Preliminary Schedule of Invited Talks

August 9th, 2010

8:40-8:50 Welcome to the workshop and the BIRS centre

8:50-9:30
Prof. D. Jerison (MIT)
Title: A gradient bound for free boundaries

9:40-10:20 Prof. R. Mazzeo (Stanford)
Title:Renormalized area of minimal surfaces in hyperbolic space

10:20:10:45
Coffee Break

10:45-11:25
Prof. M. Kowalczyk (Univ. Chile)
Title:Minimal surfaces and entire solutions of the Allen-Cahn equation

11:35-12:15
Prof. W. Meeks (Univ. Mass.)
Title: Constant mean curvature surfaces in homogeneous 3-manifolds

12:15-2:30
Lunch

2.30-3:10
Prof. E.N. Dancer (Univ. Sydney)
Title: Stable solutions on all space and applications

3:10-3.40
Coffee Break

3:40-4:20
Prof. R. Kusner (Univ. Mass.)
Title: Moduli spaces of complex projective structures and CMC surfaces

4:30-5:10
Prof. H. Matano (Univ. Tokyo)
Title: Front progagation for nonlinear diffusion equations on the hyperbolic space

5:20-6:00
Prof. C. Gui (Univ. Connecticut)
Title: Axial Symmetry of Some Entire Solutions of Nonlinear Elliptic Equations

Dinner

August 10th, 2010

8:50-9:30
Prof. P.H. Rabinowitz (Univ. Wisconsin)
Title: Hybrid solutions for a semilinear elliptic PDE

9:40-10:20
Prof. A. Malchiodi (SISSA)
Title: Asymptotically periodic solutions for the NLS and related issues

10:20:10:45
Coffee Break

10:45-11:25
Prof. P. Bates (Michigan State Univ.)
Title: Heirarchy of solutions to gradient elliptic systems with symmetry

11:35-12:15
Prof. P. Sternberg (Indiana Univ.)
Title: Global minimizers of the nonlocal isoperimetric problem in two dimensions

12:15-2:30
Lunch

2:30-3:10
Prof. N. Kapouleas (Brown Univ.)
Title: Doubling and desingularization constructions for minimal surfaces.

3:10-3.40
Coffee Break

3:40-4:20
Prof. J.M. Roquejoffre (Univ. Toulouse)
Title: 3D travelling waves for mean curvature motion and unbalanced Allen-Cahn,

4:30-5:10
Prof. A. Farina (Univ. Picardie)
Title: Phase transition, stability and symmetry

5:20-6:00
Prof. N. Ghoussoub (Univ. British Columbia)
Title: Regularity issues and Liouville theorems for 4th order equations

Dinner

August 11th, 2010

8:50-9:30
Prof. F. Pacard (Univ. Paris 12)
Title: The role of minimal and constant mean curvature surfaces in some overdetermined elliptic problem.

9:40-10:20
Prof. P. Polacik (Univ. Minnesota)
Title: Symmetry properties of nonnegative solutions of elliptic equations

10:20-10:45
Coffee Break

10:45-11:25
Prof.J. Ratzkin (Univ. Connecticut)
Title: Isoperimetric-type inequalities and eigenvalues

11:35-12:05
Prof. M. Musso (Cath. Univ. Chile)
Title: Finite-energy sign-changing entire solutions for some classical semilinear elliptic equations

Lunch
Free Afternoon

August 12th, 2010

8:50-9:30
Prof. C.S. Lin (National Taiwan Univ.)
Title: Mean field equations of Liouville type at critical parameters

9:40-10:20
Prof. Y. Tonegawa (Hokkaido Univ.)
Title: Regularity of stable phase interfaces in the Van der Waals-Cahn-Hilliard theory

10:20:10:45
Coffee Break

10:45-11:25
Prof. E. Valdinoci (Univ. Roma 3)
Title: Pointwise gradient estimates and rigidity results

11:35-12:05
Prof. Y. Sire (Univ. Paul Cezanne)
Title: Some nonlinear problems with fractional Laplacians

12:05-2:30
Lunch

2:30-3:00
Prof. S. Yan (Univ. New England)
Title: Infinitely many positive solutions for an elliptic problem with critical or super-critical growth

3:00-3.30
Coffee Break

3:30-4:00
Prof. X. Nguyen (Kansas State Univ.)
Title: Complete embedded self-translating surfaces under mean curvature flow

4:10-4:40
Prof. J. Dávila (Univ. Chile)
Title: Periodic fronts for nonlocal equations

4:50-5:20
Prof. P. Montecchiari (Univ. Pol. delle Marche).
Title: Prescribed Energy solutions of semilinear elliptic equations on cylindrical domains.

5:30-6:10 Prof. X.-F. Wang (Tulane)
Title:Thermal Insulation via Anisotropic Coatings

Dinner

August 13th, 2010

8:50-9:30
Prof. W.-X. Chen (Yeshiva)
Title: Symmetry and Regularity of Solutions for Nonlinear Systems of Wolff Type.

9:40-10:20
Prof. Congming Li (Univ. Colorado)
Title: Qualitative Analysis of Solutions to the HLS systems

10:20:10:45
Coffee Break

10:45-11:15
Prof. F. Mahmoudi (Univ. Tunis., Univ. Chile)
Title: TBA

11:25-12:05
Prof. Y. Du (Univ. New England)
Title: Spreading speed revisited-a free boundary approach

Lunch

END of Program. Thank You!

Have a Nice Trip!

Abstracts of Invited Talks at Banff Workshop (August 8–13, 2010)

Title: *Heirarchy of solutions to gradient elliptic systems with symmetry*

Prof. Peter Bates

We examine the asymptotic states of symmetric solutions to $\Delta u - gradW(u) = 0, u : R^n \to R^n$constructed by Alikakos and Fusco. Here W is equivariant under a finite reflection group and has $n + 1$ nondegenerate minima. Passing to the limit as $x \to \infty$ in certain direction gives lower dimensional solutions with symmetry. This is joint work with N. Alikakos.

Title: Symmetry and regularity of solutions for nonlinear systems of Wolff type

Prof. Wenxiong Chen

Attached

Title Stable solutions on all space and Applications

Prof. E.N. Dancer

We discuss that when we can prove all bounded linearized stable solutions of

$$-\Delta u = f(u) \text{on } R(^N)$$

are constants,we can nearly always obtain a good deal of understanding of the half space case with Dirichlet boundary conditions and the finite Morse index solutions. We then discuss the application of these results to bounded domain problems where the diffusion is small or the solutions are large. In particular,how the theory can be used to prove that the branch of positive solutions of the bounded domain problem has infinitely many bifurcations for a large number of rapidly growing nonlinearities.

Symmetry and Regularity of Solutions for Nonlinear Systems of Wolff Type.

Wenxiong Chen Congming Li

July 12, 2010

Abstract

In this talk, we will consider radial symmetry and regularity for positive solutions of the fully nonlinear integral systems involving Wolff potentials:

$$\begin{cases} u(x) = W_{\beta,\gamma}(v^q)(x), & x \in R^n; \\ v(x) = W_{\beta,\gamma}(u^p)(x), & x \in R^n; \end{cases} \tag{1}$$

where

$$W_{\beta,\gamma}(f)(x) = \int_0^\infty \left[\frac{\int_{B_t(x)} f(y)dy}{t^{n-\beta\gamma}} \right]^{\frac{1}{\gamma-1}} \frac{dt}{t}.$$

In a special case when $\beta = \frac{\alpha}{2}$ and $\gamma = 2$, system (1) reduces to

$$\begin{cases} u(x) = \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} v(y)^q dy, & x \in R^n, \\ v(x) = \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} u(y)^p dy, & x \in R^n. \end{cases} \tag{2}$$

The solutions (u, v) of (2) are critical points of the functional associated with the well-known Hardy-Littlewood-Sobolev inequality. The classification of solutions would provide the best constant in the HLS inequality.

We can also prove that the integral system (2) is equivalent to the system of partial differential equations

$$\begin{cases} (-\Delta)^{\alpha/2} u = v^q, u > 0, \text{ in } R^n, \\ (-\Delta)^{\alpha/2} v = u^p, v > 0, \text{ in } R^n. \end{cases} \tag{3}$$

And in particular when $\alpha = 2$, it reduces to the well-known Lane-Emden system. And even more particularly, when $p = q = \frac{n+2}{n-2}$, it becomes the Yamabe equation.

The symmetry is obtained by the integral form of the method of moving planes. This method is quite different from the ones for PDEs. Instead of using maximum principles, some global norms are estimated.

The regularity is established by liftings. We will mention two convenient ways to lift regularity for solutions: one by contracting operators and the other by the combined use of contracting and shrinking operators. We will focus on the latter–a new idea which has just been applied in our recent paper to establish Lipschitz continuity of positive solutions for system (1).

Usually, in order to lift the regularity of a solution from a lower to a higher space (in terms of regularity), we required that operator T be contracting in *both* spaces. However, for a nonlinear operator in certain spaces, it is sometimes very difficult, or even impossible, to prove it to be contracting, although one may still be able to show that it is "shrinking". Here we introduce a more general theorem , namely, one which requires that the operator be contracting in one space but only "shrinking" in the other. We believe this theorem will find broad applications in many other situations in nonlinear analysis.

Let V be a Hausdorff topological vector space. Suppose there are two extended norms (i.e., the norm of an element in V might be infinity) defined on V,

$$\|\cdot\|_X, \ \|\cdot\|_Y : V \rightarrow [0, \infty].$$

Let

$$X := \{v \in V : \|v\|_X < \infty\} \ \text{ and } \ Y := \{v \in V : \|v\|_Y < \infty\}.$$

We also assume that X is complete and that the topology in V is weaker than the topology of X and the weak topology of Y, which means that the convergence in X or weak convergence in Y will imply convergence in V.

Definition 1. *("XY-pair")* The pair of spaces (X, Y) described above is called an "XY-pair", if whenever the sequence $\{u_n\} \subset X$ with $u_n \to u$ in X and $\|u_n\|_Y \leq C$ will imply $u \in Y$.

In practice, we usually choose V to be the space of distributions and X, Y to be the function spaces, such as L^p spaces, Hölder spaces, Sobolev spaces, and so forth. There are many commonly used function spaces that are "XY-pairs," as will be illustrated in the remark after the theorem.

Theorem 1. *(Regularity Lifting) Suppose Banach spaces X, Y are an "XY-pair", both contained in some larger topological space V satisfying properties described above. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be closed subsets of X and Y respectively. Suppose $T : \mathfrak{X} \to X$ is contracting:*

$$\|Tf - Tg\|_X \leq \eta \|f - g\|_X, \ \forall f, g \in \mathfrak{X} \ \text{ and for some } 0 < \eta < 1; \quad (4)$$

and $T : \mathfrak{Y} \to Y$ is shrinking:

$$\|Tg\|_Y \leq \theta \|g\|_Y, \ \forall g \in \mathfrak{Y}, \ \text{ and for some } 0 < \theta < 1. \quad (5)$$

Define

$$Sf = Tf + F \quad \textit{for some } F \in \mathfrak{X} \cap \mathfrak{Y}.$$

Moreover, assume that

$$S : \mathfrak{X} \cap \mathfrak{Y} \to \mathfrak{X} \cap \mathfrak{Y}. \tag{6}$$

Then there exists a unique solution u of equation

$$u = Tu + F \quad \textit{in } \mathfrak{X},$$

and more importantly,

$$u \in Y.$$

Remark 1. In some situations, one can choose $\mathfrak{X} = X$ and $\mathfrak{Y} = Y$.

In practice, if one knows that a solution u of $u = Su$ belongs to X (usually with lower regularity), then by Theorem 1, one can lift the regularity of u up to $u \in X \cap Y$ (with higher regularity).

Remark 2. "XY-pairs" are quite common, as one can see from the following examples.

- $X = L^p(U)$ for $1 \leq p \leq \infty$, $Y = C^{0,\alpha}(U)$ for $0 < \alpha \leq 1$, and V is the space of distributions. Here U can be any subset of R^n or R^n itself.
- X is a Banach space, Y is a reflexive Banach space, and both are in some bigger topological space, V. Of course, we assume V is Hausdorff and has topology weaker than the topology of X and the weak topology of Y. Then for any $u_n \to u \in X$ and $\|u_n\|_Y \leq C$, we have $u \in Y$.

  Notice that all Hilbert spaces, such as L^2, H^1, and H^2, are reflexive Banach spaces.

Periodic fronts for nonlocal equations
Coville, J. Davila, S. Martinez

The main objective is to construct pulsating front solutions of the evolution equation

$$u_t = J * u - u + f(x,u) \qquad t \in \mathbb{R},\ x \in \mathbb{R}^N. \tag{0.1}$$

We assume that $J : \mathbb{R}^N \to \mathbb{R}$ satisfies

$$\begin{cases} J \geq 0,\ \displaystyle\int_{\mathbb{R}^N} J = 1,\ J(0) > 0, \\ J \text{ is smooth, symmetric with support contained in the unit ball,} \end{cases}$$

and that $f : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ is $[0,1]^N$-periodic in x, that is,

$$f(x+k,u) = f(x,u) \quad \forall k \in \mathbb{Z}^N,$$

and satisfies:

$$\begin{cases} f \in C^3(\mathbb{R}^N \times [0,\infty)), \\ f(\cdot,0) \equiv 0, \\ f(x,u)/u \text{ is decreasing with respect to } u \text{ on } (0,+\infty), \\ \text{there exists } M > 0 \text{ such that } f(x,u) \leq 0 \text{ for all } u \geq M \text{ and all } x. \end{cases}$$

The model example is

$$f(x,u) = u(a(x) - u)$$

where $a(x)$ is periodic C^3.

We will assume in what follows that the stationary problem

$$0 = J * u - u + f(x,u) \qquad x \in \mathbb{R}^N \tag{0.2}$$

has a positive periodic continuous solution $p(x)$. Some authors [1, 5, 6] have already identified a condition characterizing the existence of a positive stationary solution, and also proved that it is unique. This condition says that the linearized stationary operator around zero, that is,

$$-(J * \phi - \phi - f_u(x,0)\phi)$$

considered on the space of periodic continuous functions, has a negative principal eigenvalue.

Let e be a unit vector in $\mathbb{R}^N$, $c \in \mathbb{R}$ and p be the positive periodic continuous solution of (0.2). We say that a solution u to (0.1) is a periodic front propagating in the direction $-e$ with effective speed c if

$$u(t + k \cdot e/c, x) = u(t, x+k) \quad \forall t \in \mathbb{R},\ x \in \mathbb{R}^N \quad \forall k \in \mathbb{Z}^N$$

and

$$u(t,x) \to 0 \quad \text{as } t \to -\infty, \quad \text{for all } x$$
$$u(t,x) \to p(x) \quad \text{as } t \to +\infty, \quad \text{for all } x.$$

The main result is the following.

Theorem 1. *Given any unit vector $e \in \mathbb{R}^N$ there is a number $c^*(e)$ such that for $c \geq c^*(e)$ (0.1) has a pulsating front solution with effective speed c, and for $c < c^*(e)$ there is no such solution.*

Traveling waves appear naturally in homogeneous reaction diffusion equations, and were studied intensively starting with the pioneering works [7, 10]. The notion of pulsating front was introduced in [14, 15], as a generalization of traveling waves to the situation of equations with periodic inhomogeneities. The problem of finding periodic fronts in some contexts, i.e. discrete setting, or elliptic differential operators, has been addressed by many authors [2, 3, 4, 8, 9, 12, 11, 16]. The problem of findind spreading speeds for the nonlocal equation (0.1) is considered in [13].

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Title:Spreading speed revisited–A free boundary approach

Prof. Yihong Du

Much previous mathematical investigation on the spreading of population was based on the diffusive logistic equation over the entire space $\mathbb{R}^N$:

$$u_t - d\Delta u = au(1-u),\ t>0,\ x\in\mathbb{R}^N, \tag{0.3}$$

where a and d are positive constants. In the pioneering works of Fisher (1937) and Kolmogorov et al (1937), for space dimension $N=1$, traveling wave solutions have been found for (0.3): For any $|c|\geq c^* := 2\sqrt{ad}$, there exists a solution $u(t,x) := W(x-ct)$ with the property that

$$W'(y)<0 \text{ for } y\in\mathbb{R}^1,\ W(-\infty)=1,\ \ W(+\infty)=0;$$

no such solution exists if $|c|<c^*$. The number c^* is called the minimal speed of the traveling waves. c^* is also known as the spreading speed of a new population $u(t,x)$ (governed by the above logistic equation) with initial distribution $u(0,x)$ confined to a compact set of x (i.e., $u(0,x)=0$ outside a compact set): For such $u(t,x)$, it was shown by Aronson and Weinberger (1978) that

$$\lim_{t\to\infty,\ |x|\leq(c^*-\epsilon)t} u(t,x)=1, \qquad \lim_{t\to\infty,\ |x|\geq(c^*+\epsilon)t} u(t,x)=0$$

for any small $\epsilon > 0$. These results have motivated extensive research on traveling wave solutions and the spreading speed in several directions. In this talk, I will report recent joint work with Zhigui Lin and Zongming Guo, where we use the diffusive logistic model with a free boundary to describe the spreading of a new or invasive species, with the free boundary representing the expanding front. We prove a spreading-vanishing dichotomy for this model, namely the species either successfully spreads to all the new environment and stabilizes at a positive equilibrium state, or it fails to establish and dies out in the long run. Moreover, we show that when spreading occurs, for large time, the expanding front moves asymptotically at a constant speed. This asymptotic spreading speed is uniquely determined by an elliptic problem induced from the original model, and is different from c^* mentioned above.

Title: Phase transitions, stability and symmetry
Prof. A. Farina

Attached

Title: Regularity issues and Liouville theorems for 4th order equations
Prof. N. Ghoussoub

Title: *Axial Symmetry of Some Entire Solutions of Nonlinear Elliptic Equations*
Prof. Changfeng Gui

In this talk, I will present some recent results on the axial symmetry of certain entire solutions which are anisotropic. The type of solutions includes stationary solutions for nonlinear Schrodinger equation, saddle solutions and traveling wave solutions for Allen-Cahn equations.

Phase transitions, stability and symmetry

Alberto Farina

Université de Picardie Jules Verne

LAMFA, CNRS UMR 6140

Amiens, France

In 1978 E.De Giorgi [4] posed the following question : *Let* $u \in C^2(\mathbb{R}^N, [-1,1])$ *satisfy*

$$-\Delta u = u - u^3 \qquad \text{and} \qquad \frac{\partial u}{\partial x_N} > 0 \tag{1}$$

in the whole $\mathbb{R}^N$*.*
Is it true that all the level sets of u *are hyperplanes, at least if* $N \le 8$ *?*

Equivalently, De Giorgi's conjecture can be reformulated by saying that the considered solution u is **1D**, that is, it depends only on one variable (up to rotations).

De Giorgi's conjecture is settled for $N = 2, 3$ ([10],[2]). When $4 \le N \le 8$, the conjecture is still open and no counterexample is available (not even for more general semilinear equations). For $N \ge 9$, in [5] the authors have constructed a solution of (1), which is **not 1D**. This implies that the assumption $N \le 8$ in De Giorgi's conjecture cannot be removed.

The PDE in (1) is the well-known Allen-Cahn equation arising in phase transition problems [1] and a possible motivation for the conjecture is the following : let u be as in De Giorgi's conjecture, $\epsilon > 0$ and let $u_\epsilon(x) := u(x/\epsilon)$. The monotonicity assumption in De Giorgi's conjecture *seems to suggest* that :

- the level sets of u (and thus those of u_ϵ) are graphs over $\mathbb{R}^{N-1}$,
- the phase transition happens in a straight, *minimal* way.

Thus, when $\epsilon \to 0^+$, the level sets of u_ϵ are closer and closer (in a suitable way ([11],[12],[3])) to entire minimal graphs φ over $\mathbb{R}^{N-1}$, i.e. a solution of

$$-\text{div}\left[\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right] = 0 \quad \text{in} \quad \mathbb{R}^{N-1}. \tag{2}$$

Since entire minimal graphs are flat for $N - 1 \le 7$, due to Bernstein-type Theorems, it follows that the level sets of u_ϵ are close to a flat hyperplane.

Here $\quad N \le 8 \quad$ *is crucial* !

Now, since elliptic problems are somehow "rigid", we may *suspect* that once the level set $\{u_\epsilon = c\}$ is close enough to a hyperplane, it is a hyperplane itself. By scaling back, this would give that $\{u = c\}$ is a hyperplane.

Then, the level sets of u would be parallel hyperplanes and thus u would be 1D, as asked by De Giorgi's conjecture.

In the previous (heuristic) argument some gaps have to be filled :

• no minimality condition is explicitly required in De Giorgi's conjecture, so the results about the asymptotic behavior of minimizers are not directly applicable,

• the monotonicity condition does not assure, in principle, that the level sets of u are entire graphs over $\mathbb{R}^{N-1}$, so Bernstein-type results are not directly applicable,

• one would need to proof the rigidity argument.

In this talk we discuss how (and how much of) this program can be carried out. We also present the recent progress on the De Giorgi's conjecture and its generalizations, for $N \geq 4$ ([6],[7],[8],[9],[13]). To this end we introduce the notion of *stable* solution of the semilinear equation $-\Delta u = f(u)$, $f \in C^1$, and then we prove some one-dimensional symmetry results for this class of solutions. In particular, since monotone solutions and local minimizers are special cases of stable solutions, any classification result (1D symmetry result) about stable solutions immediately gives an answer to De Giorgi's conjecture.

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Title: *A Gradient Bound for Free Boundaries*

Prof. David Jerison

The gradient bound for minimal graphs of Bombieri, De Giorgi and Miranda says that every solution to the minimal surface equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$$

in B_2, the ball of radius 2, satisfies

$$\max_{B_1} |\nabla u| \leq C$$

where C depends only on dimension and the oscillation of u,

$$\max_{B_2} u - \min_{B_2} u$$

Following the celebrated results of De Giorgi and Nash, the method of Moser shows that the gradient bound leads to smoothness of solutions u.

Consider the functional

$$J_1(v) = \int_\Omega |\nabla v|^2 + \chi_{\{v>0\}}; \quad \Omega \subset \mathbf{R^n}$$

and a minimizer u of J_1 among all functions with boundary conditions $v = f$ on $\partial\Omega$. We will only discuss the one-phase case, that is, we assume $f \geq 0$, so that $u \geq 0$. Denote the positive phase of u by

$$\Omega^+(u) = \{x \in \Omega : u(x) > 0\}$$

Thus $u = 0$ in $\Omega \backslash \Omega^+(u)$. The interface between the positive phase and the zero phase is known as the *free boundary*,

$$F(u) = \Omega \cap \partial\Omega^+(u)$$

The Euler-Lagrange equations for the minimizer u are

$$\begin{aligned} \Delta u &= 0 \quad \text{on } \Omega^+(u) \\ |\nabla u| &= 1 \quad \text{on } F(u) \end{aligned}$$

The boundary condition is valid in a suitable weak sense.

In 1981, Alt and Caffarelli showed that there is a deep analogy between minimal surfaces and free boundaries $F(u)$. They proved that the free boundary has finite $(n-1)$-Hausdorff measure and is smooth except on a set of zero $(n-1)$-Hausdorff measure. On the smooth subset, the free boundary condition $|\nabla u| = 1$ is valid in the ordinary sense. Since then many results that deepen this analogy have been obtained. We will discuss a few of them.

Daniela de Silva and I showed (in J. Reine Angew. Math. 2009) that, as in the case of area-minimizing hypersurfaces, there is a critical dimension above which minimizers need not be smooth. In this talk we discuss further joint work establishing the analogue of the classical gradient bound for minimal surfaces. In particular, our bound implies that energy-minimizing free-boundary graphs are smooth in all dimensions. Our proof also gives a new way to demonstrate the classical minimal surface gradient bound as well. The way these two proofs are related reveals parallels between free boundaries and minimal surfaces that I expect to have an impact on the theory of semilinear functionals.

An application of smoothness of graph solutions in high dimensions is suggested by work of my PhD student Nikola Kamburov. He considers the functional

$$J_2(v) = \int_\Omega |\nabla v|^2 + \chi_{\{-1<v<1\}}$$

A minimizer u of J_2 (minimizing over all domains $\Omega \subset \mathbf{R^n}$ and all functions sharing the same boundary values on $\partial\Omega$) has Euler-Lagrange equations

$$\begin{aligned} \Delta u = 0 &\quad \text{on } \{-1 < u < 1\} \\ |\nabla u| = 1 &\quad \text{on } F(u) \end{aligned}$$

where this time the free boundary $F(u)$ has two pieces, namely, the interface between $\{-1 < u < 1\}$ and $u = 1$ and the interface between $\{-1 < u < 1\}$ and $u = -1$. This is a singular limit of functionals resembling de Giorgi functional, with a two-well potential. As in the case of de Giorgi's functional, work of Caffarelli and Cordoba (strengthening the results of Modica) shows that the blow-down of the set $\{-1 < u < 1\}$ is a global minimal surface.

Kamburov has constructed global minimizers in dimension 9 or more, whose level sets are graphs but not planes. His proof uses the important construction of supersolutions of Del Pino, Kowalczyk and Wei, but does not require their iteration. The regularity theory described above can be expected to lead to infinite differentiability of the level sets of Kamburov's global solutions.

Doubling and desingularization constructions for minimal surfaces.

Prof. N. Kapoulous

By a doubling construction we mean the construction of minimal surfaces based on a given minimal surface Σ in a Riemannian three-manifold as follows. Each minimal surface M constructed is smooth and can be written as the union $M = \Sigma_1 \cup \Sigma_2 \cup B$ where Σ_1 and Σ_2 approximate two copies of Σ and $B = \cup_{i=1}^{N} B_i$ is the union of N annuli approximating catenoidal bridges B_i connecting Σ_1 and Σ_2. More precisely, Σ_1 and Σ_2 are graphs of two small in C^2-norm functions ϕ_1 and ϕ_2 over $\Sigma \setminus \cap_{i=1}^{N} D_i$, the given surface with N small discs D_i removed, and the boundary of each B_i is the union of the boundary circles of Σ_1 and Σ_2 over ∂D_i. It is expected in general that as the number of bridges N tends to ∞, their size tends to 0, and M tends as a varifold to a double covering of Σ.

Title:*Minimal surfaces and entire solutions of Allen-Cahn equation*
Prof. M. Kowalczyk

This is a joint work with Manuel del Pino and J. Wei

We consider the Allen-Cahn equation

$$\Delta u + (1-u^2)u = 0 \quad \text{in } \mathbb{R}^N. \tag{0.4}$$

E. De Giorgi [3] formulated in 1978 the following celebrated conjecture:

(DG) *Let u be a bounded solution of equation* (0.4) *such that $\partial_{x_N} u > 0$. Then the level sets $[u=\lambda]$ are hyperplanes, at least for dimension $N \le 8$.*

Equivalently, under the above conditions the statement asserts the existence of $a \in \mathbb{R}^N$, $b \in \mathbb{R}$, $|a|=1$ such that u has the form

$$u(x) = w(a\cdot x - b)$$

where $w(t)$ is the unique solution of

$$w'' + (1-w^2)w = 0, \quad w(0)=0, \quad w(\pm\infty) = \pm 1,$$

namely $w(t) = \tanh(t/\sqrt{2})$. De Giorgi conjecture has been proven in dimensions $N=2$ by Ghoussoub and Gui [6] and for $N=3$ by Ambrosio and Cabré [1]. Savin [7] proved its validity for $4 \le N \le 8$ under a mild additional assumption. (DG) is a statement parallel to *Bernstein's theorem* for minimal graphs which in its most general form, due to Simons [9], states that any minimal hypersurface in $\mathbb{R}^N$, which is also a graph of a function of $N-1$ variables, must be a hyperplane if $N \le 8$. Bombieri, De Giorgi and Giusti [2] proved that this fact is false in dimension $N \ge 9$, by constructing a nontrivial entire solution to the minimal surface equation

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^8, \tag{0.5}$$

by means of the super-subsolution method. Let us write

$$x' = (x_1, \dots, x_8) \in \mathbb{R}^8, \quad u = \sqrt{x_1^2 + \cdots + x_4^2}, \quad v = \sqrt{x_5^2 + \cdots + x_8^2}.$$

The BDG solution has the form $F(x') = F(u,v)$ with the symmetry property $F(u,v) = -F(v,u)$ if $u \ge v$. In addition we can show that F becomes asymptotic to a function homogeneous of degree 3 that vanishes on the cone $u=v$. Let $\Gamma = \{x_9 = F(x')\}$ be the minimal BDG graph, and let us consider for $\alpha > 0$ its dilation $\Gamma_\alpha = \alpha^{-1}\Gamma$, which is also a minimal graph. Our result, which disproves statement (DG) in dimensions 9 or higher is the following.

Theorem 1. [4] *Let $N = 9$. For all $\alpha > 0$ sufficiently small there exists a bounded solution $u_\alpha(x)$ of equation (0.4) such that*

$$\partial_{x_9} u_\alpha(x) > 0 \quad \textit{for all } x \in \mathbb{R}^9,$$

and such that for $x = y + t\nu(\alpha y)$, where $y \in \Gamma_\alpha$ and ν is a choice of normal to Γ we have

$$u(x) = w(t) + o(1),$$

where $|t| < \frac{\delta}{\alpha}$ and $o(1) \to 0$ uniformly as $\alpha \to 0$.

Let us consider coordinates to describe points in $\mathbb{R}^9$ near Γ_α, $x = y + t\nu(\alpha y)$, $\quad y \in \Gamma_\alpha, \quad |t| < \frac{\delta}{\alpha}$. Then we choose as a first approximation $\mathtt{w}(x) := w(t + h(\alpha y))$ where h is a smooth, small function on Γ, to be determined. Looking for a solution of the form $\mathtt{w} + \phi$, the problem becomes essentially reduced to

$$\Delta_{\Gamma_\alpha}\phi + \partial_{zz}\phi + f'(w(z))\phi + E + N(\phi) = 0, \quad \text{in } \Gamma_\alpha \times \mathbb{R},$$

where $S(\mathtt{w}) = \Delta \mathtt{w} + f(\mathtt{w})$, $E = \chi_{|z|<\alpha^{-1}\delta} S(\mathtt{w})$, $N(\phi) = f(\mathtt{w}+\phi) - f(\mathtt{w}) - f'(\mathtt{w})\phi + B(\phi)$, $f(\mathtt{w}) = \mathtt{w}(1 - \mathtt{w}^2)$, and $B(\phi)$ is a second order linear operator with small coefficients, supported in $\{|z| \le \delta\alpha^{-1}\}$. Rather than solving the above problem directly we consider a projected version of it:

$$\mathcal{L}(\phi) := \Delta_{\Gamma_\alpha}\phi + \partial_{zz}\phi + f'(w(z))\phi = -E - N(\phi) + c(y)w'(z) \quad \text{in } \Gamma_\alpha \times \mathbb{R},$$
$$\int \phi(y,z)w'(z)\,dz \;=\; 0 \quad \text{for all } y \in \Gamma_\alpha, \tag{0.6}$$

A solution to this problem can be found in such a way that the size and decay rate of the error E, which is roughly of the order $\sim r(\alpha y)^{-3}e^{-|z|}$, is respected (here $r(\alpha y)$ is the distance to the origin of the projection of $\alpha y \in \Gamma_\alpha$ on $\mathbb{R}^8$). This is made precise with the use of a linear theory for the projected problem in weighted Sobolev norms and an application of contraction mapping principle. Finally h is found so that $c(y) \equiv 0$. We have $c(y)\int w'^2 dz = \int (E + N(\phi))w'\,dz$ and thus we get reduced to a (nonlocal) nonlinear PDE in Γ of the form

$$\mathcal{J}(h) := \Delta_\Gamma h + |A|^2 h = O(\alpha)r(y)^{-3} + P_\alpha(h) \quad \text{in } \Gamma, \quad h = 0 \quad \text{on } \Gamma \cap [u = v], \tag{0.7}$$

where $P_\alpha(h)$ is a small operator which includes nonlocal terms. A solvability theory for the Jacobi operator in weighted Sobolev norms is then devised, with the crucial ingredient of the presence of explicit barriers for inequalities involving the linear operator above, and asymptotic curvature estimates by Simon [8]. Using this theory, problem (0.7) is finally solved by means of contraction mapping principle. The monotonicity property follows from maximum principle applied to the linear equation satisfied by $\partial_{x_9} u$.

The method described above generalizes to the construction of new entire solutions in dimension $N = 3$ in [5]. We consider a minimal surfaces M which is complete, embedded and has finite total curvature in $\mathbb{R}^3$. We assume that M has $m \geq 2$ ends, and additionally that M is non-degenerate, in the sense that its all bounded Jacobi fields can be obtained considering rigid motions of the surface (this is known for instance for a catenoid and for the Costa-Hoffman-Meeks surface of any genus).

We prove that for any small $\alpha > 0$, and up to rigid motions, the Allen-Cahn equation has a family of bounded solutions depending on $m - 1$ parameters. Level sets of these solutions are embedded surfaces lying close to the blown-up surface $M_\alpha := \alpha^{-1}M$, with ends possibly diverging logarithmically from M_α. We prove that these solutions are L^∞-*non-degenerate* up to rigid motions, and find that their Morse index coincides with the index of the minimal surface. Our construction, and known classification results for minimal surfaces, suggest parallels of De Giorgi conjecture for general bounded solutions with finite Morse index.

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Moduli spaces of complex projective structures and CMC surfaces

Prof. Rob Kusner (dedicated to my father, David Kusner, 1932-2010)

Complete embedded constant mean curvatures (CMC) surfaces in $\mathbf{R}^3$ are highly transcendental objects [8, 3, 12, 10, 13] whose moduli spaces are understood in only a few special cases [1, 14, 9, 2, 6]. This talk (reporting on joint work [5] with Karsten Grosse-Brauckmann, Nick Korevaar and John Sullivan) discusses a surprising connection between CMC surfaces and complex projective structures, allowing us to make the former just a bit more explicit.

The well-known (see [4]) correspondence between projective structures and holomorphic quadratic differentials $q(z)\,dz^2$ via the holomorphic Hill equation

$$(\star) \qquad u_{zz} + q(z)\,u = 0,$$

guides our work: if $u_1(z), u_2(z)$ is a basis of solutions to $(\star)$, then their ratio $\frac{u_1}{u_2}$ is the developing map for a projective structure whose Schwarzian is $q(z)$, where z is a local coordinate belonging to some background projective structure. This makes the moduli space of projective structures over a fixed Riemann surface into a complex affine space modeled on the vector space of quadratic differentials.

In case of coplanar k-unduloids, CMC surfaces of genus 0 with k ends [7], the underlying Riemann surface is $\mathbf{C}$ with global coordinate z (unique up to $z \to az+b$) belonging to its standard projective structure, and $q(z)$ is a polynomial of degree $k-2$ normalized to be monic with root-sum zero. For example, the unduloids all have $q(z) = 1$ and an exponential developing map, while all triunduloids have $q(z) = z$ and developing map given by a ratio of Airy functions. For $k \geq 4$, it is not practical to solve the Hill equation $(\star)$ explicitly, so instead we perform a careful asymptotic analysis.

Each of the k ends corresponds to an asymptotic half-space in the flat metric given by $|q(z)|\,|dz|^2 \sim |dw|^2$. We use the flat half-space coordinate w to rewrite $(\star)$ as an $O(\frac{1}{w^2})$ perturbation of the constant coefficient equation. This allows us to analyze growing and decaying solutions on each half-space and show that the ratio of two independent global solutions to $(\star)$ is the developing map of a k-point projective structure: an equivalence class of the k-point spherical metrics previously used [7] to classify coplanar k-unduloids, except now two k-point metrics are equivalent if they differ by a fractional linear map (rather than an isometry) of $\mathbf{S}^2$.

Since the quotient space of fractional linear maps by isometries is a 3-ball, the moduli space of all k-point metrics – or equivalently, of all coplanar k-unduloids – is homeomorphic to the product of this ball with the space of k-point projective structures, and thus to $\mathbf{R}^{2k-3} = \mathbf{B}^3 \times \mathbf{C}^{k-3}$, where the second factor comes from realizing k-point structures by (affine) space of normalized polynomials of degree $k-2$. We already knew [6, 7] the topology of these moduli spaces for the cases $k = 3, 4$, but we had suspected that for $k \geq 5$ these spaces were not even simply connected, and thus it came as quite a surprise that they were contractible!

An interesting question we continue to explore is how this description for CMC moduli space compares with others, such as that coming from spherical metrics or, more speculatively, from the holomorphic potentials methods stemming from [3]. And although not discussed in this talk, one hopes these ideas may also be applied to give a more explicit description of minimal surfaces [11] in $\mathbf{S}^3$ which are cousins of CMC surfaces in $\mathbf{R}^3$.

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Title: Qualitative analysis of solutions to the HLS systems
Prof. Congming Li

Attached

Title: Mean field equations of Liouville type at critical parameters.
Prof. Changshou Lin

In this talk, I will discuss mean field equation with singular data at $8\pi m$, where m is an positive integer. In my previous joint paper with L.C. Wang, we have proved some sufficient and necessary condition for the existence of solution at 8 pi. In this talk, I will prove a non-existence result of mean field equation at 16 pi when the fundamental cell of a torus is a rectangle. The proof is a nice combination of classical elliptic function theory and the method of moving planes.

Title: *Asymptotically periodic solutions for the NLS and related issues*
Prof. A. Malchiodi

We consider a semilinear equations in R^n motivated by the study of the (focusing) Nonlinear Schroedinger Equation or of some chemical/biological models. Inspired by some construtions in differential geometry concerning constant mean curvature surfaces in R^3, we produce new entire solutions which decay to zero exponentially away from three half lines, and which are asymptotically periodic in these directions. We also prove that entire solutions decaying away from a line must be axially symmetric.

Qualitative Analysis of Solutions to the HLS systems

Congming Li
Univ. of Colorado at Boulder

August 9, 2010

The main object is the study of nonnegative solutions to the integral system derived from the well known Hardy-Littlewood-Sobolev inequality in both weighted and non-weighted form:

$$\begin{cases} u(x) = \frac{1}{|x|^\alpha} \int_{R^n} \frac{v(y)^q}{|y|^\beta |x-y|^\lambda} dy \\ v(x) = \frac{1}{|x|^\beta} \int_{R^n} \frac{u(y)^p}{|y|^\alpha |x-y|^\lambda} dy \end{cases} \tag{1}$$

$0 < p, q < \infty$, $0 < \lambda < n$, $\lambda \leq \overline{\lambda} \doteq \lambda + \alpha + \beta \leq n$, $\frac{1}{p+1} + \frac{1}{q+1} = \frac{\overline{\lambda}}{n}$, $\frac{\alpha}{n} < \frac{1}{P+1} \doteq 1 - \frac{1}{r} < \frac{\lambda+\alpha}{n}$, and $\frac{\beta}{n} < \frac{1}{q+1} = 1 - \frac{1}{s} < \frac{\lambda+\beta}{n}$

We begin with a short/incomplete review of the related results and problems. The main focus is on the classification of solutions in the 'critical-case' and on the Liouville type theorems in the 'subcritical case'. The well-known Land-Emden conjecture is a Liouville type 'theorem' in a special case of the 'subcritical' type HLS systems. We then present some integral type estimates – a key ingredient in deriving both classification and the Liouville type theorem. As an application, we use the special integral type estimates to derive the asymptotic expansion of the solutions at infinity as well as at the possible interior singular point.

The talk will consist with five main parts:

1: The 'Uniqueness' of Nonlinear Differential and Integral Systems: Liouville Type Theorems and Classification

2: Hardy-Littlewood-Sobolev Inequality and Its Euler-Lagrange Equations.

3: The Role of Symmetry, Integrability, and Asymptotics.

4: The Radial Symmetry: the MMP and the Regularity Lifting

5: The Asymptotic via the Optimal Integrability- a recent work.

The following local estimate will be discussed in detail:

Assume $\lambda q + \beta(q+1) > n.$, *then for any* $0 < \theta \leq \alpha$ *and* $\frac{1}{r} > \frac{\alpha-\theta}{n}$, *we have* $|x|^\theta u(x) \in L^r_{loc}(R^n)$. In particular, $|x|^\alpha u(x) \in L^r_{loc}(R^n)$ for any $r < \frac{n}{\alpha-\theta}$, which means $|x|^\alpha u(x)$ is ALMOST locally bounded.

This result shows the optimal local integrability of $|x|^\theta u(x)$. In fact, one can prove $|x|^\theta u(x) \geq \frac{C}{|x|^{\alpha-\theta}}$, which implies that

$$\||x|^\theta u(x)\|_r \geq C\|\frac{1}{|x|^{\alpha-\theta}}\|_s = \infty$$

as long as $r \geq n/(\alpha-\theta)$.

Title:*Front propagation for nonlinear diffusion equations on the hyperbolic space.*

Prof. H. Matano

In this talk I will discuss the front propagation for the Allen-Cahn equation and the KPP type equation on the hyperbolic space H^n. More precisely, I consider the equation

$$u_t = \Delta_S u + f(u), x \in \mathbb{H}^n,$$

where Δ_S stands for the Laplace-Beltrami operator. The nonlinearity f is either of the bistable type with 0 and 1 being the two stable zeros, or of the KPP type with $f(0) = f(1) = 0,\ f(s) > 0\,(0 < s < 1)$. We assume that the initial data u_0 is nonnegative and compactly supported (and $u_0 \not\equiv 0$). Our goal is:
(a) to estimate the spreading speed of the front; (b) to show that the shape of the front converges to a geodesic sphere (more precisely, the level surface $\{u = a\}$ with arbitrary $0a < 1$ becomes a smooth surface after finite time, and it stays within finite distance from an expanding geodesic sphere whose geodesic radius tends to infinity with a specific rate; (c) to show that the profile of the solution near the front area converges to that of a "horoshperical wave", whose meaning will be specified later.

This is joint work with Fabio Punzo and Alberto Tesei.

Title: *Renormalized area of minimal surfaces in hyperbolic space*

Prof.Raff Mazzeo

I will discuss joint work with Spyros Alexakis concerning an interesting functional (closely related to the Willmore functional) on the space of properly embedded minimal surfaces in H^3 with boundary at infinity an embedded curve in S^2. The corresponding Euler Lagrange equation can be regarded as a nonlinear nonlocal elliptic operator on the boundary curve. Recent progress includes an epsilon regularity theorem which controls the regularity of the boundary curve in terms of the size of the functional in the interior.

Constant mean curvature surfaces in homogeneous 3-manifolds

Prof.William H. Meeks III [1]

I will discuss some of the recent advances made in the theory of H-surfaces in a homogeneous 3-manifold, where by H-surface I mean a complete immersed surface of constant mean curvature H.

In 1951 H. Hopf proved the following beautiful uniqueness and embeddedness result for H-spheres in $\mathbb{R}^3$.

Theorem 2. *A sphere of constant mean curvature $H > 0$ in $\mathbb{R}^3$ is round of radius $\frac{1}{H}$.*

Hopf's theorem has four immediate consequences. If we denote by $\overline{\mathcal{M}}_{\mathbb{R}^3}$ the moduli space of congruency classes of spheres of varying constant mean curvature H in $\mathbb{R}^3$, then these consequences take the following forms.

1. **Uniqueness::** Two spheres of the same constant mean curvature in $\mathbb{R}^3$ are congruent.
2. **Existence/Moduli Space::** The mean curvature function $\mathbf{H}\colon \overline{\mathcal{M}}_{\mathbb{R}^3} \to (0,\infty)$ is a diffeomorphism to the interval $(0,\infty)$ of its values.
3. **Embeddedness/Alexandrov Embeddedness::** Constant mean curvature spheres in $\mathbb{R}^3$ are embedded.
4. **Index 1 and Nullity 3::** Constant mean curvature spheres in $\mathbb{R}^3$ have index 1 and nullity 3.

The first goal of this talk is to discuss appropriate generalizations of Hopf's theorem by Mira, Meeks Perez and Ros to the general setting of H-spheres in simply-connected homogeneous 3-manifolds X. I will explain how the related moduli space of varying constant mean curvature H-spheres in X is naturally a connected interval parametrized by the values of its mean curvature function and, depending on the topology and geometry of X, these constant mean curvature spheres are embedded or Alexandrov embedded and have index of stability at most one. This is work in progress and not in final form. Still I plan to give an outline of the proofs of some of the key results in this new theory and perhaps to explain some revelent aspects of the classification of metric Lie groups by Milnor.

A second important goal of the talk is to discuss some of the beautiful unsolved problems in this classical subject and the special recent results that motivate some of them.

[1]This material is based upon work for the NSF under Award No. DMS - 0703213. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the NSF.

Title: *Prescribed Energy solutions of semilinear elliptic equations on cylindrical domains.*

Prof. Piero Montecchiari,

For a given smooth domain $\Omega \subset \mathbb{R}^{n-1}$ $(n > 1)$ we are concerned with the class of semilinear elliptic problems

$$\begin{cases} -\Delta u(x,y) + W_u(y, u(x,y)) = 0, & (x,y) \in \mathbb{R} \times \Omega \\ u(x,y) = \phi(y) \text{ (or } \partial_\nu u(x,y) = 0), & (x,y) \in \mathbb{R} \times \partial\Omega \end{cases} \tag{0.8}$$

where $W : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\phi : \bar{\Omega} \to \mathbb{R}$ are regular functions. Note that W and ϕ do not explicitly depend on the variable x and one recognizes that if u is a (suitably bounded) solution of (??) then the corresponding Energy function is conserved:

$$E_u = \tfrac{1}{2}\|\partial_x u(x,\cdot)\|^2_{L^2(\Omega)} - V(u(x,\cdot)), \quad \forall x \in \mathbb{R} \tag{0.9}$$

where

$$V(u(x,\cdot)) = \tfrac{1}{2}\|\nabla_y u(x,\cdot)\|^2_{L^2(\Omega)} + \int_\Omega W(y, u(x,y))\, dy.$$

We consider the problem of finding solutions of (0.8) at prefixed values of the Energy. For that we exploit an Energy constrained variational argument giving applications to the Allen Cahn and Nonlinear Schroedinger equations.

Title: Finite-energy sign-changing entire solutions for some classical semilinear elliptic equations

Prof. M.Musso

We construct new families of finite energy sign changing solutions with dihedral symmetry for $\Delta u - u + u^p = 0$ in R^N, where $1 < p < \frac{N+2}{N-2}$. This is joint work with F. Pacard and J. Wei. We also construct sequences of sign changing solutions for some semilinear elliptic equation which is defined on S^n, with $n \geq 3$, and which is conformally invariant. The sequence of solutions we obtain have large energy and concentrate along some special submanifolds of S^n. In any dimension $n \geq 3$ we obtain sequences of solutions whose energy concentrates along $\ell \geq 1$ circles or, in dimension $n \geq 4$, which concentrate along a two dimensional torus. This is in collaboration with M. del Pino, F. Pacard and A. Pistoia.

Title : *The role of minimal and constant mean curvature surfaces in some overdetermined elliptic problem.*

Frank Pacard

I will report some recent work on the existence of domains on which one can find solutions of some semilinear elliptic equation with 0 Dirichlet data and constant Neumann data. I will describe families of solutions to this overdetermined problem and explain how they are related to minimal and constant mean curvature hypersurfaces. This is a joint work with M. del Pino and J. Wei.

Complete Embedded Self-Translating Surfaces under Mean Curvature Flow.

Prof. Xuan Hien Nguyen [2]

We describe a construction of complete embedded self-translating surfaces under mean curvature flow from desingularizing the intersection of a finite family of grim reapers in general position.

The mean curvature flow is the gradient flow of the surface area; it is the perturbation of embedded surfaces in $\mathbf{R}^3$ that moves each point on the surfaces along the normal direction with a speed proportional to the mean curvature. Self-translating surfaces (STS) are surfaces that are translated by the mean curvature flow at constant speed, so they are eternal solutions. The work is motivated by a result of Huisken and Sinestrari [3] in which they showed that STS can model the asymptotic behavior for slow forming singularities of the mean curvature flow. A detailed example of asymptotic convergence is given in Angenent and Velazquez [2].

Although the study of STS and singularities of the mean curvature flow are linked, few examples are available. Besides the two classic examples of a plane and a grim reaper $\widetilde{\Gamma} = \{(x, y, z) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbf{R}^2 \mid \mathbf{y} = -\log\cos(\mathbf{x})\}$, Altschuler and Wu [1] showed the existence of a paraboloid type self-translating surfaces that are graphs over convex domains in $\mathbf{R}^2$, with a prescribed angle of contact to the boundary cylinder. The author constructed STS by desingularizing the intersection of a grim reaper cylinder and a plane [5]. In this article, we generalize the previous result and show that we can desingularize the intersection of a finite family of grim reapers to obtain a STS.

Main result

Let us consider a finite family of grim reaper cylinders, $\{\widetilde{\Gamma}_n\}_{n=1}^{N_\Gamma}$, where each $\widetilde{\Gamma}_n$ is a translated copy of $\widetilde{\Gamma}$ of the previous paragraph. If no three $\widetilde{\Gamma}$'s intersect on the same line, and no two $\widetilde{\Gamma}$'s have the same asymptotic plane, then it is possible to find a complete embedded self-translating surface $\widetilde{\mathcal{M}}$ that is close to $\bigcup_n \widetilde{\Gamma}_n$ (in the L^∞ norm) with $\widetilde{\mathcal{M}}$ decaying exponentially to the ends of $\bigcup_n \widetilde{\Gamma}_n$ at infinity.

The result presented also applies to desingularizing intersection of vertical planes to obtain minimal surfaces, provided no three planes

[2]This work is partially supported by the National Science Foundation, Grant No. DMS-0710701.

intersect on the same line, and no two planes are parallel. Ours is a generalization of the construction of singly periodic minimal surfaces by Traizet [6].

The proof is inspired by techniques used by Kapouleas in his construction of minimal surfaces from a family of catenoids [4]. The idea is to first construct an initial surface by replacing the intersections with pieces of Scherk minimal surfaces. The exact solution will be obtained as a graph of a small function over the initial surface. The function is the solution to a nonlinear second order elliptic differential equation on the initial surface, with exponential decay at infinity. Unfortunately, the corresponding linear operator has small eigenvalues. Another difficulty comes from the fact that we need to ensure exponential decay of the solution. In the talk, we will present the main steps with pictures, underlining the difficulties encountered and how to circumvent them.

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Title:*Symmetry properties of nonnegative solutions of elliptic equations*

Prof.Peter Polacik

We consider the Dirichlet problem for a class of fully nonlinear elliptic equations on a bounded domain Ω. We assume that Ω is symmetric about a hyperplane H and convex in the direction perpendicular to H. By a well-known result of Gidas, Ni and Nirenberg and its generalizations, all positive solutions are reflectionally symmetric about H and decreasing away from the hyperplane in the direction orthogonal H. For nonnegative solutions, this result is not always true. We show that, nonetheless, the symmetry part of the result remains valid for nonnegative solutions: any nonnegative solution u is symmetric about H. Moreover, we prove that if $u \not\equiv 0$, then the nodal set of u divides the domain Ω into a finite number of reflectionally symmetric subdomains in which u has the usual Gidas-Ni-Nirenberg symmetry and monotonicity properties. We also show several examples of nonnegative solutions with a nonempty interior nodal set.

Title: *Isoperimetric-type inequalities and eigenvalues*
Prof. Jesse Ratzkin

Attached

Title: *3D travelling waves foor mean curvature motion and unbalanced Allen-Cahn*
Prof. Jean-Michel Roquejoffre

Title: *Hybrid solutions for a semilinear elliptic PDE*
Prof. Paul Rabinowitz

For certain problems, variational methods have been developed which find new solutions by gluing together mountain pass solutions (or minima). For a class of problems where both mountain pass solutions and minima coexist, we show how to glue them together, obtaining 'hybrid' solutions.

Title: Global minimizers of the nonlocal isoperimetric problem in two dimensions
Prof. P. Sternberg

Attached

Title: Regularity of stable phase interfaces in the van der Waals-Cahn-Hilliard theory
Prof. Y. Tonegawa

Isoperimetric-type inequalities and Eigenvalues

1 Introduction

The history of isoperimetric-type inequalities stretches back well over 100 years. In particular, St. Venant conjectured in the 1880's that a round beam is the strongest, among all beams with a given cross-sectional area. At approximately the same time, Rayleigh conjectured that a round drumhead makes the lowest note among all drumheads with a given area. Both these conjectures were proven in the early and middle of the last century. To make the statements more precise, we need to introduce some notation.

Let $D \subset \mathbb{R}^n$ be a bounded domain with a locally Lipschitz boundary, which satisfies a uniform cone condition. The fundamental frequency (*i.e.* the first Dirichlet eigenvalue of the Laplacian on D) $\lambda_1(D)$ is defined by

$$\lambda_1(D) = \inf \left\{ \frac{\int_D |\nabla u|^2}{\int_D u^2} : u \in W_0^{1,2}(D) \right\}. \tag{1}$$

Under the stated conditions on the domain, a minimizer ϕ exists and solves the boundary value problem

$$\Delta\phi + \lambda_1(D)\phi = 0, \qquad \phi|_{\partial D} = 0. \tag{2}$$

Similarly, the torsional rigidity $P(D)$ is defined by

$$\frac{4}{P(D)} = \inf \left\{ \frac{\int_D |\nabla u|^2}{\left(\int_D u\right)^2} : u \in L^1(D) \cap W_0^{1,2}(D) \right\}. \tag{3}$$

As is the case for the fundamental frequency, a minimizer ϕ exists and solves the boundary value problem

$$\Delta\phi + 2 = 0, \qquad \phi|_{\partial D} = 0, \tag{4}$$

and, in fact one can recover the torsional rigidity as

$$P(D) = 2\int_D u.$$

We can now state St. Venant's principle as $P(D) \le P(D^*)$, where D^* is a round ball with the same volume as D. Similarly, Rayleigh's conjecture, now known as the Faber-Krahn theorem, says $\lambda_1(D) \ge \lambda_1(D^*)$. In both cases one achieves equality only if $D = D^*$ almost everywhere. Moreover, one can prove both these inequalities by rearranging a test function for the relevant quotient.

2 Eigenvalues of domains in cones

We introduce some terminology to state a theorem regarding $\lambda_1(D)$, where D lies inside a cone. Let$\Omega \subset \mathbb{S}^{n-1}_+$ be a convex domain in the upper unit hemisphere, and let

$$\mathcal{W} = \{(r,\theta) : r > 0, \theta \in \Omega\}$$

be the cone overΩ . For $r_0 > 0$ let

$$\mathcal{S}(r_0) = \mathcal{W} \cap \mathbb{B}^n_{r_0}(0) = \{(r,\theta) : 0 < r < r_0, \theta \in \Omega\}$$

be the sector of radius r_0 in $\mathcal{W}$. Notice that, if ψ is the first Dirichlet eigenfunction of Ω , with eigenvalue μ, then

$$w(r,\theta) = r^{\alpha}\psi(\theta), \qquad \alpha = \frac{2-n}{2} + \sqrt{\left(\frac{2-n}{2}\right)^2 + \mu} \tag{5}$$

is a positive, harmonic function with zero boundary data on $\partial\mathcal{W}$.

The following theorem extends a two-dimensional result of Payne and Weinberger to all dimensions.

Theorem 1. *(-) Let $D \subset \mathcal{W}$ be a bounded, locally Lipschitz domain with a uniform cone condition, and choose r_0 so that*

$$\int_{\mathcal{S}(r_0)} w^2 = \int_D w^2.$$

Then $\lambda_1(D) \geq \lambda_1(\mathcal{S}(r_0))$, with equality if and only if $D = \mathcal{S}(r_0)$ almost everywhere.

The key tool in the proof of the theorem directly above is the following weighted isoperimetric inequality.

Proposition 2. *Let $D \subset \mathcal{W}$ be a bounded, locally Lipschitz domain with a uniform cone condition. Then*

$$\int_{\partial D} w^2 dA \geq \left[(2a+2)\int_D w^2 dV\right]^{\frac{2a+1}{2a+2}}, \qquad a = \alpha + \frac{n-2}{2} = \sqrt{\left(\frac{n-2}{2}\right)^2 + \mu},$$

with equality if and only if $D = \mathcal{S}(r_0)$ almost everywhere for some r_0.

3 Interpolation results

Tom Carroll and I have discovered a one-parameter family of variational problems which interpolate between the fundamental frequency and torsional rigidity defined above.

Definition 1. *Let if $n = 2$ let $p \geq 1$, and if $n \geq 3$ let $1 \leq p < \frac{2n}{n-2}$. For a smooth, bounded domain $D \subset \mathbb{R}^n$ define*

$$\mathcal{C}_p(D) = \inf\left\{\frac{\int_D |\nabla u|^2}{\left(\int_D u^p\right)^{2/p}} : u \in L^p(D) \cap W_0^{1,2}(D)\right\} = \inf \Phi_p(u).$$

Critical points of the functional Φ_p satisfy the well-known PDE

$$\Delta\phi + \Lambda\phi^{p-1} = 0 \tag{6}$$

for some Lagrange multiplier Λ . Standard results in PDE tell us that, in the stated range of the exponent p, a positive minimizer ϕ exists for the functional Φ_p, and it is fairly straight-forward to derive scaling laws.

Theorem 3. *(Carroll, -) If $1 \leq p < q$ then*

$$\mathrm{Vol}(D)^{2/p}\mathcal{C}_p(D) > \mathrm{Vol}(D)^{2/q}\mathcal{C}_q(D).$$

Notice that the inequality in this theorem is always strict.

Theorem 4. *(Carroll, -) Let D^* be the ball with the same volume as D. Then $\mathcal{C}_p(D) \geq \mathcal{C}_p(D^*)$, with equality if and only if $D = D^*$ almost everywhere.*

Some remarks are in order. First, we recover several inequalities relating $\lambda_1(D)$ and $P(D)$, which one can find in Polya and Szegő's book. Second, the PDE (6) is attached to a huge literature, and our primary goal is to point out another natural source of this equation. Third, we see this as an interpolation result, and so we hope that one can use the functionals Φ_p and the continuity method to derive estimates for $\lambda_1(D)$ from estimates for $P(D)$, or vice-versa.

Global minimizers of the nonlocal isoperimetric problem in two dimensions

P. Sternberg, Indiana University

Abstract

In this talk we analyze the minimization of the so-called nonlocal isoperimetric problem (NLIP) posed on the flat 2-torus. The nonlocal isoperimetric problem (NLIP) is given by

$$\text{minimize} \quad E_\gamma(u) := \frac{1}{2}\int_{\mathbb{T}^2} |\nabla u| + \gamma \int_{\mathbb{T}^2} |\nabla v|^2 \, dx, \tag{0.1}$$

over all $u \in BV(\mathbb{T}^2, \{\pm 1\})$ satisfying

$$\int_{\mathbb{T}^2} u \, dx = m$$

and v satisfying

$$-\Delta v = u - m \ \text{ in } \mathbb{T}^2 \quad \text{with} \quad \int_{\mathbb{T}^2} v \, dx = 0. \tag{0.2}$$

Here $\mathbb{T}^2$ is the flat 2-torus and the first term in E_γ computes the perimeter of the set $\{x : u(x) = 1\}$. For a specific range of m-values and for γ small, we show that the global minimizer is lamellar; that is, the set $\{x : u(x) = 1\}$ is simply a strip.

The problem (NLIP) arises, up to a constant factor, as theΓ -limit as $\varepsilon \to 0$ of the well-studied Ohta-Kawasaki sequence of functionals $E_{\varepsilon,\gamma}$ which model microphase separation of diblock copolymers, [3]:

$$E_{\varepsilon,\gamma}(u) := \begin{cases} \int_{\mathbb{T}^2} \frac{\varepsilon}{2}|\nabla u|^2 + \frac{(1-u^2)^2}{4\varepsilon} + \gamma\, |\nabla v|^2 \, dx & \text{if } u \in H^1(\mathbb{T}^2) \\ & \text{ and } \int_{\mathbb{T}^2} u \, dx = m, \\ +\infty & \text{otherwise,} \end{cases} \tag{0.3}$$

where again v satisfies (0.2). There is an extensive literature exploring the energy landscape for $E_{\varepsilon,\gamma}$ in two and three dimensions, whether posed on the flat torus (i.e. with periodic boundary conditions) or on a general domain with homogeneous Neumann data, cf. e.g. [1, 5, 6, 7, 8, 9]. The picture is quite rich and complicated, with the diffuse interface sometimes bounding one or more strips, wriggled strips, discs or ovals.

Much the same richness exists for the energy landscape of (NLIP). As such, independent of its connection to Ohta-Kawasaki, (NLIP) attracts interest as a rather canonical nonlocal perturbation of the classical isoperimetric problem. Indeed, as a model for pattern formation, (NLIP) sets up a basic competition between low surface area (the perimeter term) and high oscillation (the nonlocal term).

In three dimensions, computations reveal a wide array of stable critical points, with the free boundary $\partial\{x : u(x) = 1\}$ consisting of one or more pairs of parallel planes, one or more spheres, cylinders or even hypersurfaces resembling more exotic triply periodic constant mean curvature surfaces such as gyroids, depending on where in the (m,γ)

parameter space one looks. With few exceptions, however, rigorous proofs of stability for particular patterns are rare, and to our knowledge, there are no proofs of global or even local minimality of specific critical points. In this regard, we mention the interesting investigation of [4], in which the authors seek to show that a lamellar (striped) pattern minimizes energy for a slightly different model related to diblock copolymers. Commenting on the inherent difficulty in picking out such a pattern as the "winner" in an energy landscape full of locally minimizing competitors, the authors of [4] remark, "...when comparing a striped pattern with arbitrary multidimensional patterns we know of no rigorous results, for any system." We also note the recent work [2] on a characterization of minimizers in a related model including screened Coulomb interaction in the setting of small volume fraction. There the author shows that minimizers form a collection of nearly identical circular droplets.

Here we have chosen to focus on the two-dimensional setting of (NLIP) with γ small in order to present what is perhaps the first rigorous proof that a particular pattern is globally minimizing.

This research represents joint work with Ihsan Topaloglu.

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REGULARITY OF STABLE PHASE INTERFACES IN THE VAN DER WAALS–CAHN–HILLIARD THEORY

YOSHIHIRO TONEGAWA

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain and consider the family of van der Waals–Cahn–Hilliard energy functionals E_ε, $\varepsilon \in (0,1)$, ([2]; see also [8]) given by

$$E_\varepsilon(u) = \int_\Omega \frac{\varepsilon|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}\, dx, \tag{1}$$

where $u : \Omega \to \mathbb{R}$ belongs to the Sobolev space $H^1(\Omega) = \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)\}$ and $W : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ is a given C^3 double-well potential function with (precisely two) strict minima at ± 1 with $W(\pm 1) = 0$. In case $W(u) \notin L^1(\Omega)$, we set $E_\varepsilon(u) = +\infty$. When $\varepsilon \to 0^+$ with $E_\varepsilon(u_\varepsilon)$ remaining bounded independently of ε, it is clear (from the bound on the second term of the integral above) that u_ε must stay close to ± 1 on a bulk region in Ω and typically (i.e. in case the sets $\{u_\varepsilon \approx 1\}$ and $\{u_\varepsilon \approx -1\}$ each has measure $\geq$ a fixed proportion of the measure of Ω) there is a transition layer of thickness $O(\varepsilon)$, which we may call an "interface region" or a "diffused interface".

In the past few decades it has been established that in the presence of a uniform bound on the energy $E_\varepsilon(u_\varepsilon)$ and under natural variational hypotheses on u_ε of varying degrees of generality, for small $\varepsilon > 0$, the interface region corresponding to u_ε is close to a (weak) minimal hypersurface V of Ω (the "limit-interface" as $\varepsilon \to 0^+$) and that $E_\varepsilon(u_\varepsilon)$ approximates a fixed multiple of the $(n-1)$-dimensional area of this hypersurface. Modica ([7]) and Sternberg ([11]) established this, in the framework of Γ-convergence, for absolutely energy minimizing u_ε; they proved that the limit-interface V is locally area minimizing in that case, and hence by the well known regularity theory for locally area minimizing currents, it is smooth away from a possible closed singular set of codimension ≥ 7; Kohn–Sternberg ([5]) extended the result for locally energy minimizing u_ε. More recently, Hutchinson and I ([4]) showed that V is a stationary integral varifold if u_ε are merely critical points of E_ε. Subsequently, I ([12]) showed that whenever the u_ε are stable critical points of E_ε, the limit stationary integral varifold V is stable in the sense that V admits a generalized second fundamental form which satisfies the stability inequality.

Concerning smoothness of V when the critical points u_ε are not assumed to be energy minimizing, little is known beyond the following theorem ([12]): *Suppose that* $n = 2$, $\varepsilon_i \to 0^+$ *as* $i \to \infty$ *and that* $\{u_{\varepsilon_i}\}_{i=1}^\infty \subset H^1(\Omega)$ *is a sequence of* stable *critical points of* E_{ε_i} *with* $\sup_\Omega |u_{\varepsilon_i}| + E_{\varepsilon_i}(u_{\varepsilon_i}) \leq c$ *for all* $i \in \mathbb{N}$ *and some* $c > 0$. *Then for any* $0 < s < 1$, *a subsequence of the sequence of sets* $\{x \in \Omega : |u_{\varepsilon_i}(x)| \leq s\}$ *converges locally in Hausdorff distance to a union of non-intersecting lines.* Thus in case $n = 2$, any stable diffused interface must be close to non-intersecting lines for sufficiently small positive values of the parameter ε.

Can one make analogous conclusions in dimensions $n > 2$? Here I describe the recent results which give a satisfactory affirmative answer to this question in all dimensions. We show that under the same assumption of stability, for each fixed $s \in (0,1)$, a subsequence of the sequence of interface regions $\{x \in \Omega : |u_{\varepsilon_i}(x)| < s\}$ converges locally in Hausdorff distance to an *embedded smooth* stable minimal hypersurface if $2 \leq n \leq 7$; for $n \geq 8$, the limit stable minimal hypersurface may carry a singular set of Hausdorff dimension at most $n-8$. As mentioned before, this regularity result was known for the limit-interfaces corresponding to sequences $\{u_{\varepsilon_i}\}$ of energy minimizers since in that case the limit-interfaces are area-minimizing. The new result is that the stability hypothesis, which is much weaker than the energy minimizing assumption, suffices to guarantee the same regularity of the limit-interface.

The main reason why, in [12], the interface regularity was established only in case $n = 2$ and not for $n > 2$ was that while the structure of a stationary 1-dimensional varifold was known (due to the work of Allard and Almgren ([1])), there was no sufficiently general regularity theory available at the time for stable codimension 1 integral varifolds of arbitrary dimension. The essential new input to this problem is the recent regularity theory of Wickramasekera ([13]), which gives a necessary and sufficient geometric structural condition for a stable codimension 1 integral varifold to be regular. The limit-interfaces in question satisfy precisely this structural condition; their regularity then follows directly from the general theory of [13].

While the present work as well as the series of works mentioned above ([7, 11, 4, 12]) investigate the general character of limit-interfaces, there have been a number of articles which address the question of existence of critical points of (1) whose interface regions converge to a given minimal hypersurface. In this direction we mention the work by Pacard–Ritoré ([10]), Kowalczyk ([6]) and a number of recent joint works by del Pino, Kowalczyk, Pacard, Wei and Yang (see the recent survey paper by Pacard [9] for a complete list of references). The work of del Pino–Kowalczyk–Wei ([3]) in particular shows that singular limit-interfaces do occur in dimensions $n \geq 8$, demonstrating that the result in fact gives the best possible general dimension estimate on the singular set of a stable limit-interface.

This is a joint work with Neshan Wickramasekera of University of Cambridge.

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Department of Mathematics, Hokkaido University

Attached

Title:Some nonlinear problems with fractional laplacians
Prof. Yannick Sire

I will describe in this talk several results related to the following equation

$$(-\Delta)^s u = \lambda f(u)$$

where $s \in (0,1)$ and $(-\Delta)^s$ stands for the fractional laplacian. In a first part of the talk, I will describe symmetry properties of stable solutions of the problem, whatever the nonlinearity is. In the spitit of the De Giorgi con=jecture, I will provide several 1D symmetry results for monotone solutions i.e. : Let $u \in C^2_{\mathrm{loc}}(\mathbb{R}^{\ltimes})$ be a bounded solution with $n = 2$ and f locally Lipschitz. Suppose that

$$\partial_{y_2} u > 0. \tag{0.10}$$

Then, there exist $\omega \in \mathrm{S}^1$ and $u_o : \mathbb{R} \to \mathbb{R}$ such that

$$u(y) = v_o(\omega \cdot y)$$

for any $y \in \mathbb{R}^{\not\vDash}$.
I will give a proof based on Liouville theorem and a characterization of stable solutions.
As far as radial stable solutions are concerned, there is a value λ_* of the parameter λ such that

- for $0 < \lambda < \lambda^*$, there exists a minimal solution u_λ. In addition, u_λ is semi-stable and increasing with λ.
- for $\lambda = \lambda^*$, the function $u^* = \lim_{\lambda \nearrow \lambda^*} u_\lambda$ is a weak solution. We call λ^* the extremal value of the parameter and u^* the extremal solution.
- for $\lambda > \lambda^*$, there is no solution.

I will describe several results concerning the regularity of the extremal solution: assume $n \geq 2$ and let u^* be the extremal solution. We have that:
(a) If $n < 2(s + 2 + \sqrt{2(s+1)})$ then $u^* \in L^\infty(B_1)$.
(b) If $n \geq 2(s + 2 + \sqrt{2(s+1)})$, then for any $\mu < n/2 - 1 - \sqrt{n-1} - s$, there exists a constant $C > 0$ such that $u^*(x) \leq C|x|^{-\mu}$ for all $x \in B_1$.
In a second part of the talk, I will describe several results related to the special nonlinearity given by the power nonlinearity with the

critical exponent, namely

$$(-\Delta)^s u = u^{\frac{n+2s}{n-2s}}.$$

I will describe a concentration-compactness result for sequences in fractional Sobolev spaces and as a consequence provide a concentration result for a subcritical approximation of the previous equation.

Title: Thermal Insulation via Anisotropic Costings
Prof. Xuefeng Wang

Attached

Thermal Insulation via Anisotropic Coatings

Xuefeng Wang
Tulane University
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Of concern is the thermal insulation abilities/properties of anisotropic materials. This is motivated by significant engineering applications of nano-composite materials which, at the macroscale, commonly exhibit anisotropy. Our *physical goal* is to *efficiently* protect a conducting bodyΩ_1 from overheating by an anisotropically conducting coatingΩ_2, thin compared to the scale of the body (see Figure 1). An example of such is a space shuttle coated by a nano-insulator; another example is the inner coating of the combustion chamber of a turbine engine. One of the central questions is: *how thin can the coating be if we know how small the thermal tensor of the insulator is?*

To answer this question, we use eigen-analysis, finding that the following scaling law

$$\lim_{\delta \to 0^+} \frac{\sigma}{\delta^2} = 0 \tag{1}$$

ensures good insulation of the bodyΩ_1, where δ is the thickness of the coatingΩ_2, and σ is a parameter proportional to the thermal conductivity of the coating. On the other hand, we study directly the heat equation with both Dirichlet and Robin boundary conditions on the outer boundary of the coatingΩ_2: we identify scaling relations among the thermal tensor and thickness of the coating, and the thermal transport coefficient, such that the *effective boundary conditions* on the boundary of the bodyΩ_1 are Dirichlet, Robin and Neumann, with the last one ensuring good insulation of the body. In particular, from this point of view the scaling law

$$\lim_{\delta \to 0^+} \frac{\sigma}{\delta} = 0 \tag{2}$$

ensures effective Neumann boundary condition on $\partial\Omega_1$ and hence good insulation ofΩ_1.

We also find that all these results hold, regardless of the thermal conductivity of the coating in the directions tangent to the boundary of the body, as long as the coating is "optimally aligend"and the conductivity is bounded; thus the tangent directions can be left out, safely, of the thermal consideration, leaving room for mechanical considerations such as elasticity. By optimal alignment ofΩ_2 we mean that inΩ_2 the direction normal to $\partial\Omega_1$ is an eigenvector of the thermal tensor corresponding to its smallest eigenvalue. (In this case, σ in both Laws (1) and (2) is proportional to the thermal conductivity of the coating in the normal direction.) This allows us to take the full advantages of the anisotropy.

Very recently, we find that under the scaling law

$$\lim_{\delta \to 0^+} \frac{\sigma}{\delta^{1+\epsilon}} = 0, \quad \epsilon > 0 \tag{3}$$

the *maximal* time duration when the body is effectively well-insulated is large and at the order of $\delta^{-\epsilon}$, and in this case we establish the convergence rate of the temperature function to the solution of the limiting problem inside the body. We also find that in the case of optimally aligned coating, if the thermal conductivity of the coating in the tangent directions

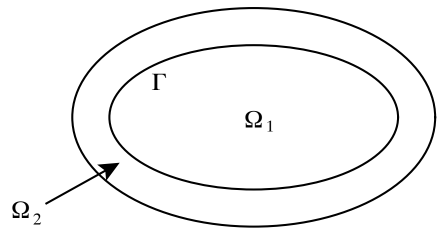

FIGURE 1. Ω_2 is uniformly thick and its thickness $= \delta$

is unbounded as $\delta \to 0$, then the effective boundary condition on the boundary of the body may be Wentzell's boundary condition.

The results described above come from the joint work with Jingyu Li, Steve Rosencrans, Guojing Zhang and Kaijun Zhang, and the on-going thesis of Cody Pond.

Infinitely many positive solutions for an elliptic problem with critical or super-critical growth

Prof. Shusen Yan

This is a joint work with J. Wei

We consider the following equation with super-critical growth:

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}}, \ u > 0 & \text{in } \mathcal{D}, \\ u = 0, & \text{on } \partial\mathcal{D}, \end{cases} \tag{0.11}$$

where $m \geq 1$ is a positive integer, $\mathcal{D}$ is a bounded domain in $\mathbb{R}^{N+m}$, and $N \geq 3$. Note that $\frac{N+2}{N-2}$ is the critical exponent in $\mathbb{R}^N$. So it is super-critical in $\mathbb{R}^{N+m}$.

Pohozaev identity shows that if $m \geq 0$, (0.11) has no solution if $\mathcal{D}$ is star-shaped. On the other hand, if $m = 0$, Bahri and Coron proved that if $\mathcal{D}$ has non-trivial homology, (0.11) has a solution. A problem raised by Rabinowitz is whether the non-triviality of the domain topology can guarantee the existence of at least one positive solution for the following problem

$$\begin{cases} -\Delta u = u^p, \ u > 0 & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{0.12}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $p > \frac{N+2}{N-2}$. This was answered negatively by Passaseo by means of an example for $N \geq 4$ and $p > \frac{N+1}{N-3}$. Our aim is to prove that (0.11) has infinitely many positive solutions if the domain $\mathcal{D}$ satisfies certain conditions. We assume that $\mathcal{D}$ is a torus-like domain satisfying the following condition:

(R): Write $y = (y^*, y^{**})$, $y^* \in \mathbb{R}^{N-1}$ and $y^{**} \in \mathbb{R}^{m+1}$. Then $y \in \mathcal{D}$ if and only if $(y^*, |y^{**}|, 0, \cdots, 0) \in \mathcal{D}$.

For any $\mathcal{D}$ satisfying (R), we look for a solution of the form $u(y) = u(y^*, |y^{**}|)$ for (0.11). Let

$$\Omega = \big\{(y^*, y_N) \in \mathbb{R}^N_+ : (y^*, y_N, 0, \cdots, 0) \in \mathcal{D}\big\},$$

where $\mathbb{R}^N_+ = \{y : \ y \in \mathbb{R}^N, \ y_N > 0\}$. Then (0.11) is transformed to the following problem:

$$\begin{cases} -div(y_N^m Du) = y_N^m u^{2^*-1}, \ u > 0, & y \in \Omega, \\ u = 0, & \text{on } \partial\Omega \cap \mathbb{R}^N_+. \end{cases} \tag{0.13}$$

To obtain a solution for (0.13), we impose the following condition on Ω:

(Ω_1): There is a $a_0 > 0$, such that for any $y \in \Omega$, $y_N \geq a_0$;
(Ω_2): For any $\theta \in (0, 2\pi)$, $(r\cos\theta, r\sin\theta, y_3, \cdots, y_N) \in \Omega$, if $(r, 0, y_3, \cdots, y_N) \in \Omega$.
(Ω_3): $y \in \Omega$ if and only if $(y_1, y_2, y_3, \cdots, -y_i, \cdots, y_{N-1}, y_N) \in \Omega$, $i = 3, \cdots, N-1$;
(Ω_4): there is $x^* \in \partial\Omega$ with $x^* = (r^*, 0, \cdots, 0, l^*)$ for some $r^* > 0$ and $l^* > 0$, such that

$$\partial\Omega \cap \{y_2 = \cdots = y_{N-1} = 0\} \cap B_\delta(x^*) = \{y_N = \psi(y_1),\ y_2 = \cdots = y_{N-1} = 0\} \cap B_\delta(x^*),$$

and

$$\Omega \cap \{y_2 = \cdots = y_{N-1} = 0\} \cap B_\delta(x^*) = \{y_N > \psi(y_1),\ y_2 = \cdots = y_{N-1} = 0\} \cap B_\delta(x^*),$$

for some C^2 function ψ and small $\delta > 0$. Moreover, r^* is either a strict local minimum point, or strict local maximum point of ψ.

Our main result in this paper can be stated as follows:

Theorem 3. *Suppose that $N \geq 5$. If Ω satisfies (Ω_1), (Ω_2), (Ω_2), and (Ω_4), then problem (0.13) has infinitely many distinct positive solutions.*

Bubbling solutions for (0.12) were obtained by Del Pino, Felmer and Musso if $p = \frac{N+2}{N-2} + \varepsilon$, where $\varepsilon > 0$ is small; and by Del Pino, Musso and Pacard if $p = \frac{N+1}{N-3} - \varepsilon$. Our result does not involve any perturbation. We proved the main theorem by constructing solutions with large number of bubbles.

Pointwise gradient estimates and rigidity results
Prof. Enrico Valdinoci

We consider bounded solutions of $\Delta u + F'(u) = 0$ in possibly unbounded proper domains whose boundary has nonnegative mean curvature.

We prove a pointwise gradient estimate of the form

$$\frac{1}{2}|\nabla u(x)|^2 \leq c_u - F(u(x)), \qquad \text{for any } x \in \Omega, \tag{0.14}$$

where

$$c_u := \sup_{r \in [0, ||u||_{L^\infty(\Omega)}]} F(r).$$

Formula (0.14), which we proved in [FV09], may be seen as an extension of the one obtained in [Mod85], where a similar result was proved in the case $\Omega = \mathbb{R}^n$. The assumptions that the solution is bounded and that the boundary has nonnegative mean curvature cannot, in general, be removed, as simple examples show.

Remarkably, any solution satisfying (0.14) enjoys some extra properties, even independently on the curvature of the domain: indeed if (0.14) is satisfied, then

$$c_u = \max\Big\{F(0),\ F\Big(||u||_{L^\infty(\Omega)}\Big)\Big\}$$
$$\text{and } c_u > F(t) \text{ for any } t \in \Big(0, ||u||_{L^\infty(\Omega)}\Big).$$

Moreover, if $F'(0) \geq 0$, then

$$c_u = F\Big(||u||_{L^\infty(\Omega)}\Big)$$
$$\text{and } c_u > F(t) \text{ for any } t \in \Big[0, ||u||_{L^\infty(\Omega)}\Big).$$

In general, the strict sign holds in (0.14), and in fact, roughly speaking,1D solutions are the exceptional ones that attains equality in (0.14). Indeed, if equality in (0.14) holds at some non-critical point, then it holds everywhere and the solution depends only on one Euclidean variable. Moreover, either Ω is a halfspace or Ω is a strip, and the solutions may be classified as well.

Extensions of these results to quasilinear PDEs are possible (see [CGS94] for the case $\Omega = \mathbb{R}^n$ and [CFV10] for proper domains)

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