The SU(3) Casson invariant of spliced sums

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1 Overview of the Field

Taubes [11] laid the groundwork for new topological invariants motivated by Chern-Simons theory by showing that the $\mathrm{SU}(2)$ Casson invariant of a homology 3–sphere M has a gauge theoretical interpretation as the Euler characteristic of $\mathcal{A}/\mathcal{G}$ in the spirit of the Poincaré-Hopf theorem, where he views the Chern-Simons invariant

$$cs(A) = \frac{1}{8\pi^2}\int_M \mathrm{tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A)$$

as a S^1-valued Morse function on $\mathcal{A}/\mathcal{G}$, $\mathcal{A}$ being the space of $\mathrm{SU}(2)$ connections on M and $\mathcal{G}$ the group of gauge transformations. Taubes realized that the Hessian of the Chern-Simons invariant and the odd signature operator coupled to the same path of $\mathrm{SU}(2)$ connections have the same spectral flow.

Using Taubes's point of view, an $\mathrm{SU}(3)$ Casson invariant τ was introduced by [1] and later refined by [3], where suitable correction terms needed to be incorporated to make the invariant independent of perturbations. In the process of understanding this new invariant, a connected sum formula was found [2], and computational tools were developed for Dehn surgeries on $(2, q)$-torus knots [6] as well as for Brieskorn spheres [10, 4, 5]. These papers provide calculation methods for homology spheres obtained by several different cut and paste methods, and the families of examples for which these have yielded calculations show intriguing patterns; they have not, however, led to a conjectural formula for a general Dehn surgeries. It is therefore important to continue exploring the behavior of the invariant under further cut and paste constructions.

2 Recent Developments and Open Problems

More recently, some results were obtained concerning the behavior of τ under the spliced sum construction. Given knots K_1 and K_2 in homology 3-spheres M_1 and M_2, respectively, the spliced sum of M_1 and M_2 along K_1 and K_2 is the homology 3-sphere obtained by gluing the two knot complements along their boundaries matching the meridian of one knot to the longitude of the other. This operation is a generalization of connected sum; indeed when K_1 and K_2 are trivial knots, the spliced sum of M_1 and M_2 along K_1 and K_2 is none other than the connected sum $M_1 \# M_2$.

Casson's invariant $\lambda_{\mathrm{SU}(2)}$, which is additive under connected sum, is also additive under the more general operation of spliced sum by Boyer and Nicas [8] and independently Fukuhara and Maruyama [9]. Recently, the $\mathrm{SU}(3)$ Casson invariant for spliced sums of a $(2, p)$ and $(2, q)$ torus knot K_1 and K_2 in the 3–sphere was computed in [7] as

$$\tau(M) = 16\,\lambda'(K_1)\;\lambda'(K_2), \qquad (1)$$

where $\lambda'(K)$ is the $\mathrm{SU}(2)$ Casson knot invariant normalized to be 1 for the trefoil.

In this Research in Teams project we set out to extend these splice sum computations to splice sums of arbitrary (p, q) torus knots. The flat moduli spaces of these knot complements, when $p, q \neq 2$, is more complicated, having a stratified structure. For this reason, the flat moduli space of such a splice sum is necessarily degenerate, and requires perturbation to obtain a finite collection of points to count to evaluate the invariant.

3 Scientific Progress Made

To reach our goal for *Research in Teams* of extending formula (1) to splice sums of arbitrary torus knots, the following steps were necessary:

1. Analyzing the $\mathrm{SU}(3)$ representation varieties of the knot complements,
2. Determining suitable perturbations and the resulting perturbed moduli spaces, and
3. Computing the spectral flow between the perturbed components.

We have completed the first step, both for (p, q) torus knots and for complements of a singular fiber in a 3-fiber Brieskorn sphere. To be more precise, we determined not just the representation varieties but also the restriction maps from the knot group representation varieties for X_1 and X_2 to the representation variety of their common boundary T^2

$$R(X_1, \mathrm{SU}(3)) \xrightarrow{r_1} R(T^2, \mathrm{SU}(3)) \xleftarrow{r_2} R(X_2, \mathrm{SU}(3)).$$

In the cases being considered, the maps r_i are not generally one to one, but have the form of slightly singular fibrations above their images, which are smooth submanifolds of $R(T^2, \mathrm{SU}(3))$, and the fibration structure only breaks down along a codimension one subset of the image. We verified that the images of r_1 and r_2 meet transversely, for arbitrary torus knots.

Suppose r_1 and r_2 have transverse images in a small neighborhood of $[\alpha_0] \in R(T^2, \mathrm{SU}(3))$. , and suppose $\alpha = \alpha_1 \cup_{\alpha_0} \alpha_2$ is nontrivial with $\mathrm{Stab}(\alpha_0) \neq T^2$, $\alpha_i \in R(X_i, \mathrm{SU}(3))$. Set

$$C = \{[\beta] \in R(M, \mathrm{SU}(3)) \mid \beta = \beta_1 \cup \beta_2 \text{ with } \beta_i \in r_i^{-1}(\alpha_0) \text{ and in same component as } \alpha_i\}.$$

We showed that $C \subset R^*(M, \mathrm{SU}(3))$ and C is diffeomorphic to a fiber product of $r_1^{-1}(\alpha_0) \times r_2^{-1}(\alpha_0)$ with $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))/\mathbf{Z}_3$, and $\chi(C) = 0$. This shows that many of the fiber products associated to many intersections contribute zero to τ.

A more complicated situation occurs when α_0 is a singular point for one or both of the fibrations r_i. C is a product of two 2–spheres, at least one of which has a reducible representation on it and is not cut out transversely. This makes it more difficult to determine how this component contributes to τ after perturbation.

We verified that, when this happens, we could perturb so that the perturbed moduli space is obtained as a fiber product is taken along smooth fibers, because, after perturbation, the singular points of r_1 don't hit the image of r_2, and vice versa. We therefore expect the components to resolve into 4 components in the specific case of $(2, q)$ torus knots. This prelimary work demonstrates that the current methods are sufficient to calculate the value of τ on fiber sums of arbitrary (p, q) torus knots. To complete the calculation, we must do some book keeping (counting the number of intersections, of the different types), and do complete the spectral flow computations necessary to evaluate the signs and the correction terms. Again, our preliminary work during the week at BIRS indicates that the current methods will handle the needed spectral flow computations.

It had been observed in [5] that the representation variety of the complement of a singular fiber in a Brieskorn sphere has a similar stratified structure to that of the representation variety for a torus knot. In the course of our work, we realized that property that the images of r_1 and r_2 are transverse holds as well in this context. We were able to determine that most of the types of representation variety components for a splice sum of Brieskorn spheres can be handled by the same techniques as we developed for splice sums of torus knots, under a mild restriction on the Seifert invariants. (Unfortunately, the restriction on Seifert invariants precludes our treating 4-fiber Brieskorn spheres as splice sums of singular fibers in 3-fiber Brieskorn spheres, so this class of 3-manifolds remains a wide open challenge, but the allowable Seifert invariants still give many examples of interesting graph manifolds.) Unfortunately, splice sums of knots in nontrivial homology spheres give rise to additional types of fiber product components (involving different orbit types of α_1, α_2 and α_0) in the flat moduli space, so these Brieskorn sphere examples will require more analysis.

4 Outcome of the Meeting

In summary, our meeting provided the three participants, who bring three distinct areas of expertise to the problem, the chance to develop common notation, and bring each other up to speed on the different aspects of the problem. Our collaboration has led to dramatic progress on all three steps identified above in calculating the $SU(3)$ Casson invariant for splice sums of arbitrary torus knots. We are currently in the process of writing up a paper with the results on splice sums of torus knots using the workshop notes as a basis.

The meeting also allowed us to understand much more deeply the steps that will be necessary to extend our calculations to splice sums of Brieskorn spheres. Because of different orbit types of the representations that show up in this case, more advanced transversality and spectral flow techniques will be necessary to complete this project, but this line of research was also advanced significantly by our meeting.

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