Topology and Measure in Dynamics and Operator Algebras

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September 8 -13, 2019

It is not a coincidence that ergodic theory and the theory of operator algebras were both born as formal disciplines under the pioneering vision of von Neumann, and indeed the two subjects have enjoyed a venerable history of interaction dating back to the seminal papers of Murray and von Neumann in the 1930s and 40s. This interaction largely respects the two distinct but interrelated streams into which each of the subjects principally branches, the one measure-theoretic (von Neumann algebras and measurable dynamics) and the other topological (C*-algebras and topological dynamics). In the early years both ergodic theory and operator algebras benefitted both separately and together from a common viewpoint of operators on Hilbert space, but the eventual development in dynamics of combinatorial and probabilistic methods, in conjunction with an emphasis on asymptotic over local structural phenomena, seemed at a certain point to cause the two to drift apart. It is certainly true that dynamics has never ceased to provide a rich source of examples and inspiration in operator algebras, but it is only the last few decades that operator algebra theory has really been able to repay its debts through the promotion of a certain kind of abstract structural viewpoint that links it up with the theory of orbit equivalence and questions of rigidity, finite approximation, and tileability in the study of general groups and their actions (what is often nowadays compiled under the rubric "measured group theory").

One of the major themes that has long guided progress in both ergodic theory and operator algebra theory is the linkage between measure and topology, not only at the direct technical level (as one can see for example in the variational principle in dynamics and in the study of traces on C*-algebras) but also at the level of analogy. This latter kind of connection, involving the identification of fruitful conceptual correspondences, has played out with spectacular consequences in the classification theory of amenable operator algebras. Indeed the celebrated work of Connes and Haagerup on the von Neumann algebra side not only gave us a classification of amenable (i.e., injective) factors that paralleled the work of Elliott on AF C*-algebras, but has also served as an invaluable resource of techniques and ideas in the much more ambitious classification program that evolved out of the AF framework, a program which in its simple unital form has now essentially been consummated, after many years of research by many hands, through a felicitous combination of papers by Gong–Lin–Niu [8], Elliott–Gong–Lin–Niu [6], and Tikuisis–White–Winter [17] (these treat the stably finite case, with the purely infinite case having been handled earlier by Kirchberg and Phillips in the 1990s).

In the C*-algebra world, counterexamples of Toms and Rørdam based on work of Villadsen showed that amenability (or, as it is more commonly known in this setting, nuclearity) by itself is insufficient for classifiability by standard invariants (K-theory and traces), and a major goal of the Elliott program has been to formulate the right abstract regularity property that for the purposes of classification would replace amenability. We now know this role to be fulfilled by the nuclear dimension of Winter and Zacharias. The idea of such a dimensional invariant based on completely positive approximation and order-zero maps was a driving force in the novel approach to classification that Winter developed in the 2000s, and an influential part of this project has been the Toms–Winter conjecture postulating the equivalence of finite nuclear dimension, $\mathcal{Z}$-stability, and strict comparison for simple separable unital nuclear C*-algebras. In addition to its technical utility in classification arguments, the property of $\mathcal{Z}$-stability, itself a kind of finite-dimensional embeddability via order-zero maps, has turned out be a considerably more flexible condition to verify in C*-crossed products on the basis of underlying dynamical data.

It is the breakthroughs of Matui and Sato on the Toms–Winter conjecture [10, 11] that bring us to the defining theme of the workshop. In this work Matui and Sato brought von Neumann algebra techniques to bear in a surprisingly direct way on problems of a C*-algebraic nature, announcing a new phase in the interplay between measure and topology within operator algebra theory. These methods and ideas have been widely influential, and in particular have paved the way to further advances on the Toms–Winter conjecture leading to the recent full confirmation of its hypothesized equivalence between finite nuclear dimension and $\mathcal{Z}$-stability in the simple separable unital nuclear case [2]. In the domain of topological dynamics, with completely independent motivation coming from problems in entropy theory, Downarowicz and Zhang showed that amenable groups can be exactly tiled using finitely many Følner shapes, yielding a refinement of the Ornstein–Weiss tileability that can be viewed as a hybridization of topological and measure-theoretic perspectives [5]. These apparently disparate developments were soon discovered to be closely related through the fulcrum of $\mathcal{Z}$-stability, with the Toms–Winter conjecture now coming to serve as the passport to classifiability for a large swath of crossed products. The aim of the workshop was to consolidate and promote all of these ideas operating at the intersection of measure and topology, and in particular to explore how they play out within the classification theory of C*-algebras and in the study of dynamical and discrete structures that represent its basic source of examples and applications.

One of the unusual and very timely opportunities that the workshop afforded was an in-depth exposition of a new approach to the stably finite classification due to José Carrión, Jorge Castillejos, Samuel Evington, James Gabe, Christopher Schafhauser, Aaron Tikuisis, and Stuart White. This work traces its origins to a prior BIRS workshop and combines von Neumann algebra methods inspired by Connes and Haagerup with the powerful KK-theoretic methodology recently developed by Schafhauser, very much exemplifying the theme of the meeting and providing a striking illustration of the simplifying power of the measure-theoretic perspective in the analysis of topological structure. The entire constellation of ideas was presented in a series of four lectures, the first by Carrión on the classification of *-homomorphisms, the second by Evington on complemented partitions of unity and uniform property Gamma (the latter being the crucial tool in [2] for establishing the equivalence of of finite nuclear dimension and $\mathcal{Z}$-stability in the Toms–Winter conjecture), the third by Gabe highlighting the similarities with the purely infinite classification and speculating on the possibility of a truly unified theory, and the fourth by Castillejos on applications of complemented partitions of unity and uniform property Gamma. Talks on nuclear dimension were delivered by Stuart White and Wilhelm Winter, the first on obtaining estimates in the presence of $\mathcal{O}_\infty$-stability and the second on a relativization to sub-C*-algebras. In the non-unital domain, Huaxin Lin reported on classification and range results in the non-unital case obtained in collaboration with George Elliott, Guihua Gong, and Zhuang Niu [7] and with Guihua Gong.

These presentations on classification were augmented by a discussion of some of the ramifications of uniform property Gamma for the study of crossed products in the talks of Hung-Chang Liao on the small boundary property and almost finiteness and of Zhuang Niu on radius of comparison and mean dimension for actions of $\mathbb{Z}^d$ and groups of subexponential growth [12, 13]. These represent two approaches to the question of deciding when a free minimal action of an amenable group on a compact metrizable space yields a classifiable crossed product, a problem which can be traced back to the early days of classification theory as one of its principal sources of motivation and about which there is now some optimism of finally being able to attain a more or less general conclusion. In a complementary direction, Karen Strung showed us how many classifiable C*-algebras that may not be expressible as crossed products can nevertheless be realized dynamically by using the equivalence relation that naturally arises from the breaking of orbits in a $\mathbb{Z}$-action [14]. This brings us more into line with the measure-theoretic perspective of von Neumann algebras, where orbit breaking is already directly built into the structure of the crossed product, and opens up a wellspring in the search for model algebras in classification. Viewing actions on C*-algebras as objects in their own right, Gábor Szabó presented a categorical framework, anchored in the relation of cocycle conjugacy, for establishing equivariant versions of the core components and statements of the Elliott classification program [16]. Coming back to the question of how discrete data transfers into C*-algebra structure, but now in the context of k-graph C*-algebras, Elizabeth Gillaspy discussed the relation between Morita equivalence and moves on the graph, with an eye towards a longer-term effort to establish the kind of classification theorem that Eilers, Restorff, Ruiz, and Sørensen obtained in the ordinary graph framework. On the groupoid front, Charles Starling presented an equivalent set of conditions for the simplicity of a C*-algebra associated to an étale groupoid, resolving a question about the simplicity of a non-Hausdorff example coming from the Grigorchuk group [3], while Volodymyr Nekrashevych demonstrated that the dynamic asymptotic dimension of the groupoid of germs associated to a locally expanding self-covering of a compact space is equal to the covering dimension of the space, and in particular is finite, which brings these specimens under the compass of classification.

We also obtained a glimpse into rigidity phenomena, a topic which looks to be one of the next major horizons in C*-algebra theory following a long and impressive line of development over the last two decades in the domains of von Neumann algebras and measure-preserving dynamics. In this direction we heard talks by Hanfeng Li on entropy and Shannon orbit equivalence [9], by Yuhei Suzuki on "tight" inclusions of C*-algebras [15], and by Mehrdad Kalantar on representation rigidity for groups. The representation theme was also taken up by Kristin Courtney, who spoke on amalgamated free products of residually finite-dimensional C*-algebras [4], and Rufus Willett, who spoke on stability questions connected to index theory and on their relevance to classification theory.

All in all, the workshop gave us a vivid picture of C*-algebra theory at a historic crossroads, on the crest of spectacular advances which have not only resulted in definitive classification theorems but also delivered powerful new tools that promise to usher in new chapters in the study of regularity, homogeneity, and rigidity in dynamics and operator algebras, both within and beyond the realm of amenability.

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