New trends on Structural Graph Theory
September 5–10, 2010

ABSTRACTS

Recent research

Boris Alexeev

I am currently working on an induced subgraph characterization of double-split graphs. (Recall that double-split graphs are one of the "basic classes" in the proof of the strong perfect graph theorem.)

On Tuza's Conjecture

Jessica McDOnald

Tuza conjectured that for every graph G, the maximum size of a set of edge disjoint triangles is at most twice the minimum size of a set of edges meeting every triangle. Krivelevich proved that this result is true if we replace the set of edges with a fractional set and asked if any further improvement on this bound might be possible. We achieve an essentially best possible bound relating these two parameters. We also consider a related problem for weighted graphs. This is joint work with Guillaume Chapuy, Matt DeVos, Bojan Mohar, and Diego Scheide.

Growth in Cubes of Graphs

Matt DeVos

Let A be a generating set of a finite multiplicative group. A number of important problems in additive number theory and group theory concern the increase in size from $|A|$ to $|A^k|$. This can be stated in terms of graphs by comparing the number of edges in the Cayley Graph generated by A compared with the number in the Cayley Graph generated by A^k. Motivated by this connection, Hegarty considered this problem in more general graphs. He proved that $e(G3) > (1 + c)e(G)$ for every connected regular graph of diameter at least 3 and asked what the best possible constant is in this theorem. We show that $c = 3/4$ is best possible. Joint with Stephan Thomasse.

Recent research

Zdenek Dvorak

Recently, I was considering various problems related to 4-color theorem, as well as alternative approaches to its proof (with only a little progress, unfortunately). Examples:
– Grotzsch conjecture regarding the 3-edge-colorability of subcubic planar graphs
– characterization of 5-critical graphs with crossing number 1
– characterization of non-4-colorable Eulerian triangulations of bounded genus
– for each cubic planar graph G with a fixed number of half-edges incident with the outer face and each precoloring psi of these half-edges, we can consider the number c(G,psi) of the 3-edge-colorings of G that extend psi. Can we characterize the possible vectors c(G,.)? Using Kochol's technique and block-reducibility arguments, we can obtain a linear program constraining these vectors; can this characterization be strenghtened?

A new proof of the graph removal lemma

Jacob Fox/Massachusetts Institute of Technology

Let H be a fixed graph with h vertices. The graph removal lemma states that every graph on n vertices with $o(n^h)$ copies of H can be made H-free by removing $o(n^2)$ edges. We give a new proof which avoids Szemerédi's regularity lemma and gives a better bound. This approach also works to give improved bounds for the directed and multicolored analogues of the graph removal lemma. This answers questions of Alon and Gowers.

Immersion in digraphs and related problems

Alexandra Fradkin/Princeton University

I have recently been working on some problems involving immersion in digraphs. A digraph H is immersed in a digraph G if the vertices of H are injectively mapped to the vertices of G, and the edges of H are mapped to edge-disjoint (directed) paths of G. Immersion in digraphs seems to be much less well behaved than in undirected graphs. For example, immersion gives a well-quasi-order for graphs, but not for digraphs. Not surprisingly then, immersion containment can be tested in polynomial time in graphs, but not in digraphs (in general). However, for some (fixed) digraphs H, the question "can H be immersed in G?" for an input graph G can be answered in time polynomial in the size of G. Classifying such digraphs H seems to be a non-trivial problem. This is something that I have been working on with Paul Seymour, Maria Chudnovsky, and Ilhee Kim. For example, we have proved that both the path with k changes of direction and the cycle with k changes of direction fall into the class of digraphs mentioned above. On the other hand, there are digraphs H on just two vertices for which the problem is NP-complete (e.g. let H be a digraph on vertices u and v with two edges from u to v and two edges from v to u or with one edge from u to v, one edge from v to u and a loop at each of u and v).

One problem closely related to immersion is the k edge-disjoint paths problem. For fixed k, this can be solved in polynomial time in undirected graphs and is NP-complete in digraphs even when $k = 2$. However, with Paul Seymour we showed that for all fixed k the problem becomes polynomial-time solvable once you restrict to digraphs with bounded independence number. This generalizes a theorem of Bang-Jensen, who proved that the problem is polynomial in tournaments with $k = 2$.

The concept of "cutwidth" for digraphs plays a major role in the algorithm for solving the k-edge-disjoint paths problem. We say that a digraph has cutwidth at most k if its vertices can be ordered $v_1, \ldots, v_n$ in such a way that for $1 \leq i \leq n$, there are at most k edges $v_h v_j$ with $h \leq i < j$. Digraphs with bounded independence number and bounded cutwidth can easily be handled algorithmically by dynamic programming.

In view of the usefulness of cutwidth, algorithmic questions relating to computing cutwidth in digraphs are of interest. Cutwidth is very easy to compute in tournaments: just order the vertices of a tournament by increasing outdegrees and compute the cutwidth of that enumeration. We can also compute cutwidth within a constant factor (in polynomial time) in semi-complete digraphs (these are simple digraphs that can be obtained from tournaments by adding edges). It would be interesting to see if this can also be done for digraphs with bounded independence number.

The stable set polytope of claw-free graphs

Anna Galluccio, Istituto di Analisi dei Sistemi ed Informatica "Antonio Ruberti" Consiglio Nazionale Ricerche, Viale Manzoni 30 00185Roma

In 1965 Edmonds provided the first complete description of the polyhedron associated with a combinatorial optimization problem: the matching polytope. As the matching problem is equivalent to the stable set problem over line graphs, many researchers tried to generalize Edmonds' result by considering the stable set problem over a super class of line graphs: the claw-free graphs. However, as testified also by Grötschel, Lovász, and Schrijver in [1], "*in spite of considerable efforts, no decent system of inequalities describing $STAB(G)$ for claw-free graphs is known*".

We prove that the stable set polytope $STAB(G)$ of a claw-free graph with stability number at least four and with no 1-join is completely described by the following inequalities: nonnegativity, rank, lifted 5-wheel inequalities and some special inequalities named *multiple geared inequalities*. The proof is based on the polyhedral properties of a new graph operation (2-*clique-bond composition*) that generalizes the generalized 2-join used by Chudnovsky and Seymour in their decomposition of claw-free graphs [2].

We also noticed that the polyhedral structure of $STAB(G)$ strongly depends on the buiding blocks of G. So, we called $\mathcal{W}$-perfect the graph whose stable set polytope is described by: nonnegativity, rank, and lifted 5-wheel inequalities, and we showed that claw-free graphs obtained as 2-clique-bond composition of fuzzy line graphs with three different types of three-cliqued graphs (*fuzzy antihat graphs*) are $\mathcal{W}$-perfect.

It would be interesting to know which claw-free graphs with stability number three among those listed by Chudnovsky and Seymour in [2] are $\mathcal{W}$-perfect.

It is still an open problem to identify which inequalities are facet defining for $STAB(G)$ when G is fuzzy circular interval.

This work is joint with C. Gentile e P. Ventura (IASI-CNR).

[1] M. Grötschel and L. Lovász and A. Schrijver, *Geometric algorithms and combinatorial optimization*, Springer Verlag, Berlin, 1988,

[2] M. Chudnovsky and P. Seymour, *Claw-free graphs V: Global structure*, Jour. Comb. Th. B 98 (2008), 1373-1410.

Matroid Minors

Bert Gerards, Centrum Wiskunde & Informatica and Maastricht University

Over the last decade Jim Geelen, Geoff Whittle and I work on generalizing Robertson and Seymour's Graph Minor Theory to matroids representable over finite fields. Early this year we completed the description of structure of minor-closed classes of GF(4) representable matroids.

Group-labelled Graphs and Matroids

Tony Huynh, CWI Amsterdam

Let Γ be a group. A Γ-labelled graph is an oriented graph with its edges labelled from Γ, and is thus a generalization of a signed-graph. My PhD thesis generalizes the main result from Graph Minors XIII. For any finite abelian group Γ, and any fixed Γ-labelled graph H, we give a polynomial-time algorithm to test if an input Γ-labelled graph G has an H-minor. This is joint work with Jim Geelen.

I am also finishing off some work on Γ-labelled graphs in surfaces, which leads to some interesting questions concerning homeomorphims of (first) homology into Γ. This is joint work with Jim Geelen and Bert Gerards.

With Stefan van Swan, I will soon be thinking about inequivalent representations of matroids over finite fields of non-prime order.

An open problem that I like is whether there exists a polynomial-time algorithm that determines if a graph has a cycle of length 0 (mod k), where k is fixed.

Representations of non-degenerate even cycle matroids

Bertrand Guenin, joint work with Irene Pivotto and Paul Wollan

A signed graph is a representation of an even cycle matroid M if the cycles of M correspond to the even cycles of the signed graph. Two signed graphs are equivalent if they are related by Whitney flips and signature exchanges. An even cycle matroid is degenerate if it is the projection of a graphic matroid. We show that an even cycle matroid which contains a non-degenerate fixed size minor has a bounded number of inequivalent representations. For instance, even cycle matroids which contain R_{10} as a minor have at most 6 non-equivalent representations. We also derive new operations that preserve even cycles in signed graphs.

Displaying blocking pairs in signed graphs

Bertrand Guenin, joint work with Irene Pivotto and Paul Wollan

A signed graph is a pair (G, Σ) where G is a graph and Σ is a subset of the edges of G. A cycle C in G is even (resp. odd) if $E(C) \cap \Sigma$ is even (resp. odd). A blocking pair in a signed graph is a pair of vertices $\{x, y\}$ such that every odd cycle in (G, Σ) intersects at least one of the vertices x and y. Blocking pairs arise in a natural way in the study of even cycle matroids on signed graphs as well as signed graphs with no odd K_5 minor. In this article, we characterize when the blocking pairs of a signed graph can be represented by 2-cuts in an auxiliary graph. We discuss the relevance of this result to the problem of characterizing signed graphs with no odd K_5 minor and determing when two signed graphs represent the same even cycle matroid.

Small models of minors in dense graphs.

Gwenaël Joret, Université Libre de Bruxelles

A well-known result of Mader shows that there is a minimum function $f(t)$ such that every graph with average degree at least $f(t)$ contains a K_t-minor. This function $f(t)$ is well understood: For instance, it is known that $f(t) \in \Theta(t\sqrt{\log t})$, and $f(t)$ has been determined for small values of t; see [1] for a survey of this topic.

I worked with S. Fiorini, D. O. Theis, and D. R. Wood on a variant of the above setting. The variant is obtained by furthermore requiring that the K_t-minor be ‘small’. This idea is evident when $t = 3$. A graph contains a K_3-minor if and only if it contains a cycle. Every graph with average degree at least 2 contains a cycle, whereas every n-vertex graph with average degree at least 3 contains a cycle of length $O(\log n)$. That is, high average degree forces a short cycle, which can be thought of as a small model of K_3.

We proved in [2] that there exists a function g such that every n-vertex graph with average degree at least $g(t)$ contains a K_t-model with $O_t(\log n)$ vertices. (For fixed t, the logarithmic upper bound is best possible, since for all $d \geq 3$ there are d-regular n-vertex graphs with girth $\Omega_d(\log n)$.) We showed that $g(t) = 2^{t-1} + \varepsilon$ will do, where ε is any positive real. It is an open problem to determine if this exponential bound in t could be reduced to a polynomial bound.

We also determined the best possible density function for $t \leq 4$. For instance, average degree at least $4 + \varepsilon$ guarantees a K_4-model with $O_\varepsilon(\log n)$ vertices, while average degree 4 is not enough (as shown by the square of an n-vertex cycle). For $t = 5$, there are triangulations of the torus where every K_5-model has linear size, showing that average degree 6 does not yield a small K_5-model. However, we conjecture that average degree $6 + \varepsilon$ will do, similarly as for the $t = 4$ case.

References

[1] ANDREW THOMASON. Extremal functions for graph minors. In *More sets, graphs and numbers*, vol. 15 of *Bolyai Soc. Math. Stud.*, pp. 359–380. Springer, Berlin, 2006.

[2] SAMUEL FIORINI AND GWENAËL JORET AND DIRK O. THEIS AND DAVID R. WOOD. Small minors in dense graphs. `http://arxiv.org/abs/1005.0895`.

Computing the excluded minors of the tree-structures over nearly-embeddable graphs

Rohan Kapadia, University of Waterloo

Robertson and Seymour proved that for each graph H, any graph without H as a minor can be built starting with graphs embedded in surfaces in which H cannot be embedded, adding vortices and apex vertices, and taking clique-sums. Seymour has made a short algorithm that decomposes a graph with no H minor into these ingredients; it requires being able to test whether a given graph has this structure. This motivates a question we are currently working on, which is to find the excluded minors for the class of all minors of graphs with this structure. According to a theorem of Adler, Grohe and Kreutzer, the set of excluded minors for a class is computable given two things: an explicit bound on the tree-width of its minimal excluded subgraphs, and a definition, in monadic second-order logic, of the graphs of bounded tree-width in the class.

The first can be found by showing that the minimal excluded subgraphs do not have a large grid minor. For the second, we are working on making a dynamic programming algorithm to recognize these graphs.

This is joint work with Jim Geelen.

Matching Structure of Symmetric Bipartite Graphs and a Generalization of Pólya's Problem

Naonori Kakimura, University of Tokyo

A bipartite graph is said to be *symmetric* if it has symmetry of reflecting two vertex sets. This talk discusses matching structure of symmetric bipartite graphs. We first apply the *Dulmage-Mendelsohn decomposition* to a symmetric bipartite graph. The resulting components, which are matching-covered, turn out to have symmetry. We then decompose a matching-covered bipartite graph via an *ear decomposition*, which is a sequence of subgraphs obtained by adding an odd-length path repeatedly. We show that, if a matching-covered bipartite graph is symmetric, an ear decomposition can retain symmetry by adding no more than two paths.

As an application of these decompositions to combinatorial matrix theory, we present a natural generalization of Pólya's problem. We introduce the problem of deciding whether a rectangular $\{0,1\}$-matrix has a signing that is totally sign-nonsingular or not, where a rectangular matrix is *totally sign-nonsingular* if the sign of the determinant of each submatrix with the entire row set is uniquely determined by the signs of the nonzero entries. We show that this problem can be solved in polynomial time with the aid of the matching structure of symmetric bipartite graphs. In addition, we provide a characterization of this problem in terms of excluded minors.

Packing six T-joins in plane graphs

Daniel Král', joint work with Zdeněk Dvořák and Ken-ichi Kawarabayashi

Let G be a plane graph and T an even subset of its vertices. It has been conjectured that if all T-cuts of G have the same parity and the size of every T-cut is at least k, then G contains k edge-disjoint T-joins. The case $k = 3$ is equivalent to the Four Color Theorem, and the cases $k = 4$, which was conjectured by Seymour, and $k = 5$ were proved by Guenin. We settle the next open case $k = 6$. As a corollary we obtain that every 6-regular 6-edge-connected plane multigraph is 6-edge-colorable.

A few colouring and structure problems

Andrew D. King, Columbia University, IEOR Department

Here are two general questions that I am interested in exploring at the workshop. Both would belong in the graph colouring discussion group.

- **Hitting all maximal cliques of a given size.**

  If a graph has $\omega > \frac{2}{3}(\Delta+1)$, then there is a stable set intersecting every maximum clique. One reason this is nice is because it gives you the condition $\omega \leq \frac{2}{3}(\Delta+1)$ for a minimal counterexample to any bound of the type $\chi \leq \lceil \beta(\Delta+1) + (1-\beta)\omega \rceil$ for any $\beta \in [\frac{1}{2}, 1)$ on any hereditary class of graphs. (Reed's ω, Δ, χ conjecture proposes that this holds for all graphs and $\beta = \frac{1}{2}$.)

  But what can you say about *maximal* cliques? Certainly the best you can hope for is that any graph has a stable set hitting every maximal clique of size $> \frac{2}{3}(\Delta+1)$, but the central lemma in the proof for maximum cliques breaks down in the setting of maximal cliques. Can it be proven in another way, or disproven? This problem is related to independent transversals and, of course, graph colouring.

- **Do all triangle-free graphs satisfy $\chi \leq \frac{\Delta}{2} + 2$?**

  A seemingly harmless special case of Reed's ω, Δ, χ conjecture is the conjecture that any graph with $\omega = 2$ has $\chi \leq \frac{\Delta}{2} + 2$. This is particularly attractive because it is a special case of bull-free graphs, for which Chudnovsky has a structure theorem by which the broader conjecture might be attacked. Furthermore, the asymptotics of χ are known for triangle-free graphs: Johansson proved that $\chi \leq \frac{9\Delta}{\log \Delta}$.

Kostochka also has two important related results: Every triangle-free graph satisfies $\chi \leq \frac{2}{3}\Delta + 2$, and every graph with girth $\geq 4(\Delta+2)\log\Delta$ satisfies $\chi \leq \frac{\Delta}{2} + 2$.

All this notwithstanding, the problem on triangle-free graphs is open for $\Delta = 5$ and the easier case $\Delta = 6$. Can we at least prove that a graph with maximum degree 6 and girth ≥ 5 is 5-colourable? It would be possible to prove the entire conjecture for triangle-free graphs by bounding the chromatic number for triangle-free graphs with maximum degree up to about 20, then using a Lovász-type partitioning approach to colour a triangle-free graph with higher maximum degree. I think this would be a great result but quite difficult.

Recent Research

Tereza Klimosova

We searched for the minimal forbidden minors for the class of apexes of partial 2-trees. We found 19 of them but it still remains to analyze a special type of graphs with path-width three.

Currently I am working on immersions. It is known that average degree $\Omega(k^2)$ guarantees K_k as a topological minor and average degree $\Omega(k\sqrt{\log k})$ is sufficient for K_k as a minor. We are trying to find similar condition for immersions.

Hadwiger number of small graphs

A. Kostochka, University of Illinois

The following 3 results on Hadwiger number of small graphs were obtained recently.

1) (Joint with T. Boehme and A. Thomason) Motivated by Hadwiger's conjecture, we say that a coloring of a graph is *over-dominating* if every vertex is joined to a vertex of each other color and if, for each pair of color classes C_1 and C_2, either C_1 has a vertex adjacent to all vertices in C_2 or C_2 has a vertex adjacent to all vertices in C_1.

We show that a graph that has an over-dominating coloring with k colors has a complete minor of order at least $2k/3$ and that this bound is asymptotically best possible.

2) (Joint with T. Boehme and A. Thomason) We prove that every graph G with n vertices and chromatic number k has Hadwiger number, $h(G)$, at least $(4k-n)/3$. This result gives better lower bounds for $h(G)$ when n is not far from k. For example, if $k \geq 2n/3$, then it yields $h(G) \geq 5k/6$.

3) (Joint with J. Balogh) Recently, there was a series of papers bounding from below Hadwiger number of graphs with small independence number. They were mostly of the kind $h(G) \geq \frac{n}{2(\alpha(G)-c)}$. Then Fox proved the bound

$$h(G) \geq \frac{n}{(2-c)\alpha(G)} \tag{1}$$

for $c \sim 0.0174 \sim \frac{1}{57.5}$. We elaborate and extend the ideas of Fox to prove (1) for $c = \frac{1}{20}$.

Algorithms for disjoint rooted paths and other problems

Zhentao Li, McGill University

We gave a linear time algorithm for determining if an input graph contains a K_5-minor. This algorithm is then modified to obtain a linear time algorithm to determine if vertex disjoint paths exist from input vertices s_1 to t_1 and s_2 to t_2. The parity version of the problem is also studied. Here, the path from s_1 to t_1 and

s_2 to t_2 must also be of a specified parity. We provide a polynomial time algorithm to solve this problem and modify the algorithm to solve other related problems. Different parts of these results is joint work with subsets of Bruce Reed, Ken-ichi Kawarabayashi and Rohan Kapadia.

Currently, work is being done on the structure theorem for quasi-line graphs.

How many edges are needed to force a K_t-immersion?

Kevin G. Milans, University of South Carolina

In a multigraph, the *lifting operation* replaces incident edges uv and vw with a new edge joining u and w. Just as the edge contraction operation leads to the concept of a minor, the lifting operation leads to the concept of an *immersion*. A graph G contains an H-immersion if H can be obtained from a subgraph of G by a sequence of lifting operations. Robertson and Seymour proved that graphs are well-quasi-ordered by immersion containment. Bollobás and Thomason, and independently Komlós and Szemerédi proved that there is a constant c such that if G has average degree ct^2, then G contains a K_t-subdivision and hence a K_t-immersion. The author is not aware of any subquadratic bound on the average degree needed to force a K_t-immersion and believes such a result would be interesting. In fact, DeVos, Kawarabayashi, Mohar, and Okamura conjectured that each graph with minimum degree $t-1$ contains a K_t-immersion and proved the conjecture for $t \leq 7$.

Additionally, the author is interested in several of the proposed workshop problems: the Erdös–Hajnal Conjecture, coloring perfect graphs, recognizing graphs with odd holes, and questions on digraph minors.

Packing minors half-integrally

Sergey Norin, Princeton University

Given graphs G and H, a k-half-integral packing of H-minors in G is a collection of subgraphs $G_1, G_2, ..., G_k$ of G such that each vertex of G belongs to at most two of them, and each G_i contains H as a minor. We prove a conjecture of Thomas, showing that the Erdos-Posa property holds for half-integral packing of H-minors. That is, for every graph H there exists a function $f_H(k)$ such that every graph G either contains a k-half-integral packing of H minors or a set X of at most $f_H(k)$ vertices such that $G - X$ has no H minor.

Hyperbolic surface subgraphs of doubles of free groups

Sang-il Oum, Department of Mathematical Sciences, KAIST, Daejeon, Korea

In 1993, Gromov asked whether every one-ended word-hyperbolic group contains a hyperbolic surface group. We prove that every one-ended double of a free group contains a hyperbolic surface group, if the amalgamating set of words contains each generator the same number of times, or the free group has rank two. Why is this related to graph theory? Kim and Wilton [*Polygonal words in free groups*, submitted, 2009] made Tiling Conjecture on tiling of closed surfaces and proved that Tiling Conjecture implies an affirmative answer to Gromov's question for doubles of free groups. We formulate the following graph-theoretic conjecture equivalent to Tiling conjecture and prove it for regular graphs and 4-vertex graphs.

Conjecture 1 *Let $G = (V, E)$ be a non-acyclic graph with a fixed point free involution $\mu : V \to V$ and a bijection $\sigma_v : \delta(v) \to \delta(\mu(v))$ for each vertex v such that there are $\deg(v)$ edge-disjoint paths from v to $\mu(v)$. Then G admits a nonempty list of cycles such that for each adjacent pair of edges e and f incident with v, the number of cycles in the list containing both e and f is equal to the number of cycles in the list containing both $\sigma_v(e)$ and $\sigma_v(f)$. Moreover if G is connected and $|V(G)| \geq 4$, then the list can be assumed to have a cycle of length at least* 3.

This is a joint work with Sang-Hyun Kim.

Spanning closed walks in 3-connected plane graphs.

Kenta Ozeki (National Institute of Informatics, Japan).

A well-known Tutte theorem states that every 4-connected planar graph has a Hamilton cycle, and there exist infinitely many 3-connected non-Hamiltonian plane graphs. Therefore, in 3-connected plane graphs, several researchers have tried to find some "good" structures which are close to Hamilton cycles. For example, every 3-connected plane graph has a 3-tree (a spanning tree with maximum degree at most three) by Barnette, and a 2-walk (a spanning closed walk which passes each vertex at most twice) by Gao and Richter. Moreover, Nakamoto, Oda and Ota showed that every 3-connected plane graph of order n has a 3-tree such that the number of vertices of degree three is at most $\frac{n-7}{3}$. Similarly to this result, consider a spanning closed walk such that the number of vertices that are taken more than once is bounded. This seems a good problem, but no one succeeded to give an answer to it. Recently, we show the following. This is a joint work with K. Kawarabayashi (National Institute of Informatics).

Theorem 1 *Let G be a 3-connected plane graph of order n. Then G has a spanning closed walk such that the number of vertices that are taken more than once is at most $\frac{1}{3}(n-4)$.*

The coefficient $\frac{1}{3}$ of n is best possible. However, we do not know whether we can improve the walk obtained by Theorem 1 to a 2-walk. This is still open problem.

Even pairs and the circular chromatic number of K_4-free graphs with no odd holes

Yori Zwols, McGill University

An *odd hole* in a graph is an induced cycle of odd length at least five. We show that every imperfect K_4-free graph with no odd hole either is one of two basic graphs, or has an even pair or a clique cutset. We use this result to show that every K_4-free graph with no odd hole has circular chromatic number strictly smaller than 4. We also exhibit a sequence $\{G_n\}$ of such graphs with $\lim_{n\to\infty} \chi_c(G_n) = 4$.