Crossing Numbers Turn Useful

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1 An Overview of Crossing Numbers

A *graph* G represents a relation between pairs of items. The items are commonly called *vertices* and the set of vertices is denoted $V(G)$. A *relation* is a pair of vertices $\{v_1, v_2\}$. Each relation is called an *edge*, and the set of all edges is denoted $E(G)$. For example, $G = K_5$ is the graph with 5 vertices, every pair of which are together in an edge. This is called the *complete graph* of order 5.

Graphs are important models in many contexts because of their generality. For example, the vertices may represent people and the edges represent when a pair of people are friends. Analysis of this abstracted graph can reveal an underlying structure of the social relationship. Or the vertices may represent processors in a computer network and the edges represent communication networks. The analysis of this graph can reveal the connectivity of the underlying network.

An important class of graphs are those that can be drawn on a plane so that edges do not cross. Continuing the application where the graph represents a computer network, such a graph can be laid out on a circuit board so that communication channels do not cross, so no insulation is needed to avoid electrical shorts. Graphs so drawn without edge crossings are called *planar graphs*.

Not every graph is planar; for example, the graph K_5 described above is not planar. In this case the next best thing would be to draw the graph G in the plane with as few crossings as possible. This minimum taken over all drawings is called the *crossing number of* G, denoted $cr(G)$.

The problem of minimizing crossings when drawing a graph was first raised by Paul Turán. He tells [27] of how he posed the problem while in a forced labor camp in World War II. Here there were a set of n kilns making bricks and m railroad terminals to ship the bricks. Each kiln was connected to each terminal by a rail line. When two lines crossed cars carrying bricks could derail, creating extra work. Turán's idea was to lay out the camp so as to minimize the number of crossing tracks. Here the graph is denoted $K_{n,m}$ (there are n kilns related to each of m terminals, called a *complete bipartite graph*), so in modern terms he was asking for $cr(K_{n,m})$.

Despite the simple nature of the problem not much is known about this parameter. For example, neither the crossing number of the complete graph $cr(K_n)$ or the crossing number of the complete bipartite graph $cr(K_{n,m})$ is known exactly, or even their asymptotic trend.

Applications of the crossing number include VLSI circuit layouts as described above. In 1983 Leighton [17] proved the that the area needed to represent a layout of an electric circuit is closely related to the crossing number of the underlying graph. Another application of crossing number is in the visualization of graphs. Minimizing the number of edge crossings is desirable, albeit competing with other properties such as symmetry and the shape of the edges. Purchase [20] says "... *reducing the number of edge crossings is by far* *the most important aesthetic, while minimizing the number of bends and maximizing symmetry have a lesser effect."*

2 A Closer Look at Crossing Numbers

We turn to a more specific description of problems involving crossing numbers.

2.1 Three classes of graphs

Our first question is about the crossing number of the complete graph.

Conjecture 2.1 $cr(K_n) = \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$.

The formula comes from the optimal known drawings with few crossings. Conjecture 2.1 is true for $n \leq 12$ and is known to be an upper bound for general n. Proving that the formula is a lower bound is much harder. An important partial result would be to establish the asymptotic behavior of the crossing number:

Question 2.2 *Determine*

$$\lim_{n\to\infty} \frac{cr(K_n)}{n^4}.$$

We turn next to the crossing number of complete bipartite graphs.

Conjecture 2.3 $cr(K_{n,m}) = \lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$.

Equality was thought to be established by Zarankiewicz [29], but a flaw in his proof was detailed by Guy [15]. Again, the formula is known to be an upper bound as given by constructions.

The third class of interesting graphs is the *Cartesian product of two cycles* $C_n \times C_m$. The vertex set of this graph is the product of two cyclic groups $Z_n \times Z_m$ with edges joining pairs where one coordinate is the same and the other coordinate differs by one.

Conjecture 2.4 $cr(C_n \times C_m) = n(m-2)$ *where* $n \geq m$.

These graphs embed on the torus. They were originally examined by Harary, Kainen, and Schwenk [16] who showed these crossing numbers grow with $\min\{n, m\}$. In particular, this showed that the crossing numbers of toroidal graphs can be arbitrarily large when drawn in the plane. Again, drawings illustrate that $n(m-2)$ is an upper bound on the crossing number. Once more, demonstrating the lower bound is harder.

These are of course three specific classes of graphs. They are important as they give rise to other natural questions. For example, the crossing number of the complete graph leads to the crossing number of a graph and its minors [22, 23], the crossing number of the complete bipartite graph is important in the theorem of set systems [29], and the crossing number of the product of cycles leads to the number of intersections in meshes of curves [21]. The common theme is that a question about a specific class of graphs can evolve into questions about the general structure of graphs.

2.2 The Geometric Crossing Number

Heretofore we have been examining arbitrary drawings in the plane. A common restriction is to require that edges be line segments. A *geometric drawing* represents vertices by points in the plane with edges being straight-line segements between their ends. By convention, no edge is allowed to pass through another vertex. The *geometric crossing number* $\bar{cr}(G)$ is the minimum number of crossings over all geometric drawings. Note that the geometric crossing number is at least the crossing number, since we are taking the minimum over a smaller collection of drawings.

Perhaps surprisingly, the geometric crossing number can be strictly greater than the crossing number. In particular $\bar{cr}(K_8) = 19 > 18 = cr(K_8)$. This is the smallest complete graph for which the difference is strict. Lovász et al. invoked [28] to show that $cr(K_n)$ and $\bar{cr}(K_n)$ differ by $\Omega(n^4)$. In contrast:

Conjecture 2.5 $\bar{cr}(K_{n,m}) = cr(K_{n,m})$.

We would like to present a conjecture for the exact value of $\bar{cr}(K_n)$, but its value is unclear. Upper bounds are demonstrated by examples, and there is a long history of these that give sequentially smaller number of crossings. Attention has focused on the following (see [3]) :

Conjecture 2.6 *Find*

$$\lim_{n\to\infty} \frac{\bar{cr}(K_n)}{\binom{n}{4}}.$$

This limit is called q_* for reasons discussed in the next section. It is known to exist and that

$$.379972 < \frac{277}{729} \leq q_* \leq \frac{83247328}{218791125} < .380488.$$

Observe that the difference between the upper and lower bounds is quite small (or, from another view, quite large).

2.3 Sylvester's Four-Point Problem

An important motivation of the geometric crossing number is *Sylvester's Four-Point Problem* from geometric probability (see [25], with [26] for a more rigorous statement). Let R be an open set in the plane with finite area. What is the probability $q(R)$ that four points chosen randomly from R form a convex quadrilateral? It is known [7] that if R is convex, then $q(R)$ is minimized when R is a disk and maximized when R is a triangle. Let q_* denote the infimum of $q(R)$ over all R. Scheinerman and Wilf [24] showed this was closely related to the geometric crossing number $\bar{cr}(K_n)$, specifically, that

$$q_* = \lim_{n\to\infty} \frac{\bar{cr}(K_n)}{\binom{n}{4}}.$$

2.4 Lower Bounds on the Crossing Number

The literature abounds with examples of upper bounds for the crossing numbers of classes of graphs. These are given by drawings which realize specific classes of graphs. However, establishing lower bounds is much more difficult. There are two main techniques.

Theorem 2.7 *Let G be a simple graph with n vertices and m edges. Then*

1. (Euler Bound): $cr(G) \geq m - 3n + 6$*; and*
2. (Crossing Lemma Bound): *If $m > 4n$, then* $cr(G) \geq \frac{m^3}{64n^2}$.

The first bound comes from the observation that the largest number of edges in a simple planar graph is $3n - 6$. Hence each edge beyond this must be involved in at least one crossing. The second bound is harder to prove (see [5, 17, 19]), but is much stronger for dense graphs.

2.5 *k*-Edges

Consider a set P of n points in general planar position, that is, no three are colinear. We consider these as the vertices of a geometric drawing of K_n. A *k-edge* is a line through two of the points with exactly k points on one side. Choosing the smaller side we assume that $k \leq \lfloor n/2 \rfloor - 2$. A $(\leq k)$*-edge* is a j-edge with $j \leq k$. Denote the number of $(\leq k)$-edges by $E_{\leq k}$. A remarkable theorem by Lovász, Vesztergombi, Wagner and Welzl [18] and independently by Ábrego and Fernández-Merchant [2] shows

$$\bar{cr}(P) = \sum_{k=0}^{\lfloor n/2 \rfloor - 2} (n - 2k - 3)\, E_{\leq k}(P),$$

so that a lower bound on $E_{\leq k}(P)$ determines a lower bound on the crossing number of that point set. References [2, 18] also show that $E_{\leq k}(P) \geq 3\binom{k+2}{2}$ for any point set k. These two inequalities combine to prove that

$$\bar{cr}(K_n) \geq \frac{3}{8}\binom{n}{4} + \Theta(n^3).$$

Hence the constant q_* of Sylvester's Four-Point Problem is at least $3/8$. A more subtle analysis can improve the bound on $E_{\leq k}(P)$, resulting on an improved estimate for q_*, as well as on the exact calculation of $\bar{cr}(K_n)$ for $n \leq 27$ [1] (see also [4]).

3 Workshop Presentation Highlights

The workshop was characterized by three factors:

1. The emphasis on short, interactive presentations;
2. a flexible schedule; and
3. the willingness of the participants to interact.

To emphasize the first point the entire first day was devoted to short individual presentations. These were originally intended to be 5-minutes presenting an interesting problem or a report on recent research. These quickly turned into more interactive events with the audience asking for further information, mentioning relations with other problems, and offering generalizations. This back-and-forth exchange was greatly appreciated. The format was especially helpful since it introduced each participant to the others and let everyone know who was interested in what sort of problems.

The flexible schedule was also very helpful. As the success of the individual presentations became apparent we quickly expanded the time for them from Monday morning only through the afternoon session as well. The extra time was well spent. We also had our first problem session Tuesday morning. We worried about it being redundant given the extensive discussions on the first day, but were proved wrong. A night's reflection led to some expansions on the first day's talks and the led participants to think of other ideas.

The first major presentation was by Pedro Ramos. This was rescheduled from later in the week because it provided some important background on issues raised in the introductory discussions. Dr. Ramos began with a review of the current state of the rectilinear crossing number, focusing on its relation with Sylvester's 4-point constant and with $(\leq k)$-edges. Specific details about this topic were given in Section 2.5.

The next two major presentations were by Dan Cranston and Peter Hliněný. Dr. Cranston spoke on "Crossings, colorings and cliques", in which he addressed results related to Albertson's conjecture described in Section 4.4. Dr. Hliněný took crossing numbers off of the plane, introducing a new embedding-density parameter *stretch* for a graph embedded on an arbitrary surface. This can be used for lower bounds on the planar crossing number.

The final two presentations were given by Eva Czabarka and Markus Chimani. Dr. Czabarka spoke on lower bounds for crossing numbers. She raised the question of finding the optimal Crossing Lemma type bound. For more information on this the reader is referred to Section 4.3. Dr. Chimani spoke on approximation algorithms calculating the crossing number. In particular he discussed vertex- and edge-insertation algorithms and gave several clever examples. For details see Section 4.7.

The format provided several times for group discussions. Tuesday we as a group decided on several topics on which to focus (for details see Section 4). We then broke into smaller groups to brainstorm on specific projects. One "requirement" set by the organizers was that groups had to meet in Max Bell so that individuals were available for quick questions between groups. This requirement was agreed to, adhered to, and worked perfectly to ensuring the hoped-for frequent collaboration.

Late Thursday we met in full session to report on our group discussions and offer some further problems. Among the other open problems discussed were:

1. Is computing the crossing number was APX-hard? That is, does there exist a polynomial-time algorithm to approximate the crossing number within a factor of $1 + \epsilon$?

2. A 2-page drawing is a drawing with all vertices on a line, and with no edges crossing that line. The 2-page crossing number of a graph is the minimum number of crossings in a 2-page drawing of the graph. Conjecture 2.1 can be achieved by a 2-page drawing of K_n as well. Is the 2-page crossing number of K_n equal to its crossing number?

3. Is the sequence $\{cr(K_n)\}_{n=1}^{\infty}$ convex?

4. Does every optimal drawing of K_n contain an edge that is in no crossing? (This is false for general graphs.)

5. How does the crossing number change upon the deletion of a random subset of edges?

4 Workshop Research Highlights

We next give some specifics of problems discussed and discoveries made during the meeting.

4.1 The 3-Cut Problem

Suppose that a graph G has a cut-set of 3 edges whose deletion leaves graphs G_1, G_2. Let $\bar{G}_1$, $\bar{G}_2$ denote the graphs resulting by contracting the other subgraph to a single point. How is $cr(G)$ related to $cr(\bar{G}_1)+cr(\bar{G}_2)$? During this workshop Jesús Leaños, Markus Chimani, and Drago Bokal proved the following:

Theorem 4.1 $cr(G) = cr(\bar{G}_1) + cr(\bar{G}_2)$.

The proof, while too technical to include here, has already been written up [8]. Bruce Richter praised the effort as an example of the best type of collaboration, saying "There are three independent important new ideas in the proof, one created by each of the three authors."

4.2 Crossing Numbers of Periodic Graphs

Consider a graph T with two disjoint distinguished sets of vertices $\ell_1, \ldots, \ell_k$ and $r_1, \ldots, r_k$. Build a graph T^n from n disjoint copies T_j of T by adding edges between each r_i in the j^{th} copy to ℓ_i in the $j+1^{st}$ copy for $j = 1, \ldots, n-1$. The graph can be visualized by drawing T_j in a rectangle, or *tile*, with the vertices ℓ_i on the left boundary and r_i on the right boundary, placing the tiles next to each other left-to-right along a line, and adding in the connecting edges between adjacent tiles. The drawing suggests that the crossing number of T^n might be linear in n. Confirming this intuition is more difficult. In fact, the first goal is to show that determining $lim_{n\to\infty} cr(T^n)/n$ is computable.

This problem was proposed by Bruce Richter. A large group of people, led by Zdenek Dvorak and Bojan Mohar worked on the problem. Eventually, the group hammered out a proof that the limit was indeed computable, with the details still to be filled in.

4.3 Optimizing Density Lower Bounds

Drago Bokal, Mojča Bračič, Éva Czabarka and László Székely have been working recently on optimizing the density bounds for the crossing number of graphs, as density bounds can be better for a subgraph than for the whole graph. Density bounds come in two shapes. The first is a linear lower bound from the Euler formula, which has variations for genus and girth conditions. The second is a Crossing Lemma type bound, which is nonlinear, and in most of the cases comes by a bootstrapping from the linear bound. (There are exceptions, however, which came from the bisection width or other methods.) To be more specific, a Crossing Lemma type lower bound is like m^3/n^2, where $n = n(H)$ denotes the number of vertices and $m = m(H)$ is the number of edges in an induced subgraph H of G. The group conjectures that if $d_1 \geq d_2 \geq \cdots \geq d_{n(G)}$ is the degree sequence of the graph G, realized by vertices $v_1, v_2, ..., v_n$, and $H_i = G|_{\{v_1,\ldots,v_i\}}$, then

$$m^3(G)/n^2(G) \geq \max_{1\leq i\leq n} m^3(H_i)/n^2(H_i) = \Omega\left(\frac{m^3(G)/n^2(G)}{n^{2/3}}\right),$$

as a construction suggests the correctness of this suboptimal algorithm.

The optimization of an Euler-type lower bound of the form $\alpha\, m(H) - \beta\, n(H)$ for the crossing number of a graph is equivalent to finding the optimal solution in the Dual Program, which is equivalent in finding the orientation $\vec{G}$ of G where we minimize the following quantity:

$$\sum_{\substack{v\in V(G)\\ d^+_{\vec{G}}(v)\geq 3}} (d^+_{\vec{G}}(v) - 3) = m(G) - 3n(G) + \sum_{\substack{v\in V(G)\\ d^+_{\vec{G}}(v)<3}} (3 - d^+_{\vec{G}}(v)). \tag{1}$$

In a particular case, de Fraysseix and Ossona de Mendez [13] showed that

Corollary 4.2 *If G is a graph such that for each subgraph H $\frac{m(H)}{n(H)} \leq 3$, then G has an orientation where the indegree of any vertex is at most 3. In particular, planar graphs have such orientation.*

The authors solved (1) and then realized that an early work of Frank and Gyárfás [12] essentially solves the minimization problem in (1) and implies Corollary 4.2, except for the crossing number application.

4.4 Albertson's Conjecture and Convexity

Albertson's Conjecture states that the crossing number of a graph with chromatic number r is at least that of $cr(K_r)$. Barát and Tóth [9] showed that the conjecture is true for $r \leq 16$ and is also true up to a multiplicative constant. A group including Dan Cranston, Jozsef Balogh, Drago Bokal, Eva Czabarka, and László Székely worked on this conjecture, hoping to expand the work of Albertson, Cranston and Fox [6].

This group made positive progress in that they showed several approaches would not work. Their work led to a new conjecture that the sequence is *convex*: that is, $cr(K_{n+1}) + cr(K_{n-1}) \geq 2cr(K_n)$. The conjecture *should* be true, but is expected to be difficult as even the asymptotic behavior of $cr(K_n)$ is only known up to a multiplicative constant.

4.5 The Crossing Number of Twisted Planar Tiles

The following problem was posed by Bojan Mohar. Let T be a graph embedded (without crossings) in a tile. Let $\ell_1, \ldots, \ell_n$ be the vertices top-to-down order on the left edge of the tile and $r_1, \ldots, r_m$ be those in order on the right. The goal is to *twist the tile*, that is, to draw (possibly with crossings) the graph in the tile so that $\ell_1, \ldots, \ell_n$ appear on the left in top-to-down order but $r_1, \ldots, r_m$ appear on the right in down-to-top order. How can we calculate the crossing number of this twisted tile $\tilde{T}$?

If there exist k disjoint paths from the left to right edge of T, then the crossing number to $\tilde{T}$ must be at least $\binom{k}{2}$. However, this bound need not be tight. A group of participants examined this problem made some interesting progress on this problem, refining the bound mentioned above, but the exact answer remains elusive.

4.6 *k*-Edges in Drawings

Consider a planar geometric (straight-line) drawing of a complete graph. Recall from Section 2.5 that the number of $(\leq k) - edges$ is related to the crossing number of the drawing. However, the concept has not been applied to classic crossing number problems when the drawings are not geometric.

During the workshop a group around Gelasio Salazar, Silvia Fernandez, Pedro Ramos, and Oswin Aichholzer generalized the underlying concepts to topological graphs. Several promising properties of this new concept have already been obtained. Together with Bernardo Ábrego they are currently investigating the crossing number of the complete graph.

4.7 Approximating Crossing Numbers

A very interesting problem is to approximate the crossing number for *sparse* graphs; those with relatively few edges compared to vertices. The general idea is to find a large planar subgraph, then extend a planar drawing of this subgraph by adding in the few remaining edges without creating too many crossings. There is a polynomial-time algorithm to determine how to insert an edge to minimize the number of resulting crossings [14] , and it is known [10] that this algorithm provides a constant approximation factor for almost all graphs with $\Delta(G-2)/2$. Similar work [11] gives similar results for vertex insertion.

At the workshop a group investigated if the results above can be generalized for inserting a fixed number of edges. Some interesting and promising process was made.

5 Closing Comments from the Organizers

The meeting was very successful. We put together a diverse group by all acounts. We had 20 participants: 7 from the US, 3 from Canada, 2 from Germany, 2 from Mexico, 2 from the Czech Republic, 2 from Slovenia, 1 from Austria, and 1 from Spain. Of those 20 participants, 3 were female researchers, and 3 were junior researchers (who obtained their Ph.D.'s in 2007 or later).

The combination of a location where we were free to concentrate on research all day, the flexibility of the schedule, the size of the group, and choice of topics all contributed to its success. To this end we thank the staff and directorship at the BIRS facility for all of their help. Equally if not more importantly, the organizers thank the participants for their willingness to participate and to contribute to this collaboration.

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APPENDIX A: PARTICIPANTS

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Bokal, Drago	*University of Maribor*
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Székely, László	*University of South Carolina*
Tóth, Csaba D.	*University of Calgary*

APPENDIX B: SCHEDULE

Sunday

16:00	Check-in begins (Front Desk - Professional Development Centre)
17:30–19:30	Buffet Dinner, Sally Borden Building
20:00	Informal gathering in 2nd floor lounge, Corbett Hall

Monday

7:00–8:45	Breakfast
8:45	Welcome by BIRS Station Manager, Max Bell 159
9:00	Introductory short presentations on your research (in alphabetical order)
10:00	Coffee Break, 2nd floor lounge, Corbett Hall
10:30	Continued introductory short presentations
11:30–13:30	Lunch
13:00	Guided Tour of The Banff Centre; meet in Corbett Hall
14:00	Group Photo: front steps of Corbett Hall
14:15	Continued introductory short presentations
15:00	Coffee Break, 2nd floor lounge, Corbett Hall
15:30	Continued introductory short presentations
16:30	Break
17:30–19:30	Dinner

Tuesday

7:00–9:00	Breakfast
9:00	Lectures: Pedro Ramos
10:00	Coffee Break, 2nd floor lounge, Corbett Hall
10:30	Problem Session
11:30-13:30	Lunch
13:30	Lectures: Dan Cranston, followed by Peter Hlineny
15:00	Coffee Break, 2nd floor lounge, Corbett Hall
15:30	Discussion Groups
16:30	Break
17:30–19:30	Dinner
19:30	Outing to Banff Hot Springs

Wednesday

7:00–9:00	Breakfast
9:00	Lectures: Eva Czabarka, followed by Markus Chimani
10:00	Coffee Break, 2nd floor lounge, Corbett Hall
10:30	Discussion Groups
11:30–13:30	Lunch
13:30	Free Afternoon – Enjoy Banff
17:30–19:30	Dinner

Thursday

7:00–9:00	Breakfast
9:00	Discussion Groups
10:00	Coffee Break, 2nd floor lounge, Corbett Hall
10:30	Discussion Groups
11:30–13:30	Lunch
13:30	Discussion Groups
15:00	Coffee Break, 2nd floor lounge, Corbett Hall
15:30	Second Problem Session: latest ideas, and results from the week
16:30	Break
17:30–19:30	Dinner

Friday

7:00–9:00	Breakfast
9:00	Discussion Groups
10:00	Coffee Break, 2nd floor lounge, Corbett Hall
10:30	Closing Remarks, followed by Informal Discussions
11:30–13:30	Lunch
12:00	Checkout