list, some care was taken to ensure a good blend of junior, emerging and senior researchers. There was time for discussion before and after the talks. The first day of the workshop coincided with the Remembrance Day regional holiday and a minute of silence was observed by participants midway through one of the scheduled lectures.

The workshop's more senior participants played an important role in facilitating these interactions. They participated directly in the lectures and discussed research related topics during the other scheduled and unscheduled portions of the workshop. As one final indication of broader impacts, we mention that already within the 6-8 weeks following the workshop, a handful of the workshop's participants have posted to the arXiv preprint server works which intersect with their contributions to the workshop.

References

[1] N. Addington and R. Thomas. Hodge theory and derived categories of cubic fourfolds. *Duke Math. J.*, 163(10):1885–1927, 2014.

[2] B. Antieau, A. Auel, C. Ingalls, D. Krashen, and M. Lieblich. Period-index bounds for arithmetic threefolds. *Invent. Math.*, 216(2):301–335, 2019.

[3] B. Antieau, D. Krashen, and M. Ward. Derived categories of torsors for abelian schemes. *Adv. Math.*, 306:1–23, 2017.

[4] M. Artin and D. Mumford. Some elementary examples of unirational varieties which are not rational. *Proc. London Math. Soc. (3)*, 25:75–95, 1972.

[5] A. Auel, M. Bernardara, and M. Bolognesi. Fibrations in complete intersections of quadrics, Clifford algebras, derived categories, and rationality problems. *J. Math. Pures Appl. (9)*, 102(1):249–291, 2014.

[6] M. Ballard, D. Deliu, D. Favero, M. U. Isik, and L. Katzarkov. Resolutions in factorization categories. *Adv. Math.*, 295:195–249, 2016.

[7] M. Ballard, D. Favero, and L. Katzarkov. A category of kernels for equivariant factorizations and its implications for Hodge theory. *Publ. Math. Inst. Hautes Études Sci.*, 120:1–111, 2014.

[8] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010.

[9] A. Bondal and D. Orlov. Reconstruction of a variety from the derived category and groups of autoequivalences. *Compositio Math.*, 125(3):327–433, 2001.

[10] A. Căldăraru. The Mukai pairing. II. The Hochschild-Kostant-Rosenberg isomorphism. *Adv. Math.*, 194(1):34–66, 2005.

[11] P. Cascini and V. Lazić. New outlook on the minimal model program, I. *Duke Math. J.*, 161(12):2415–2467, 2012.

[12] D. Chan and C. Ingalls. The minimal model program for orders over surfaces. *Invent. Math.*, 161(2):427–452, 2005.

[13] D. Chan and R. S. Kulkarni. del Pezzo orders on projective surfaces. *Adv. Math.*, 173(1):144–177, 2003.

[14] D. Chan and R. S. Kulkarni. Numerically Calabi-Yau orders on surfaces. *J. London Math. Soc. (2)*, 72(3):571–584, 2005.

[15] C. H. Clemens and P. A. Griffiths. The intermediate Jacobian of the cubic threefold. *Ann. of Math. (2)*, 95:281–356, 1972.

[16] A. Corti and V. Lazić. New outlook on the minimal model program, II. *Math. Ann.*, 356(2):617–633, 2013.

[17] I. V. Dolgachev and Y. Hu. Variation of geometric invariant theory quotients. *Inst. Hautes Études Sci. Publ. Math.*, (87):5–56, 1998.

[18] C. F. Doran, J. Kostiuk, and F. You. The Doron-Harder-Thompson conjecture for toric complete intersections. *Preprint*, 2019.

[19] M. Green and R. Lazarsfeld. Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville. *Invent. Math.*, 90(2):389–407, 1987.

[20] P. A. Griffiths. On the periods of certain rational integrals. I, II. *Ann. of Math. (2)*, 90:496–541, 1969.

[21] Christopher D. Hacon. A derived category approach to generic vanishing. *J. Reine Angew. Math.*, 575:173–187, 2004.

[22] B. Hassett. Some rational cubic fourfolds. *J. Algebraic Geom.*, 8(1):103–114, 1999.

[23] G. Hochschild, B. Kostant, and A. Rosenberg. Differential forms on regular affine algebras. *Trans. Amer. Math. Soc.*, 102:383–408, 1962.

[24] Y. Hu and S. Keel. Mori dream spaces and GIT. *Michigan Math. J.*, 48:331–348, 2000.

[25] D. Huybrechts. *Fourier-Mukai Transforms in Algebraic Geometry*. Oxford University Press, 2006.

[26] Y. Kawamata. Equivalences of derived catgories of sheaves on smooth stacks. *Amer. J. Math.*, 126(5):1057–1083, 2004.

[27] A. Kuznetsov. Derived categories of cubic fourfolds. In *Cohomological and geometric approaches to rationality problems*, pages 219–243. Birkhäuser Boston, Inc., Boston, MA, 2010.

[28] M. Lieblich and M. Olsson. Derived categories and birationality. *Preprint*, 2020.

[29] L. Lombardi and M. Popa. Derived equivalence and non-vanishing loci II. In *Recent advances in algebraic geometry*, volume 417, pages 291–306. Cambridge Univ. Press, Cambridge, 2015.

[30] S. Mukai. Semi-homogeneous vector bundles on an Abelian variety. *J. Math. Kyoto Univ.*, 18(2):239–272, 1978.

[31] S. Mukai. Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves. *Nagoya Math. J.*, 81:153–175, 1981.

[32] N. Nakayama. *Zariski-decomposition and abundance*. Mathematical Society of Japan, Tokyo, 2004.

[33] D. Orlov. Derived categories of coherent sheaves and triangulated categories of singularities. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, volume 270, pages 503–531. Birkhäuser Boston, Inc., Boston, MA, 2009.

[34] G. Pareschi and M. Popa. Regularity on abelian varieties. I. *J. Amer. Math. Soc.*, 16(2):285–302 (electronic), 2003.

[35] G. Pareschi and M. Popa. Regularity on abelian varieties. II. Basic results on linear series and defining equations. *J. Algebraic Geom.*, 13(1):167–193, 2004.

[36] G. Pareschi and M. Popa. Regularity on abelian varieties III: relationship with generic vanishing and applications. In *Grassmannians, moduli spaces and vector bundles*, volume 14 of *Clay Math. Proc.*, pages 141–167. Amer. Math. Soc., Providence, RI, 2011.

[37] M. Popa. Derived equivalence and non-vanishing loci. In *A celebration of algebraic geometry*, volume 18 of *Clay Math. Proc.*, pages 567–575. Amer. Math. Soc., Providence, RI, 2013.

[38] M. Popa and C. Schnell. Derived invariance of the number of holomorphic 1-forms and vector fields. *Ann. Sci. Éc. Norm. Supér. (4)*, 44(3):527–536, 2011.

[39] M. Reid. Decomposition of toric morphisms. In *Arithmetic and geometry, Vol. II*, volume 36, pages 395–418. Birkhäuser Boston, Boston, MA, 1983.