QUASISYMMETRIC FUNCTIONS

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Thirty-two talks were given at the Banff International Research Station as part of a five-day workshop on the theme of "Quasisymmetric Functions". This workshop brought together researchers using these functions in different ways to make further connections between the work, foster new developments, and enhance the common understanding of the applications.

Quasisymmetric functions are power series of bounded degree in variables $x_1, x_2, \ldots$ (say) which are shift invariant in the sense that $x_1^{a_1} \cdots x_k^{a_k}$ and $x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}$ have the same coefficient for any strictly increasing sequence of positive integers $i_1 < \cdots < i_k$.

For over a century, symmetric functions have played a major role in mathematics with applications in algebraic topology, combinatorics, representation theory, and geometry. Quasisymmetric functions are extensions of symmetric functions that are becoming of comparable importance. They were first used by Stanley around 1970 in the theory of P-partitions, though he did not consider them per se. In the 1980's Ira Gessel defined quasisymmetric functions, developed many of their basic properties, and applied them to permutation enumeration.

Quasisymmetric functions have developed into a powerful tool in many areas of mathematics today. Their first algebraic interpretations were as Frobenius characteristics of the representations of the 0-Hecke algebra and as the dual to Solomon's descent algebra of the symmetric group. Their applications have since exploded in many directions. We have outlined some of these topics below.

One issue that became clear over the course of the meeting was that there needed to be some standardization of notation for various common concepts in this field. As a result, on the last day of the workshop, participants created the wikipedia page
`http://en.wikipedia.org/wiki/Quasisymmetric_function`.
This page has been further updated many times since then.

Date: December 21, 2011.

1. Research Directions

1.1. Drew Armstrong spoke on "The bounce statistic on the root lattice". The Frobenius series of diagonal harmonics can be expressed elegantly in terms of the Bergeron-Garsia nabla operator: ∇e_n. He described Haglund's "bounce" statistic as a statistic on the root lattice of type A. This leads to a (conjectural) combinatorial interpretation of the Hilbert series of $\nabla^m e_n$ for all integers m (including negative integers).

1.2. Andrew Crites spoke on "Pattern avoidance and affine permutations", based on joint work with Sara Billey. Affine permutations are an infinite generalization of classical permutations. In this talk, he discussed how pattern avoidance can be adapted to affine permutations. In particular, we can enumerate how many affine permutations avoid a given pattern. He also showed how one can apply pattern avoidance to the study of the corresponding affine Schubert varieties, and conclude with a characterization of the rationally smooth Schubert varieties in terms of pattern avoidance.

1.3. Jia Huang spoke on "0-Hecke algebra actions on coinvariants and flags". A classical result of P.A. MacMahon says that two important permutation statistics, the inversion number and the major index, are equidistributed over permutations that have a fixed descent set for their inverse. This talk explained algebraic manifestations of this result, related to the action of the 0-Hecke algebra for the symmetric group on coinvariant algebras and complete flags of subspaces over finite fields.

1.4. Marcelo Aguiar spoke on "The Kazhdan-Lusztig polynomial of a quasisymmetric function", based on joint work with Louis Billera. We introduce 'universal' Kazhdan-Lusztig polynomials. The universal property of the Hopf algebra of quasisymmetric functions, combined with an extension of Stanley's theory of P-kernels, leads to the construction of generalized Kazhdan-Lusztig polynomials in the setting of Hopf algebras. As an application, we obtain formulas for Kazhdan-Lusztig polynomials of Bruhat intervals (of Billera and Brenti) and for g-polynomials of Eulerian posets (of Bayer and Ehrenborg) in a unified manner.

1.5. Kyle Petersen spoke on "An affine descent module", based on joint work with Marcelo Aguiar. Given a finite Coxeter group W, Solomon's descent algebra is the subalgebra of the group algebra with basis given by sums of elements of W with a common descent set. One way to describe Solomon's descent algebra is via the (invariant algebra of the) semigroup of faces of the Coxeter arrangement. This connection is due to Bidigare, and is related to work on random walks on hyperplane arrangements and card shuffling by various subsets of Bidigare, Billera, Brown, Diaconis, Hanlon, Rockmore. In this work we consider an affine Coxeter group and use a semigroup structure on faces of the "Tits cone with boundary" to find a module over Solomon's descent algebra. This module is a refinement of Cellini's commutative descent algebra; it is given by the span of sums of elements of W with a common "affine" descent set.

1.6. Matthew Hyatt spoke on "Eulerian quasisymmetric functions for the type B Coxeter group and other wreath product groups". Eulerian quasisymmetric functions were introduced by Shareshian and Wachs in order to obtain a q-analog of Euler's exponential generating function formula for the Eulerian numbers. They are defined via the symmetric group, and applying the stable and nonstable principal specializations yields formulas for joint distributions of permutation statistics. We consider the wreath product of the cyclic group with the symmetric group, also known as the group of colored permutations. We use this group to introduce colored Eulerian quasisymmetric functions, which are a generalization of Eulerian quasisymmetric functions. We derive a formula for the generating function of these colored Eulerian quasisymmetric functions, which reduces to a formula of Shareshian and Wachs for the Eulerian quasisymmetric functions. We show that applying the stable and nonstable principal specializations yields formulas for joint distributions of colored permutation statistics, which generalize the Shareshian-Wachs q-analog of Euler's formula, formulas of Foata and Han, and a formula of Chow and Gessel.

1.7. Kurt Luoto spoke on "Noncommutative Schur functions", based on joint work with Christine Bessenrodt and Stephanie van Willigenburg. We introduce what we call noncommutative Schur functions, which are dual to the quasisymmetric Schur functions. We describe a Littlewood-Richardson rule for them, and discuss some additional properties and applications of noncommutative and skew quasisymmetric Schurs, such as a morphism of algebras from the Poirier-Reutenauer tableau algebra to NSym. We also discuss the "colored" generalization of these algebras and functions, and one role that is played by the noncommutative Schurs in representation theory.

1.8. Steph van Willigenburg spoke on "Qusisymmetric Schur functions", based on joint work with Christine Bessenrodt, Jim Haglund, Kurt Luoto, and Sarah Mason. . In this talk we introduce quasisymmetric Schur (QS) functions, which partition Schur functions in a natural way. Furthermore, we show how these QS functions also refine many well known combinatorial Schur function properties.

1.9. Peter McNamara spoke on “Equality questions for P-partition quasisymmetric functions: preliminary report ”, based on joint work with Ryan Ward. Considerable recent attention has been given to the problem of determining necessary and sufficient conditions for two skew shapes to yield the same skew Schur funciton. In a recently initiated joint project with Ryan Ward, we consider the more general question of equality of P-partition quasisymmetric functions. Finding necessary and sufficient conditions for equality in this case is likely out of reach; he reported on conditions that are necessary or sufficient.

1.10. Christophe Reutenauer spoke on “a Hopf algebra on double posets with a Hopf self-pairing and a Littlewood-Richardson rule”, based on joint work with Claudia Malvenuto. Let **D** be the set of isomorphism types of finite double partially ordered sets, that is sets endowed with two partial orders. On Fix this: (BZ mathbf D) we define a product and a coproduct, together with an internal product, that is, degree-preserving. With these operations Fix this: (BZ mathbf D) is a Hopf algebra. We define a symmetric bilinear form on this Hopf algebra: it counts the number of pictures (in the sense of Zelevinsky) between two double posets. This form is a Hopf pairing, which means that product and coproduct are adjoint each to another. The product and coproduct correspond respectively to disjoint union of posets and to a natural decomposition of a poset into order ideals. Restricting to special double posets (meaning that the second order is total), we obtain a notion equivalent to Stanley’s labelled posets, and a Hopf subalgebra already considered by Blessenohl and Schocker. The mapping which maps each double poset onto the sum of the linear extensions of its first order, identified via its second (total) order with permutations, is a Hopf algebra homomorphism, which is isometric and preserves the internal product, onto the Hopf algebra of permutations, previously considered by the two authors. Finally, the scalar product between any special double poset and double posets naturally associated to integer partitions is described by an extension of the Littlewood-Richardson rule.

1.11. Victor Reiner spoke on “P-partitions revisited”, based on joint work with Valentin Feray. We explain a new product formula which q-counts by major index the linear extensions of certain posets. This generalizes the Bjorner-Wachs major index q-hook formula for forests, and comes from re-examining the well-known semigroup ring of P-partitions.

1.12. Sami Assaf spoke on "From quasi-symmetric to Schur positive via dual equivalence", based on joint work with none. In this talk, we'll show how to prove the Schur positivity of a quasi-symmetric function by defining a family of involutions satisfying certain simple axioms. We'll also present a formula for the Schur expansion in terms of the involutions. We'll conclude with applications of this method to well-known classes of functions.

1.13. Nantel Bergeron spoke on "What about quasi-symmetric functions in non-commutative variables?", based on joint work with Marcelo Aguiar, Carlos André, Carolina Benedetti, Nantel Bergeron, Zhi Chen, Persi Diaconis, Anders Hendrickson, Samuel Hsiao, I. Martin Isaacs, Andrea Jedwab, Kenneth Johnson, Gizem Karaali, Aaron Lauve, Tung Le, Stephen Lewis, Huilan Li, Kay Magaard, Eric Marberg, Jean-Christophe Novelli, Amy Pang, Franco Saliola, Lenny Tevlin, Jean-Yves Thibon, Nathaniel Thiem, Vidya Venkateswaran, C. Ryan Vinroot, Ning Yan, Mike Zabrocki. Recently at a AIM a workshop on supercharacter and symmetric functions in non-commutative variables we found and analogue of Frobenius map between super character theory of upper triangular matrices (over finite fields F_2) and symmetric functions in non-commutative variables. This is an Hopf algebra morphism. This solved (in a new twisted way) a long standing problem of realizing the Hopf algebra of symmetric functions in non-commutative variables as the "grothendick Hopf algebra" of a tower of algebra. This has just been submitted as a paper with 27 authors (all the participants!) We were successful for this one but my next step is to do the same for quasi-symmetric functions in non-commutative variables. Accordingly, he presented briefly the result - Presented a quasi-symmetric function in non commutative variables - and posed the problem.

1.14. Jason Bandlow spoke on "Combinatorics of symmetric functions in non-commutative algebras", based on joint work with Jennifer Morse. In the paper 'Noncommutative Schur functions and their applications', Fomin and Greene describe a collection of families of symmetric polynomials. For every algebra A whose generators and relations satisfy certain axioms, there exists a family of symmetric polynomials, indexed by the elements of A. These polynomials are defined by their quasisymmetric expansion, and are shown to be Schur positive. Examples include Schur polynomials, stable Grothendieck polynomials, Stanley symmetric polynomials, and, in a modification of this approach due to Lam, the affine Stanley polynomials (closely related to dual k-Schur polynomials). He discussed ongoing work with Jennifer Morse, focused on unifying the combinatorics of these families.

1.15. Adriano Garsia spoke on "Recent progress on the Shuffle Conjecture", based on joint work with Angela Hicks, Mike Zabrocki and Guoce Xin. In recent work Jim Haglund, Jennifer Morse and Mike Zabrocki introduce a new statistic on Parking Functions, the "*Diagonal Composition*" of its Dyck path. That is the composition that gives the lengths of the intervals between successive diagonal hits of the Dyck path. They discovered that Nabla applied to certain modified Hall-Littlewood functions indexed by compositions yield the weighted sum of the corresponding Parking Functions by area, dinv and Gessel Quasi-Symmetric function. This led them to conjecture several super-refinements of the Shuffle conjecture. In this talk we present the work of Haglund-Morse-Zabrocki and successive joint work with Angela Hicks, Mike Zabrocki and Guoce Xin, where some special cases of the Haglund-Morse-Zabrocki conjecture have been established.

1.16. Francesco Brenti spoke on "Parabolic Kazhdan-Lusztig R-polynomials for tight quotients of the symmetric groups". We study the parabolic Kazhdan-Lusztig R-polynomials for the tight quotients of the symmetric groups. More precisely, we obtain explicit combinatorial product formulas for these polynomials. As an application of our results, we derive combinatorial closed product formulas for certain sums and alternating sums of classical Kazhdan-Lusztig R-polynomials.

1.17. Margaret Readdy spoke on "Balanced and Bruhat graphs", based on joint work with Richard Ehrenborg. The cd-index is a noncommutative polynomial which compactly encodes the flag vector data of the face lattice of a polytope, and more generally, of an Eulerian poset. There is a simple yet powerful coalgebraic structure on the cd-index which enables one to understand how the cd-index of a polytope changes under geometric operations and proves non-trivial inequalities among the face incidence data. We consider a general class of labeled graphs which satisfy a balanced condition and develop the cd-index. As a special case, this work applies to Bruhat graphs arising from the strong Bruhat order on a Coxeter group and gives straightforward proofs of recent results of Billera and Brenti. She also indicated various ongoing projects with Billera, Ehrenborg and Hetyei.

1.18. Michelle Wachs spoke on "Eulerian quasisymmetric functions (Part I)", based on joint work with John Shareshian. We discuss a class of symmetric functions that we introduced a few years ago in order to obtain, through principal specialization, a q-analog of Euler's formula for the exponential generating function of the Eulerian polynomials. This q-analog allowed us to determine the joint distribution of major index and excedance number on the symmetric group S_n. These symmetric functions are called Eulerian quasisymmetric functions because they are defined as sums of fundamental quasisymmetric functions associated with permutations having a given number of excedances. We call the polynomials in q obtained by specializing the Eulerian quasisymmetric functions "q-Eulerian polynomials". The Eulerian quasisymmetric functions have proved to be a useful tool for establishing properties of the q-Eulerian polynomials and their cycle type refinements, such as the cyclic sieving phenomenon of Stanton, Reiner and White (joint work with Sagan) and unimodality (joint work with Henderson). The Eulerian quasisymmetric functions are also interesting in their own right. They have appeared in various guises in the literature. They are enumerators for multiset derangements (Askey and Ismail) and other classes of words (Gessel, Stanley). They are Frobenius characteristics of representations of the symmetric group on homology of a certain poset (Shareshian and Wachs) and on cohomology of a certain toric variety (Procesi, Stanley).

1.19. John Shareshian spoke on "Eulerian quasisymmetric functions (Part II)", based on joint work with Michelle Wachs. See above.

1.20. Jennifer Morse spoke on "Background and Open Problem Session on k-Schur functions and connections to quasisymmetric functions". This was an informal brainstorming session to formulate interesting problems relating the k-Schur and quasisymmetric function theories. This briefly included an overview with some definitions and properties of k-Schur functions to jump-start the discussion.

1.21. Angela Hicks spoke on "A Parking Function Bijection Suggested by the Haglund-Morse-Zabrocki Conjecture". A recent sharpening of the "shuffle conjecture" by Jim Haglund, Jennifer Morse, and Mike Zabrocki suggest several combinatorial conjectures about the parking functions. In particular, we discuss a bijective map on the parking functions implied by the commutativity properties of the modified Hall-Littlewood polynomials that appear in their conjecture.

1.22. Ira Gessel spoke on " Descents, peaks, and shuffles of permutations, and noncommutative symmetric functions", based on joint work with none. If we look at all shuffles of two permutations p and q on disjoint letters then the distribution of the number of descents in the shuffles depends only on the number of descents of p and q, as shown by Richard Stanley, using P-partitions. It follows from the theory of P-partitions that some other permutation statistics also have this property: the descent set, the major index, and the joint distribution (des, maj). From John Stembridge's theory of enriched P-partitions it follows that the peak set and the number of peaks also have this property. Any such permutation statistic with this property that depends only on the descent set corresponds to a quotient of the algebra of quasi-symmetric functions, or equivalently, to a subcoalgebra of the dual coalgebra of noncommutative symmetric functions. He showed that the joint distribution of descents and peaks has this property using noncommutative symmetric functions, and describe the corresponding quasi-symmetric quotient algebra.

1.23. Francesco Brenti spoke on "Parabolic Kazhdan-Lusztig R-polynomials for tight quotients of the symmetric groups", based on joint work with none. We study the parabolic Kazhdan-Lusztig R-polynomials for the tight quotients of the symmetric groups. More precisely, we obtain explicit combinatorial product formulas for these polynomials. As an application of our results, we derive combinatorial closed product formulas for certain sums and alternating sums of classical Kazhdan-Lusztig R-polynomials.

1.24. Sam Hsiao spoke on "Quasisymmetric functions, Type B shuffling, and shifted tableaux". Following ideas of Stanley [Generalized riffle shuffles and Quasisymmetric functions, Ann. Comb. 5 (2001)], we consider applications of signed quasisymmetric functions to generalized type B ("face-up face-down") riffle shuffling. The main result is a probabilistic interpretation of Schur Q functions in terms of the Sagan-Worley insertion algorithm for shifted tableaux. We also discuss bounds on the distance to stationary of descent sets after repeated type B shuffling.

1.25. Greg Warrington spoke on "Quasisymmetric expansions of symmetric functions", based on joint work with Eric Egge and Nick Loehr. This talk has two parts, both relating to the fundamental quasisymmetric functions, Q_α. In the first part we give a combinatorial Q-expansion for Schur function plethysms of the form $s_\lambda[s_\mu]$. In the second part we modify Egecioglu and Remmel's combinatorial realization of the inverse Kostka matrix for the purpose of converting a Q-expansion to a Schur expansion.

1.26. Saul Blanco spoke on "Shortest path poset of Bruhat intervals", based on joint work with Louis J. Billera. The Bruhat interval [u,v] is endowed with rich topological and combinatorial structure; for instance, it is Gorenstein*. On the other hand, not much is known of the remaining u-v paths in the Bruhat graph B(u,v) of [u,v]. Consider the poset SP(u,v) of shortest u-v paths in B(u,v). SP(u,v) and [u,v] have similarities; for example, if there is only one rising chain (using the reflection order) in SP (u, v), then SP (u, v)) is also a Gorenstein*. Further properties of SP (u, v) were discussed.

1.27. Jim Haglund spoke on "A Polynomial Identity for the Hilbert Series of Diagonal Harmonics", based on joint work with none. A special case of Haiman's identity for the character of the space of diagonal harmonics under the action fo the symmetric group yields a formula for the bigraded Hilbert series as a sum of rational functions in q,t. In this talk, he indicated how a summation identity of Garsia and Zabrocki for Macdonald Pieri coefficients can be used to transform Haiman's formula for the Hilbert series into an explicit polynomial in q,t with integer coefficients. An equivalent formulation expresses the Hilbert series as the constant term in a certain multivariate Laurent series.

1.28. Sarah Mason spoke on "New bases for quasisymmetric functions", based on joint work with Jeffrey Remmel. Haglund, Luoto, Mason, and van Willigenburg introduced a basis for quasisymmetric functions called the quasisymmetric Schur functions which are generated combinatorially through fillings of composition diagrams in much the same way as Schur functions are generated through reverse column-strict tableaux. We introduce a new basis for quasisymmetric functions called the row-strict quasisymmetric Schur functions which are generated combinatorially through fillings of composition diagrams in much the same way as Schur functions are generated through row-strict tableaux. We describe the relationship between this new basis and other known bases for quasisymmetric functions, as well as its relationship to Schur polynomials. We obtain a refinement of the omega transform operator as a result of these relationships. We also describe a method to combine the column-strict and row-strict quasisymmetric Schur functions to obtain a quasisymmetric refinement of the (k,l)-hook Schur functions.

1.29. Sami Assaf spoke on "A user's guide to dual equivalence". Dual equivalence graphs can and have been used to give combinatorial proofs of the Schur positivity of functions expressed in terms of quasi-symmetric functions. In this talk, we

1.30. Sami Assaf spoke on "A user's guide to dual equivalence". Dual equivalence graphs can and have been used to prove the Schur positivity of functions expressed in terms of quasi-symmetric functions. In this talk, she presented several streamlined versions of this theory that make it easier to apply to new classes of functions. She also gave an explicit formula for the resulting Schur expansion along with as many applications as time allows.

1.31. Mark Skandera spoke on "Quantum immanants and the dual canonical basis", based on joint work with Justin Lambright. Abstract: Standard submodules of the type A Hecke algebra are defined in terms of cosets of the symmetric group and may be classified as double-parabolic, single-parabolic, or nonparabolic (in decreasing generality). Each has a Kazhdan-Lusztig basis consisting of some subset of the nonparabolic Kazhdan-Lusztig basis. Dual to these modules and bases are certain submodules of the quantum polynomial ring, also defined in terms of cosets, and their dual canonical bases. In the nonquantum setting, the general double-parabolic dual canonical basis elements may be expressed rather simply in terms of nonparabolic basis elements. While this fact does not quantize so easily, we show that one can use some tricks to obtain similar expressions.

1.32. Michelle Wachs spoke on "Eulerian Quasisymmetric Functions (Part I)", based on joint work with John Shareshian. We discuss a class of symmetric functions that we introduced a few years ago in order to obtain, through principal specialization, a q-analog of Euler's formula for the exponential generating function of the Eulerian polynomials. This q-analog allowed us to determine the joint distribution of major index and excedance number on the symmetric group S_n. These symmetric functions are called Eulerian quasisymmetric functions because they are defined as sums of fundamental quasisymmetric functions associated with permutations having a given number of excedances. We call the polynomials in q obtained by specializing the Eulerian quasisymmetric functions "q-Eulerian numbers". The Eulerian quasisymmetric functions have proved to be a useful tool for establishing properties of the q-Eulerian numbers and their cycle type refinements, such as the cyclic sieving phenomenon of Stanton, Reiner and White (joint work with Sagan) and unimodality (joint work with Henderson). The

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