# Algebraic Graph Theory

## Banff International Research Station

## 24–29 April 2011

### Meeting Rooms

All lectures will be held in Max Bell 159 (Max Bell Building accessible by walkway on 2nd floor of Corbett Hall). LCD projector, overhead projectors and blackboards are available for presentations. Note that the meeting space designated for BIRS is the lower level of Max Bell, Rooms 155–159. Please respect that all other space has been contracted to other Banff Centre guests, including any Food and Beverage in those areas.

### Meals

Breakfast (Buffet): 7:00–9:30 am, Sally Borden Building, Monday–Friday
Lunch (Buffet): 11:30 am–1:30 pm, Sally Borden Building, Monday–Friday
Dinner (Buffet): 5:30–7:30 pm, Sally Borden Building, Sunday–Thursday
Coffee Breaks: As per daily schedule, 2nd floor lounge, Corbett Hall

**Please remember to scan your meal card at the host/hostess station in the dining room for each meal.**

# Schedule

## Sunday

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| 4:00 | check-in begins |
| 5:30 – 7:30 | dinner |
| evening | informal gathering in Corbett Hall lounge |

## Monday

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| 8:45 – 9:00 | introduction and welcome by BIRS station manager |
| 9:00 – 10:00 | **Willem Haemers**<br>*Universal adjacency matrices with two eigenvalues* |
| 10:00 – 10:30 | coffee break |
| 10:30 – 11:00 | **Sylvia Hobart**<br>*Coherent configurations, subset bounds, and the Erdos-Renyi graph* |
| 11:00 – 11:30 | **Jason Williford**<br>*Nonexistence Conditions for Coherent Configurations* |
| 11:30 – 1:00 | lunch |
| 1:00 – 2:00 | guided tour of Banff Centre |
| 2:00 – 2:05 | group photo |
| 2:05 – 3:05 | **Eric Moorhouse**<br>*Open Problems Concerning Automorphism Groups of Projective Planes* |
| 3:05 – 3:35 | **Sebastian Cioabă**<br>*On a conjecture of Brouwer regarding the connectivity of strongly regular graphs* |

## Tuesday

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| 9:00 – 10:00 | **William Martin**<br>*Bounding the size of codes and designs in association schemes* |
| 10:00 – 10:30 | coffee break |
| 10:30 – 11:00 | **Ameera Chowdhury**<br>*On a conjecture of Frankl and Füredi* |
| 11:00 – 11:30 | **Frédéric Vanhove**<br>*Erdős-Ko-Rado problems in polar spaces* |
| 11:30 – 1:30 | lunch |
| 2:00 – 3:00 | **Rick Wilson**<br>*A zero-sum Ramsey-type problem and the Smith form of certain incidence matrices* |
| 3:00 – 3:30 | **Karen Meagher**<br>*TBA* |

## Wednesday

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|---|---|
| 9:00 – 10:00 | **Martin Roetteler** |
| | *Quantum state generation: adversaries, algorithms, applications* |
| 10:00 – 10:30 | coffee break |
| 10:30 – 11:00 | **Simone Severini** |
| | *A comprehensive theory of noncontextuality in physics* |
| 11:00 – 11:30 | **Aidan Roy** |
| | *Complex spherical designs and nonsymmetric association schemes* |
| 11:30 – 1:30 | lunch |
| afternoon | free time |

## Thursday

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|---|---|
| 9:00 – 9:30 | **Ferenc Szöllősi** |
| | *MUBs, MASAs and Complex Hadamard matrices* |
| 9:30 – 10:00 | **Ada Chan** |
| | *Complex Hadamard Matrices and Strongly Regular Graphs* |
| 10:00 – 10:30 | coffee break |
| 10:30 – 11:00 | **Graham Farr** |
| | *Transforms, minors and generalised Tutte polynomials* |
| 11:00 – 11:30 | **Kerri Morgan** |
| | *Galois Groups of Chromatic Polynomials* |
| 11:30 – 1:30 | lunch |
| 2:00 – 3:00 | **Simeon Ball** |
| | *On subsets of a finite vector space in which every subset of basis size is a basis* |
| 3:00 – 3:30 | coffee break |
| 3:30 – 4:00 | **Boyd Worwawannotai** |
| | *Dual polar graphs and the quantum affine algebra* $U_q(\mathfrak{sl}_2)$ |
| 4:00 – 4:30 | **Jae-ho Lee** |
| | $T$*-modules for* $Q$*-polynomial distance-regular graphs with a Delsarte clique* |

## Friday

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|---|---|
| 9:00 – 10:00 | **Chris Godsil** |
| | *TBA* |
| 10:00 – 10:30 | coffee break |
| 10:30 – 11:00 | **Shonda Gosselin** |
| | *Paley uniform hypergraphs* |
| 11:00 – 11:30 | **Matt DeVos** |
| | *Eigenvalues of* $(3, 6)$*-Fullerines* |
| 11:30 – 1:30 | lunch |
| 12:00 | checkout |

# Abstracts

**Simeon Ball**, Universitat Politécnica de Catalunya

### *On subsets of a finite vector space in which every subset of basis size is a basis*

In this talk we consider sets of vectors $S$ of the vector space $\mathbb{F}_q^k$ with the property that every subset of $S$ of size $k$ is a basis.
The classical example of such a set is the following.
**Example** (Normal Rational Curve) The set

$$S = \{(1, t, t^2, \ldots, t^{k-1}) \mid t \in \mathbb{F}_q\} \cup \{(0, \ldots, 0, 1)\},$$

is a set of size $q+1$. It is easily shown that $S$ has the required property by checking that the $k \times k$ Vandermonde matrix formed by $k$ vectors of $S$, has non-zero determinant.
For $q$ even and $k = 3$, one can add the vector $(0, 1, 0)$ to $S$ and obtain an example with $q + 2$ vectors. For these parameters, such a set of $q + 2$ vectors is called a *hyperoval*, and these have been studied extensively. There are many examples of hyperovals known which are not equivalent (up to change of basis and field automorphisms) to the example above. The only other known examples of size $q + 1$ is an example of size 10 in $\mathbb{F}_9^5$, due to Glynn, and an example in $\mathbb{F}_{2^h}^4$ due to Hirschfeld.
The following conjecture exists in various areas of combinatorics. It is, known as the main conjecture for maximum distance separable codes, the representability of the uniform matroid in matroid theory, the embeddability of the complete design in design theory and Segre's arcs problem in finite geometry.
**Conjecture** A set of vectors $S$ of the vector space $\mathbb{F}_q^k$, $k \leq q - 1$, with the property that every subset of $S$ of size $k$ is a basis, has size at most $q + 1$, unless $q$ is even and $k = 3$ or $k = q - 1$, in which case it has size at most $q + 2$.
I shall present a proof of the conjecture for $q$ prime and discuss the non-prime case. I shall also explain how to then prove the following theorem, which is a generalisation Segre's "oval is a conic" theorem in the case $k = 3$.
**Theorem** If $p \geq k$ then a set $S$ of $q + 1$ vectors of the vector space $\mathbb{F}_q^k$, with the property that every subset of $S$ of size $k$ is a basis, is equivalent to the Normal Rational Curve example, where $q = p^h$.

**Ada Chan**, York University

### *Complex Hadamard Matrices and Strongly Regular Graphs*

An $n \times n$ matrix $W$ is *type II* if $WW^{(-)T} = nI$, where $W^{(-)}$ is the Schur inverse of $W$. If the entries of $W$ have absolute value 1, then $W$ is a *complex Hadamard matrix*.
In this talk, we show that there are only five families of parameters for which the strongly regular graphs give complex Hadamard matrices. Using the Nomura algebras of these complex Hadamard matrices, we see that they do not arise from Diţă's construction.

**Ameera Chowdhury**, University of California San Diego

*On a conjecture of Frankl and Füredi*

Frankl and Füredi conjectured that if $\mathcal{F} \subset 2^X$ is a non-trivial $\lambda$-intersecting family of size $m$, then the number of pairs $\{x, y\} \in \binom{X}{2}$ that are contained in some $F \in \mathcal{F}$ is at least $\binom{m}{2}$ [P. Frankl and Z. Füredi. A Sharpening of Fisher's Inequality. *Discrete Math.*, 90(1):103-107, 1991]. We verify this conjecture in some special cases, focusing especially on the case where $\mathcal{F}$ is additionally required to be $k$-uniform and $\lambda$ is small.

**Sebastian Cioabă**, University of Delaware

*On a conjecture of Brouwer regarding the connectivity of strongly regular graphs*

A $(v, k, \lambda, \mu)$-strongly regular graph (SRG for short) is a finite undirected graph without loops or multiple edges such that (i) it has $v$ vertices, (ii) it is regular of degree $k$, (iii) each edge is in $\lambda$ triangles, (iv) any two nonadjacent points are joined by $\mu$ paths of length 2. The connectivity of a graph is the minimum number of vertices one has to remove in order to make it disconnected (or empty).

In 1985, Brouwer and Mesner used Seidel's characterization of strongly regular graphs with eigenvalues at least $-2$ to prove that the vertex-connectivity of any $(v, k, \lambda, \mu)$-SRG equals its degree $k$. Also, they proved that the only disconnecting sets of size $k$ are the neighborhoods $N(x)$ of a vertex $x$ of the graph.

A natural question is: what is the minimum number of vertices whose removal will disconnect a $(v, k, \lambda, \mu)$-SRG into non-singleton components? In 1996, Brouwer conjectured that this number is $2k - \lambda - 2$. In this talk, I will report some progress on this problem.

This is joint work with Kijung Kim and Jack Koolen (POSTECH, South Korea).

**Matt DeVos**, Simon Fraser University

*Eigenvalues of $(3, 6)$-Fullerines*

An ordinary Fullerine is a cubic planar map where all faces have size 5 or 6. Such graphs may be realized as carbon molecules, and information about their spectrum has physical significance. Here we consider $(3, 6)$-Fullerines, where the faces may have sizes only 3 or 6. We show that these graphs may be realized as Cayley Sum graphs, and we use this to resolve a conjecture of Fowler, which asserts that all eigenvalues of such graphs come in pairs $\{x, -x\}$ except for the four exceptional values $\{3, -1, -1, -1\}$. This is joint work with Goddyn, Mohar, and Samal.

**Graham Farr**, Monash University

*Transforms, minors and generalised Tutte polynomials*

We introduce a family of transforms that extends graph- and matroid-theoretic duality, and includes trinities and so on. Associated with each such transform are $\lambda$-minor operations, which extend deletion and contraction in graphs. We establish how the transforms interact with our generalised minors, extending the classical matroid-theoretic relationship between duality and minors: $(M/e)^* = M^* \setminus e$. We also introduce some generalised Tutte-Whitney polynomials based on these minor operations.

**Chris Godsil**, University of Waterloo

*TBA*

**Shonda Gosselin**, University of Winnipeg

*Paley uniform hypergraphs*

The well known Paley graphs are constructed on finite fields. These graphs are vertex-transitive, self-complementary, and have many other interesting properties. We introduce the Paley graph construction, and then use Paley's algebraic technique to construct some uniform hypergraphs with properties analogous to those of the Paley graphs. We examine the automorphism groups of these hypergraphs, and we show how the edge set of a complete uniform hypergraph can be decomposed into Paley uniform hypergraphs.

**Willem Haemers**, Tilburg University

*Universal adjacency matrices with two eigenvalues*

Consider a graph $G$ on $n$ vertices with adjacency matrix $A$ and degree sequence $(d_1, \ldots, d_n)$. A universal adjacency matrix of $G$ is any matrix in $\mathrm{Span}\, A, D, I, J$ with a nonzero coefficient for $A$, where $D = \mathrm{diag}(d_1, ..., d_n)$ and $I$ and $J$ are the identity and all-ones matrix, respectively. Thus a universal adjacency matrix is a common generalization of the adjacency, the Laplacian, the signless Laplacian and the Seidel matrix. We investigate graphs for which some universal adjacency matrix has just two eigenvalues. The regular ones are strongly regular, complete or empty, but several other interesting classes occur.

**Sylvia Hobart**, University of Wyoming

*Coherent configurations, subset bounds, and the Erdos-Renyi graph*

Many combinatorial objects can be used to construct coherent configurations. I will talk about using a result on subsets of a coherent configuration to bound the size of cocliques in the Erdos-Renyi graph, and suggest other applications.
This is joint work with Jason Williford.

**Jae-ho Lee**, University of Wisconsin

*$T$-modules for $Q$-polynomial distance-regular graphs with a Delsarte clique*

Let $G$ denote a $Q$-polynomial distance-regular graph with a Delsarte clique $C$. We fix a $x$ in $C$. Using $x$ and $C$ we construct a two-dimensional distance partition, and it turns out this partition is equitable. This partition naturally gives a vector space $W$ that is a module for both the subconstituent algebra $T(x)$ and the subconstituent algebra $T(C)$. In this talk, we give a detailed description of the space $W$.

**William Martin**, Worcester Polytechnic Institute

*Bounding the size of codes and designs in association schemes*

The term "Delsarte theory" has been used to describe the study of codes and designs in association schemes. The famous linear programming bound is still the most powerful general-purpose technique to establish the non-existence of error-correcting codes in the Hamming scheme as well as t-designs in the Johnson scheme. Taking this as a point of departure, the goal of the talk is to collect examples where researchers have been able to do better than the LP bound. Examples include the use of Terwilliger algebras as well as the geometry of the standard module in a variety of applications. Hopefully this incomplete survey will lead to further discussion and new techniques.

**Karen Meagher**, University of Regina

*TBA*

## **Eric Moorhouse**, University of Wyoming

*Open Problems Concerning Automorphism Groups of Projective Planes*

There is no shortage of open problems in the study of projective planes! I will mention some problems which highlight the connections between the finite and the infinite case:

(a) Given a group of automorphisms (i.e. collineations) of a projective plane, must the number of point orbits equal the number of line orbits? In the finite case, yes; the problem is open in the infinite case.

(b) What can be said about projective planes in which every quadrangle generates a proper subplane? All known finite examples are classical (i.e. Desarguesian).

(c) Does there exist an infinite plane with automorphism group $G$ such that for every $k \geq 1$, $G$ has only finitely many orbits on $k$-tuples of points? The problem is open even for $k = 4$.

## **Kerri Morgan**, Monash University

*Galois Groups of Chromatic Polynomials*

The chromatic polynomial $P(G, k)$ gives the number of proper colourings of a graph $G$ in at most k colours.

We say $P(G, k)$ has a chromatic factorisation, if $P(G, k) = P(H_1, k)P(H_2, k)/P(K_r, k)$ for graphs $H_1$ and $H_2$ and some $r \geq 0$. Any clique-separable graph, that is, a graph that can be obtained by identifying an $r$-clique in $H_1$ with an $r$-clique in $H_2$, has a chromatic factorisation. A graph is said to be strongly non-clique-separable if it is neither clique-separable nor chromatically equivalent to any clique-separable graph. We give an overview of some of our results on chromatic factorisations of strongly non-clique-separable graphs.

We then give a summary of the Galois groups of chromatic polynomials of degree at most 10. We are interested in identifying the relationships between graphs that have chromatic polynomials with the same Galois group. Some families of graphs that have chromatic polynomials with the same Galois group are given.

**Martin Roetteler**, NEC Laboratories America

*Quantum state generation: adversaries, algorithms, applications*

We introduce a new quantum adversary method to prove lower bounds on the query complexity of quantum state generation problems. This encompasses both, the computation of partial or total functions and the preparation of target quantum states. There had been hope for quite some time that quantum state generation might be a route to tackle the Graph Isomorphism problem. We show that for the related Index Erasure problem our method leads to tight lower bound on the query complexity, showing that a simple reduction to quantum search is optimal. This closes an open problem first raised by Shi [FOCS'02]. Our approach is based on (i) a generalization of the additive and multiplicative adversary methods and (ii) a representation-theoretic way to exploit symmetries of the underlying problem. We construct adversary matrices from the Bose-Mesner algebra of an association scheme that is defined implicitly by the Index Erasure problem.
On the algorithmic side, we present a new technique to tackle quantum state generation which can be thought of as a quantum analog of the classical rejection sampling method (von Neumann, 1951). We analyze the running time of our algorithm using semidefinite programming. As an application, we derive a new quantum algorithm for the hidden shift problem for an arbitrary Boolean function whose running time is obtained by "water-filling" its Fourier spectrum.
Joint work with Andris Ambainis, Loick Magnin, Maris Ozols, and Jeremie Roland and based on `arXiv:1103.3017` and `arXiv:1103.2774`.

**Aidan Roy**, University of Waterloo

*Complex spherical designs and nonsymmetric association schemes*

**Simone Severini**, University College London

*A comprehensive theory of noncontextuality in physics*

We show that there is a family of inequalities associated to each compatibility structure of a set of events (a graph), such that the bound for noncontextual theories is given by the independence number of the graph, and the maximum quantum violation is given by the Lovasz theta-function of the graph, which was originally proposed as an upper bound on its Shannon capacity. Probabilistic theories beyond quantum mechanics may have an even larger violation, which is given by the fractional packing number. We discuss the sets of probability distributions attainable by noncontextual, quantum, and generalized models; the latter two are shown to have semidefinite and linear characterizations, respectively. The implications for Bell inequalities are discussed. In particular, we show that every Bell inequality can be recast as a noncontextual inequality within this family. This provides a tool to single out experiments with a large classical-quantum gap and candidates for loophole-free Bell tests, and a unified framework to discuss alternatives to and theories beyond quantum mechanics. Related material can be accessed at `http://www.homepages.ucl.ac.uk/~ucapsse`

**Ferenc Szöllősi**, Central European University

*MUBs, MASAs and Complex Hadamard matrices*

A complex Hadamard matrix is a square matrix $H$ of order $n$ with the property $|h_{ij}| = 1$ satisfying $HH^* = nI$. In this talk we give an overview of complex Hadamard matrices and relate them to various objects including graphs, block designs, Maximal Abelian $*$-Subalgebras of the $n \times n$ matrices and Mutually Unbiased Bases of $\mathbf{C}^n$.

**Frédéric Vanhove**, Universiteit Gent

*Erdős-Ko-Rado problems in polar spaces*

Consider a vector space $V(n, q)$ over a finite field of order $q$, equipped with a non-degenerate quadratic, alternating or Hermitian form. The associated classical finite polar space consists of the totally isotropic subspaces with respect to that form, and its rank $d$ is the dimension of the maximal totally isotropic subspaces, also known as *maximals*.
We will consider *Erdős-Ko-Rado* or simply *EKR sets of maximals*: subsets of maximals, no two of which intersecting trivially. It is our aim to both determine the maximum size for such subsets, and classify those of that size. A very simple construction of an EKR set of maximals consists of all maximals through one fixed isotropic 1-space (or "point").
The Erdős-Ko-Rado problems for subsets or for subspaces of a vector space can be seen as problems in the Johnson and Grassmann graphs, respectively, and this problem can very similarly be seen as one in the so-called dual polar graphs. Here, we consider subsets of vertices, no two of which at maximal distance $d$.
A combination of eigenvalue techniques and purely geometric arguments is used to deal with this problem. The difficulty of the problem depends on the type of polar space under consideration.
The main obstacles and ideas behind the approach will be discussed during the talk. If time permits, some possible alternative approaches and some open problems will also be discussed.

**Jason Williford**, University of Wyoming

*Nonexistence Conditions for Coherent Configurations*

A coherent configuration can be viewed as a combinatorial generalization of the orbitals of a group acting on a set. A similar generalization of the orbitals of a transitive group is called an association scheme. Many interesting objects in finite geometry, design theory, and coding theory can be described as association schemes and coherent configurations. In this talk I will discuss some of the ways the nonexistence results on commutative association schemes extends to coherent configurations. This includes recent work on extending the absolute bound. This is joint work with Sylvia Hobart.

**Rick Wilson**, California Institute of Technology

*A zero-sum Ramsey-type problem and the Smith form of certain incidence matrices*

This work is motivated in part by a problem introduced by Y. Caro and N. Alon. Given a graph $G$ with the number of edges divisible by an integer $m$, what is the least integer $R(G,m)$ so that if $n \geq R(G,m)$ and the edges of the complete graph $K_n$ are colored with integers modulo $m$, there exists a subgraph $G'$ of $K_n$ that is isomorphic to $G$ and so that the sum of the colors on the edges of $G'$ is 0 modulo $m$? Caro answered this question completely when $m = 2$ and $G$ has an even number of edges. Perhaps surprisingly, $R(G,2)$ is either $k$, $k+1$, or $k+2$, where $k$ is the number of vertices of $G$, and "almost always" equal to $k$.

The problem can be stated for $t$-uniform hypergraphs as well as graphs (the case $t = 2$). This author evaluated $R(H,2)$ when $H$ is the complete $t$-uniform hypergraph on $k$ vertices, and proved that $R(H,2) \leq k+t$ for any $t$-uniform hypergraph on $k$ vertices with an even number of edges.

Given a $t$-uniform hypergraph $H$ and an integer $n$, we consider the incidence matrix $N$ or $N(H,n)$ whose rows are indexed by the $t$-subsets of an $n$-set $X$ and whose columns correspond to all isomorphic copies of $H$ in the complete $t$-uniform hypergraph on vertex set $X$. The zero-sum Ramsey-type problem asks how large must $n$ be to ensure that every vector in the $Z$-module generated by the rows of $N$ has a coordinate which is zero modulo $m$. When $m = 2$, we are asking how large must $n$ be so that the vector of all 1's is *not* in the binary code generated by $N$.

Given $H$ and $n$, we investigate the Smith form (or diagonal forms) of $N$. This is joint work with Tony Wong. (A diagonal form is known when $n \geq k+t$ where $k$ is the number of vertices of $H$.) We can determine a diagonal form for $N$ when $H$ is any simple graph. This lets us reprove Caro's result and also answer the question of when the vector of all 1's is in the $p$-ary code generated by $N$. Our results for $t$-uniform hypergraphs, though not complete, lend evidence to the conjecture that $R(H,2)$ is "almost always" the number $k$ of vertices of $H$.

**Boyd Worwawannotai**, University of Wisconsin

*Dual polar graphs and the quantum affine algebra $U_q(\mathfrak{sl}_2)$*