L-packets

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1 Overview of the Field

Although the basic idea is more than 40 years old, the machinery that attaches a full L-function $L(s, \pi)$ to an automorphic representation π of a reductive algebraic group G over a number field F is still largely conjectural.

A long series of major papers by Goro Shimura has initiated a lot of questions about the arithmeticity of special values of L-functions. In the case of $\mathrm{GL}(2)$, one first uses the coefficients in the Fourier expansion of modular forms to state and (at least in some cases) answer the questions: does this set of coefficients generate a number field, and can we use it to describe an action of the group of $\mathbb{Q}$-automorphisms of $\mathbb{C}$? In order to generalize to higher rank groups, the Fourier expansion is replaced by the finite part of the associated automorphic representation, and the problems of arthimeticity have been translated in terms of questions about models for such representations. Such problems are still open in full generality. One major idea from the beginning of the theory is the so-called Shimura-Taniyama-Weil conjecture, which is an interpretation of Artin L-functions in terms of L-functions coming from modular forms or higher rank automorphic forms. A partial proof of this conjecture has been given by Andrew Wiles [38] and has led to the proof of the Fermat's last theorem (the proof includes a joint paper by Taylor-Wiles [33]). Furthermore, the full conjecture (for elliptic curves) has now been proved by Breuil, Conrad, Diamond and Taylor ([10]).

The modern version of the general picture rests on finding the elusive Langlands group L_F and on proving the general functoriality conjecture. However, in one very important case — that of endoscopic transfer — the picture is now becoming very clear. Following the pioneering work of Jean-Pierre Labesse and Robert Langlands on SL(2) [25] and of Langlands, Diana Shelstad [26] and Robert Kottwitz [22] on transfer factors, this case can be attacked using the trace formula and its stabilization.

Thanks to Jim Arthur's work, major progress has now been made toward finding a decomposition of the discrete spectrum

$$L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A}_F)) = \bigoplus_{\phi\in\Phi(G/F)} L^2_{\text{disc}}[\phi]$$

into subspaces indexed by certain continuous homomorphisms

$$\phi : L_F \to {}^L G$$

called Langlands parameters, where ${}^LG$ is the Langlands group for G. Using ideas that trace back to Weil, these parameters in turn define L-functions, so the strategy to pass from automorphic representations to full L-functions crucially depends on obtaining detailed information about the subspaces $L^2_{\text{disc}}[\phi]$ corresponding to Langlands parameters.

Arthur's work includes detailed conjectures about these subspaces, such as formulas for the multiplicities of automorphic representations appearing in them. The collection of automorphic representations appearing in any one space $L^2_{\text{disc}}[\phi]$ is commonly called an L-packet (or an Arthur packet) of automorphic representations.

The general idea, due to Langlands, is that we can extract information for one group if we can compare its trace formula to the stable trace formula of other groups; here we are still in a basic case of such a comparison, the endoscopic case. To make such a comparison, one has to consider a stable trace formula: the trace formula of one reductive group (not necessarily connected) is said to be stabilized, if (very roughly speaking) it is invariant under conjugation by elements of the group after algebraic extension of the base field, and if it is expressed in terms of the stabilization of the trace formula of its endoscopic groups. In this context endoscopic groups have to be understood as quasi-split reductive groups with L-group obtained as centralizers of semi-simple elements in the dual group. Arthur has remarked that such a program can also give information for the smaller endoscopic groups if we have enough information for the bigger group.

Moreover, it was also known from Langlands and his collaborators that the spectral side of the stabilized trace formula contains two kinds of information: global coefficients, which have to be interpreted as global multiplicities, and stable distributions which are of a local nature. Global functoriality is clearly encoded by the local functoriality of these stable distributions.

A large part of Arthur's work is concerned with the local part of this problem. The hypothetical Langlands group will come with embeddings of local Langlands groups $L_{F_v} \hookrightarrow L_F$ for every place v of F, where $L_{F_v} = W'_{F_v} \times \text{SU}(2)$ if v is non-Archimedean and $L_{F_v} = W_{F_v} \times \text{SU}(2)$ if v is Archimedean. Every Langlands parameter $\phi : L_F \to {}^LG$ thus gives rise to local Langlands parameters $\phi_v : L_{F_v} \to {}^LG$.

At the heart of the local conjectures lies the local reciprocity map (or, the Local Langlands Correspondence)

$$\text{rec}_{G(F_v)} : \Pi(G(F_v)) \to \Phi(G(F_v)),$$

from equivalence classes of admissible representations of $G(F_v)$ onto local Langlands parameters. The local conjectures require that this function satisfy very specific properties, in large part derived from the functoriality conjecture of Langlands. The fibres of this function are called local L-packets (or, local Arthur packets) of admissible representations.

The application of the trace formula and the twisted trace formula to the number theory and specially to the construction of Galois representations attached to some Shimura varieties, is one of the goals of the theory.

This is a very long story iniated by Ihara with the Selberg trace formula, and continued by Langlands and Kottwitz. The method consists in a comparison of the Grothendieck-Lefschetz fixed point formula with the Arthur-Selberg trace formula. In [21] Kottwitz was able to solve the problem with the hypothesis that the Shimura variety is compact and without endoscopy. Even with such striking restrictions, the theory has already given major results such as the proof of the local Langlands conjecture for the general linear group in the version of Harris and Taylor ([18]). More recently, the Sato-Tate conjecture has been proved using the same kind of methods suitably generalized by Harris-Taylor with a lot of collaborators (Clozel, Barnet-Lamb, Gee, Geraghty, Shepherd-Barron ...) (see [9],[14],[16],[32])

2 Recent Developments and Open Problems

In 1990, J. Rogawski in a pioneering book, described the discrete spectrum for the quasi-split unitary group in three variables using the Langlands' program. But the fact that the group is relatively small allows him to solve some problems by hand, or by using theta series. Since that time much effort was made, mainly by Arthur, on establishing the stabilized trace formula, for a general connected reductive group.

The "fundamental lemma" which asserts that functoriality for unramified representations is compatible with the geometric transfer is crucial, and it was finally proven by Ngo ([29]) after the work of a lot of people who reduced the statement to a geometric one. Fortunately it was already known thanks to Waldspurger's work that this fundamental lemma implies other transfer properties: we apply the trace formula to a function of a reductive group and we want to compare this stabilized trace formula with the trace formula of endoscopic groups, so we have to transfer the given function to functions on the endoscopic groups.

Besides this technical but fundamental result, the recent development which allows real progress is the work of Arthur about the statement of the trace formula, the proof of an invariant trace formula ([3], [4]) and its stabilization ([5], [6], [7]). This is a colossal work because a lot of difficulties have to be solved. At the beginning (for example for a compact group) the trace formula computes the trace of the "good" funtions on the space of automorphic functions (necessarily square integrable because we assume the compactness) in terms of integrals of the functions on the orbit for the group acting by conjugation on itself, the so-called orbital integrals. But at the end of Arthur's work we have an equality between the trace of the functions acting on the space of square integrable automorphic forms in terms of a sum of orbital integrals but we have a number of other terms which are weighted traces and weighted integral orbitals. So it was necessary to define these terms before we can prove the equality.

From the beginning it was known to the specialists and of course to Langlands himself that the needed trace formula will be in the more general context of a twisted trace formula. The connected reductive group is replaced by a component of a non-connected reductive group or even better by a torsor under a reductive connected group (with some very mild assumptions). The statement of the trace formula in such a general situation was the object of the so called "morning seminar" held in Princeton in 1983 with notes by Clozel, Labesse and Langlands. Most of Arthur's formulation generalizes but new problems of convergence appear even to prove the noninvariant trace formula. And these notes are still to be written in a definitive way (this is work in progress, mainly by Labesse with contribution of Waldspurger). The way to pass from this statement of the trace formula to an invariant version has been written by Arthur himself. And there remains to stabilize the invariant trace formula which is only work in progress.

The stabilization of the twisted trace formula is needed in Arthur's work in the special case of the connected component which is not the identity component for the semi-direct product of $\mathrm{GL}(n)$ with its exterior automorphism. It is also needed for the non-identity component of $O(2n)$. Other cases will have applications, for example a connected component in the semi-direct product of a reductive group over a finite Galois extension of a number field with the Galois group of this extension.

Another point needed for the stabilisation of the trace formula is the weighted fundamental lemma, and here recent progress almost solves the question. The fundamental lemma itself is not enough for the full stabilization of the trace formula because of the extra terms in the formula which are weighted terms. Chaudouard and Laumon in [12] and [13] have generalized Ngo's method of [29] to obtain the weighted form of the lemma. Whereas Ngo computes the cohomology of the fibre of the Hitchin fibration, Chaudouard and Laumon define the truncated Hitchin fibration and prove that the weight defined by Arthur is obtained by this truncation. As an open question, there remains to write down a generalization needed for the twisted trace formula. These generalizations are explained by Waldspurger in [36]

New results and perspectives that are linked to L-packets include the Gross-Prasad conjecture which predicts, in special cases, properties of the restriction of an automorphic representation to a reductive subgroup in terms of L-packets. In their pioneer work, Gross and Zagier give a formula to compute the derivative of a L-function at the center of symetry when this L-function vanishes at this point. The Gross-Prasad conjecture is a large generalization of this formula. The first generalization is to look at higher rank pairs of groups, analogous to the basic case of an orthogonal group for an m dimensional space and the subgroup obtained as the stabilizer of an anisotropic vector. Gross, Prasad and Wee Teck Gan have a long list of groups to which the Gross-Prasad conjecture must apply. The conjecture aims to explain the local and global obstruction in order that an automorphic representation of the small group occurs in the restriction of a representation of the bigger group. For the moment the conjecture is formulated under the assumption that the representations are cuspidal and the local components are in an L-packet which contain a generic representation (a representation with a Whittaker model). The local obstruction is a fine result which relies on the local Langlands parameter, and in the case of $SO(m), SO(m-1)$ it has been proved by Waldspurger that the Arthur's theory allows one to identify the local obstruction. The global obstruction is about the non-nullity of some L-function at the center of symmetry. There is a vey nice conjectural formula of Ichino and Ikeda ([19]) to relate the central value of the L-function to a product of local integrals linked to the local obstruction. But such a formula is only proved in very special cases. Thus we have a lot of open questions, including the formulation of a general Gross-Prasad conjecture without the assumption of genericity.

The Gross-Prasad conjecture is just one of many directions where the recent progress on the trace formula gives a new push. We want also to direct attention to the consequences of the new understanding of Langlands packets for the theory of deformations of representations, and in particular to the construction of the eigenvarieties and to the construction of Galois representation, as explained for example in the "book project" of M. Harris and his collaborators (see Harris' web page).

3 Presentation Highlights

The BIRS *Workshop on L-packets* attracted 40 participants from 6 countries (14 from U.S.A., 11 from Canada, 10 from France, 3 from Japan, 1 from Israel and 1 from India) for 16 high-level lectures reflecting the best information available regarding L-packets as they appear in the arithmetic Langlands Program. The workshop was tightly focussed on the following objectives:

(1) to review the current status of Arthur's conjectures as they pertain to the structure, stability and parametrization of L-packets both global and local, Archimedean and non-Archimedean;

(2) to carefully examine examples to gather evidence to support (or refute) the conjectures;

(3) to think about arithmetic implications; and

(4) to develop strategies that might lead to progress on the conjectures.

To that end, James Arthur provided participants with an advance copy of his forthcoming book *The Endoscopic Classification of Representations: Orthogonal and Symplectic Groups* and gave four lectures on its contents. A recording of these lectures is available from the BIRS website.

Without a doubt, the highlight of the workshop was the following announcement by Arthur, which can also be found on page 468 of the workshop version of his book.

Theorem 1 (Arthur)**.** *The full local Langlands correspondence for the split groups SO*$(2n+1)$ *and Sp*$(2n)$ *is now proved, as is a slightly weaker form of the correspondence for the quasi-split groups SO*$(2n)$*.*

Arthur also announced a generalization to all the interior forms of such groups, which introduces new ideas. In a tempered Langlands packet, the classification is done using characters of the component group of the centralizer in the L-group of the parameter of the packet, trivial on the center of the L-group. For an interior form, we have to compute the centralizer of the parameter in the simply-connected cover of the semi-simple part of the L-group. Arthur defines a representation of the inverse image of the center of the semi-simple part of L-group in the cover which depends on the interior form. The Langlands parameters for tempered representations are the pair consisting of one L-morphism from the Weil-Deligne group to the L-group, and an irreducible representation of the centralizer of this morphism in the cover which restricts on the inverse image of the center to the prescribed representation.

This local result is proved simultaneously with its global analogs which describe the "Tannakean" decomposition of the space of square integrable automorphic forms. In case of a classical group, Arthur has given a very concrete version of this decomposition. Let us explain what it is in the easiest case of quasi-split groups, or a product of such groups. So fix n and consider the set of groups, denoted $\mathcal{E}ll_{2n+1}$, consisting of the product $H_{m,m'} := SO(2m+1) \times Sp(2m')$ with m, m' positive or null integers, such that $m + m' = n$ over a fixed number field, k. We decompose the set of square integrable irreducible automorphic representations of $H_{m,m'}$ in (disjoint) packets, saying that two such representations are in the same packet if they are equivalent almost everywhere. The L-group of $H_{m,m'}$ is naturally a subgroup of the L-group of $\mathrm{GL}(2n+1)$; for the connected component it is just the natural inclusion of $Sp(2m, \mathbb{C}) \times SO(2m'+1, \mathbb{C})$ in $\mathrm{GL}(2n+1, \mathbb{C})$. So for any finite place v, of the number field k, there is an unramified Langlands transfer from the set of unramified irreducible representations of $H_{m,m'}(k_v)$ in the set of unramified representations of $\mathrm{GL}(2n+1)(k_v)$. So we can say that a packet (as above) of square integrable automorphic representations transfers to $GL(2n+1)$ if there is an irreducible automorphic representation, Π, of $\mathrm{GL}(2n+1)$, which contributes to the discrete part of the twisted trace formula of $\mathrm{GL}(2m+1)$ and such that for almost all places v of k, the local component Π_v is an unramified transfer of the local component of any representation in the packet which is also unramified at this place. Using the strong multiplicity one theorem for $\mathrm{GL}(2n+1)$, this definition gives at most one representation. And we have:

Theorem 2 (Arthur)**.** *For any m, m' as above, any packet of square integrable irreducible representation of $H_{m,m'}$ transfers to* $\mathrm{GL}(2n+1)$*. And reciprocally, for any irreducible automorphic representation of* $\mathrm{GL}(2n+1)$*,* Π*, which has trivial central character, there is exactly one element $H_{m,m'} \in \mathcal{E}ll_{2n+1}$ and a packet of square integrable representations of $H_{m,m'}$ which transfers to* Π.

In fact the discrete part of any trace formula contains the contribution of the square integrable representations as well as the contribution of some square integrable representations of Levi subgroups; the same is true for twisted trace formula and in the theorem Π is assumed to come from the group and not from a Levi subgroup.

In the theorem the unicity is particularly important; moreover Arthur gives exactly the pair m, m' when Π is fixed using property of the L-function of Π. This can be explained easily if we assume that Π is cuspidal (with trivial central character): because Π appears in the twisted trace formula, Π is isomorphic to its contragredient Π^* and the L-function $L(\Pi \times \Pi, s)$ defined by Jacquet and Shalika ([20]) has a pole at $s = 1$. In this case the $H_{m,m'}$ relative to Π satifies $mm' = 0$ and we have $m = n$ exactly if $L(\Pi, \wedge^2, s)$ has a pole at $s = 1$.

Arthur's results also explain what happens at every place of k. Let us now fix Π, m, m' as in the preceding theorem. So Π determines a unique packet of irreducible square integrable representation of $H_{m,m'}$; the packet is infinite (except in very special cases) but let us fix v a place of k and look at the set of all the local components at the place v when the representation of $H_{m,m'}$ varies in the packet. This set is finite and a suitable linear combination of the representations in this set has, as twisted transfer, the twisted trace of Π_v. Let us say that the coefficient in this linear coefficient is the local multiplicity (forget a serious difficulty in normalizing the sign). Then we have the global multiplicity formula:

Theorem 3 (Arthur)**.** *The multiplicity of an irreducible square integrable representation of $H_{m,m'}$ (as above) in the discrete spectrum of this group is the product of the global multiplicity, which is one for the element in $\mathcal{E}_{2n+1}$, by all the local multiplicities.*

So a negative multiplicity is impossible and this is a constraint for the set of local components described with precision in Arthur's work.

We can explicitly compute the local multiplicity only at the non-archimedean places ([28]) and the result is ± 1. At the archimedean places, Adams, Barbasch and Vogan have described a long time ago what the local packets should be and given an interpretation of the local multiplicity in a geometric way ([1]). But it is not known that these packets are really those packets appearing in Arthur's construction and it does not seem to be easy to compute the multiplicity. Fortunately, Adams and Johnson ([2]) have done a very explicit construction for the representations with cohomology (with coefficients) with for example all the Langlands' parameters known thanks to the work of Vogan and Zuckerman ([34]). Adams has proved that in this special case, both constructions, those of [1] and [34], do indeed coincide. In the description of Adams and Johnson the local multiplicities are also ± 1. So in the case at hand, which is very important for arithmetic applications, it is natural to conjecture that the multiplicity in the discrete spectrum of $H_{m,m'}$ will be one.

As Arthur shows, the same method also applies to $\mathrm{GL}(2n)$. Now the elements of $\mathcal{E}_{2n}$ are product of quasi-split special orthogonal groups $SO(2m) \times SO(2m'+1)$ with $m + m' = n$; this allows us to also understand the orthogonal groups in even dimension. The new difficulty here is the fact that there is an exterior automorphism for such a group (if $m \neq 0$) which comes from an element in $O(2m)$ not in $SO(2m)$. The method of proof by Arthur is to realize the elements of $\mathcal{E}_{2n+1}$ and $\mathcal{E}_{2n}$ as the elliptic endoscopic groups for the non-identity component $GL(2n+1)\theta$ (where θ is the exterior automorphism) and $GL(2n)\theta$ and look at each contribution in the stabilization of the twisted trace formula coming from the elliptic endoscopic groups. But in the second case, the exterior automorphism of one $SO(2m)$ is an automorphism of the corresponding endoscopic group and it is not possible to separate a representation and its image by the automorphism. This explains the difficulty for even orthogonal groups. Moreover in the case of an even orthogonal group, the global multiplicity is one or two, but Arthur has made explicit the computation. We also have to avoid the condition on the central character of Π (with the above notations) because this cannot be achieved by twisting by a character.

During the conference, there was also explained a crucial point needed in Arthur's method: the spectral transfer factor. The endoscopy and the more general twisted endoscopy, in the local situation, relate linear combinations of traces of unitary representations of two different groups; roughly speaking, the spectral transfer factors are the coefficients in these linear combinations. Diana Shelstad explained how her idea to define geometric transfer factors can be transposed to define these spectral transfer factors and Paul Mezo uses her definition and result to prove a priori the wanted equality for tempered representations at the archimedean places (in almost the full generality needed). Contrary to the p-adic case, in the archimedean cases we have a formula for the trace of a tempered representation thanks to Harish-Chandra in the connected case and Duflo-Bouaziz (see [11]) in the non-connected case. So the equality to be proved is a concrete one and it is this problem that Mezo solves.

4 Scientific Progress Made

Speaker: James Arthur (University of Toronto)

Title: Representations of orthogonal and symplectic groups

Abstract: Suppose that G is a connected, quasisplit orthogonal or symplectic group over a number field F. I shall review the statements of theorems that classify automorphic representations of G. The proof of these theorems rests on an extended argument that is ultimately based on the comparison of trace formulas, specifically, the stabilization of the trace formula for G, and the conditional stabilization of the twisted trace formula for GL(N). I shall try to give some overview of the proof, insofar as this is feasible in the time available. If possible, I would also like to describe some of the basic implications of the theorems.

Finally, I hope to add a few remarks on how the classication would extend to inner forms of G.

Speaker: Mahdi Asgari (Oklahoma State University)

Title: Counting Cusp Forms on Classical Groups

Abstract: I will discuss some work in progress, joint with Werner Mueller, trying to establish Weyls law with remainder for classical groups. This extends results of Lapid-Mueller on GL(N) and is an application of the Arthur Trace Formula. Without remainder terms, Weyls law is now known in a rather general setting thanks to results of Lindenstrauss-Venkatesh, following a great number of earlier results.

Speaker: Thomas Haines (University of Maryland)

Title: Test functions for Shimura varieties with arbitrary level

Abstract: I will explain a conjecture (joint with Kottwitz) which predicts which test functions are plugged into the twisted orbital integrals at p in the Langlands-Kottwitz expression for the Lefschetz formula of a Shimura variety having arbitrary level at p. This conjecture is phrased in terms of L-parameters, but can be stated unconditionally and proved in several cases, including a $\Gamma_1(p)$-level situation (joint with Rapoport). It also gives rise to a further conjecture on endoscopic transfer of the stable Bernstein center, for which some results are also now available.

Speaker: Atsushi Ichino (Kyoto University)

Title: Formal degrees and local theta correspondence

Abstract: The formal degree conjecture expresses the formal degree of a discrete series representation of a reductive group over a local field in terms of the adjoint gamma factor of its Langlands parameter. This conjecture is supported by various examples but is still open for classical groups. On the other hand, it is expected that the theta correspondence realizes a functorial lift. We prove the expected behavior of formal degrees under the theta correspondence. If time permits, we also discuss a relation with the Siegel-Weil formula. This is joint work with Wee Teck Gan.

Speaker: Tasho Kaletha (IAS)

Title: Simple wild L-packets Abstract: We will review recent explicit constructions of supercuspidal L-packets for general classes of reductive p-adic groups, focusing in particular on the L-packets which consist of simple supercuspidal representations. Those representations were constructed in a recent paper of Gross and Reeder, in which the authors also define a class of Langlands parameters, called simple wild, and conjecture that these should correspond to the simple supercuspidal representations. Starting from a simple wild parameter, we will show how to construct a finite set of simple supercuspidal representations. Each step of this construction will be explicit and computable. Moreover, we will show that this finite set satisfies many properties expected from an L-packet it provides a stable character, contains a unique generic representation for a fixed Whittaker datum, and admits a description in terms of the Langlands dual group.

Speaker: Erez Lapid (Hebrew University)

Title: Whittaker-Fourier coefficients for the metaplectic group

Abstract: The (1st generation) descent method of Ginzburg-Rallis-Soudry provides an explicit realization of the generic element of an L-packet of a classical group in terms of its functorial lift to GL(n). Using it, the (special case of the) Ichino-Ikeda conjecture for Whittaker-Fourier coefficients reduces to a local statement. We study the latter in the case of the metaplectic group $\mathrm{Mp}(n)$. This gives a new approach already for the classical case n = 1 (proved by Waldspurger around 1980). Joint work with Zhengyu Mao.

Speaker: Wen-Wei Li (Institut de Mathematiques de Jussieu)

Title: Towards a stable trace formula for metaplectic groups and its applications

Abstract: Due to the recent progress on the stable trace formula, our understanding of the representations of reductive groups has advanced significantly. However, such techniques had not been systematically applied to non-algebraic coverings of connected reductive groups, for example the metaplectic twofold covering for Sp(2n), which appear naturally in number theory. Based on some ideas of Adams, I will discuss some progress in this direction and the consequences.

Speaker: Paul Mezo (Carleton University)

Title: Twisted spectral transfer for real groups

Abstract: Kottwitz and Shelstad generalized the framework of endoscopy to include twisting by group automorphisms or central characters. This generalization contained conjectural identities between orbital integrals, constituting a transfer from functions on a group to functions on one of its endoscopic groups. This geometric transfer has recently been proven by Shelstad. Dual to geometric transfer is spectral transfer, which is a collection identities between characters of L-packets of a group and one of its endoscopic groups. We show how some work of Bouaziz, Duo and Shelstad may be adapted to the twisted endoscopy of real reductive groups in order to achieve spectral transfer.

Speaker: Dipendra Prasad (Tata Institute of Fundamental Research)

Title: Relative Local Langlands

Abstract: The aim of the lecture will be formulate a precise conjecture about which representations of G(K) have G(k) invariant linear form, for G a reductive algebraic group over a local field k with K a quadratic extension of k. A special case (for K = k+k) will give a precise information on the contragredient of a representation of G(k), extending the recipe of [MVW] for classical groups. This is part of a joint work with Ananadavardhanan.

Speaker: Gordan Savin (Utah)

Title: Functoriality via matching of Hecke algebras

Abstract: Let G and G' be two split reductive groups acting on a minimal representation (r, V) as a dual pair. Let H and H ' be the Hecke algebras corresponding to hyperspecial compact subgroups K and K' of G and G' respectively. Assume that there should be a functorial transfer of representations from G to G' which, for unramified representations, is given by a natural homomorphism f from H' to H . In this lecture I will explain how to prove (in many cases) that r(T) = r(T ') on the subspace of K K' fixed vectors in V for every T ' in H ' and T = f (T '). This is a joint work with Mike Woodbury.

Speaker: Diana Shelstad (Rutgers)

Title: Some results in endoscopic transfer

Abstract: We follow the theme of stabilization, and start with Arthurs paradigm for the invariant trace formula, geometric side = spectral side, in the case G = S L(2) over a number field. A simple canonical sign, an adelic transfer factor, provides a measure of instability in the invariant trace formula from the geometric side. If we write a good product formula, over all places, for the factor then we can find another simple canonical sign, an adelic spectral transfer factor, giving a spectral interpretation of instability. This o?ers some motivation for a more recent look at endoscopy, twisted or not, for general connected, reductive G defined over R, for extended groups (K-groups) even. Internal motivation is that a structure for tempered spectral factors comes almost for free once the geometric transfer factors have been defined. We will discuss some of the theorems and describe tools used in their proof that may be helpful as well for an approach to some questions at the infinite places in Langlands program for stable transfer (or stable-stable transfer for emphasis that it lies beyond endoscopy). We also discuss results, some only partial, useful in Arthurs endoscopic classification for classical groups.

Speaker: Eric Urban (Columbia University)

Speaker: Jean-Loup Waldspurger (Institut de Mathématiques de Jussieu)

Title: Some local preparations for the stabilization of the local twisted trace formula

On donne à la suite de Labesse [23] et [24], une nouvelle présentaiton de l'endoscopie tordue sur un corps local. Cette présentation permet de définir plus canoniquement les espaces endoscopiques, la notion de transfert, ainsi que l'action du groupe d'automorphismes d'une donnée endoscopique. Considérons l'application qui, à une fonction sur un espace tordu, associe la collection de ses transferts relatifs à toutes les données endoscopiques elliptiques. On énonce une proposition, si le corps de base est non archimédien, affirmant l'injectivité de cette application et caractérisant son image et on donne une proposition analogue dans le cas archimédien mais en utilisant les K-groupes tordus.

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