# Subtle Transversality: a celebration of Allen Hatcher's sixty-fifth birthday (09w2138)

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## 1 Introduction

Allen Hatcher, probably most famous for his proof of the Smale Conjecture [6], has made significant contributions in a variety of areas centered on low-dimensional and geometric topology. Author of a widely used and highly respected text on algebraic topology, as well as numerous other books in progress, he is also admired as an expositor for his careful attention to essential ideas, intuition, and motivation.

In keeping with its stated objectives, the workshop presented current research in diverse areas in which Allen has worked, including (but not limited to) the study of spaces or complexes of low-dimensional topological objects. In addition to celebrating Allen's work and rich history of collaboration, the aim of the workshop was to enhance interaction among researchers with similar interests and share the most recent advances, approaches, and methods.

In the following section, we give an overview of the major topics and themes addressed by the workshop. Specific results and conjectures are left to the subsequent section. Throughout the discussion, $M^n$ denotes a manifold of dimension $n$, $S^n$ denotes the sphere of dimension $n$, $S_g$ denotes the closed, orientable surface of genus $g$, $H_g$ denotes a handlebody of genus $g$, and $F_g$ denotes the free group on $g$ generators.

## 2 Overview of Major Topics and Themes

### 2.1 Operad actions on the space of long knots.

A *long knot* is an embedding of $\mathbb{R}^1$ in $\mathbb{R}^3$ that agrees with a fixed linear embedding outside of a ball. Concatenation of long knots gives a monoid structure on the space of long knots. This structure extends to an action of the *little* 2*-cubes operad* of configurations of edge-aligned 2-cubes. These notions generalize to an action of the little $j+1$-cubes operad on the space of long $j$-planes in $\mathbb{R}^n$. [1]

For the classical case ($\mathbb{R}^1 \hookrightarrow \mathbb{R}^3$), the 2-cubes operad action is closely related to the Jaco-Shalen-Johannson decomposition of the knot complement by incompressible tori (see the following subsection for a brief definition of incompressibility). Examination of the JSJ decomposition leads to the definition of another operad, called the *splicing* operad, acting on the space of long knots. [2]

## 2.2 3-manifold complexity and normal surfaces.

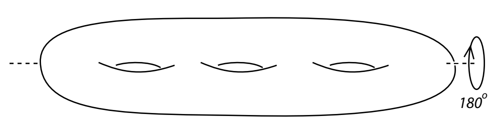
A natural notion of complexity for a 3-manifold is the minimal number of tetrahedra needed to triangulate it. A partial census of 3-manifolds based on this notion of complexity reveals interesting patterns, particularly with respect to geometric structure. For example, all manifolds of complexity up to and including five are spherical. The least complex hyperbolic manifold has complexity nine. [11]

A major tool in the study of 3-manifolds is consideration of two-dimensional objects embedded or immersed in them, such as surfaces and foliations. To give meaningful information, these objects must be "essential" in some way; for example, an embedded (two-sided) surface should be *incompressible* in the sense that any simple closed curve on the surface that bounds a disk in the manifold bounds a disk on the surface itself. For a triangulated manifold, it is natural to consider those surfaces, called *normal* surfaces, that are transverse to the edges and intersect each tetrahedron in one or more (triangular or quadrilateral) disks. Analysis of normal surfaces, via consistency conditions on the faces of the tetrahedra, provides the standard general combinatorial and algorithmic approach to identifying essential surfaces in the manifold, although the necessary calculations are often difficult to implement. (Other approaches have been used to excellent effect in specific classes of manifolds by Hatcher and others.)

## 2.3 The Torelli group.

The *mapping class group* of a manifold $M$, MCG($M$), is the group of isotopy classes of orientation-preserving self-homeomorphisms of $M$. The mapping class group of a closed, orientable surface has a symplectic representation based on the intersection form on first degree integral homology (giving the algebraic intersection number of pairs of oriented closed curves). Since orientation-preserving self-homeomorphisms of $S_g$ clearly preserve this form, and since any automorphism of $H_1(S_g)$ preserving the intersection pairing can be realized by a self-homeorphism of $S_g$, the mapping class group MCG($S_g$) surjects onto the symplectic group Sp($2g, \mathbb{Z}$). The kernel of this surjection, $\mathcal{I}_g$, is called the *Torelli* group. In other words, the Torelli group is the subgroup of the mapping class group that acts trivially on homology. In some sense, the Torelli group is the most mysterious part of MCG($S_g$), since the symplectic linear representation of MCG($S_g$) gives no information about it.

A subgroup called the *symmetric* Torelli group provides one approach to studying the full Torelli group. Consider an involution of the surface that acts trivially on homology; such an involution is called *hyperelliptic.* (See figure. Note that a hyperelliptic involution gives a two-fold branched covering of $S^2$ by $S_g$ with $2g + 2$ branch points.) The *symmetric Torelli group* $\mathcal{SI}_g$, with respect to a fixed hyperelliptic involution $i$, is the subgroup of $\mathcal{I}_g$ that commutes with $i$; any two such subgroups are conjugate.




## 2.4 The stable homology of mapping class groups.

For $g \geq 2$, the mapping class group is intimately related to the space of hyperbolic metrics (equivalently, complex structures) on $S_g$. The topological group Diff$^+(S_g)$ of orientation-preserving self-diffeomorphisms of $S_g$ acts on the space $\mathcal{H}(S_g)$ of all hyperbolic metrics on $S_g$; the quotient is the classical moduli space of Riemann surfaces homeomorphic to $S_g$. If one first takes the quotient of $\mathcal{H}(S_g)$ by the action of Diff$_0^+(S)$, the path component of the identity, one obtains the *Teichmüller space* of $S_g$, $\mathcal{T}(S_g)$, which may be thought of as the space of "marked" Riemann surfaces homeomorphic to $S_g$. The quotient of $\mathcal{T}(S_g)$ by the action of the MCG($S_g$) then gives the moduli space.

Both $\mathcal{H}(S_g)$ and $\mathcal{T}(S_g)$ are contractible ($\mathcal{T}(S_g) = \mathbb{R}^{6g-6}$); therefore, $MCG(S_g) =$ Diff$^+(S_g)$/Diff$_0(S_g)$ is homotopy equivalent to Diff$^+(S_g)$. Recently, Madsen and Weiss computed the rational homology of the stable moduli space of Riemann surfaces, proving a well-known conjecture of Mumford, by computing the stable homology of MCG($S_g$) [10]. Analogous recent results of Galatius, Hatcher, and Wahl reveal the stable homology of mapping class groups of related spaces. [4] [7] (Hatcher's most recent paper is in progress.)

### 2.5 $\Gamma_1$-structures and foliations.

As defined by A. Haefliger, a $\Gamma_1$-structure $\xi$ on a manifold consists of a line bundle $\nu$ over $M$, called the *normal bundle* to $\xi$, and the germ of a co-dimension-one foliation $F$ along the zero section, transverse to the fibers. Essentially, $\Gamma_1$ structures are models for singular foliations on $M$ of co-dimension one.

A $\Gamma_1$-structure is *regular* if $F$ is also transverse to the zero section, in which case the pull-back of $F$ to $M$ is a foliation without singularities. Given a $\Gamma_1$-structure, one would like to know if it is homotopic (in the natural sense) to a regular one and, if so, to understand the structure of the induced foliation.

For a 3-manifold $M^3$, one such structure is an *open book*. An *open book decomposition* of $M^3$ is a fibration over the circle of the complement of a link $B$ in $M^3$ - the *binding* - for which each fiber - a *page* - is the interior of an embedded surface with boundary $B$. By spiraling the pages around a tubular neighborhood of $B$ and filling this neighborhood with Reeb foliations, one obtains an *open book foliation*.

## 3 Presentation Highlights

**Ryan Budney** discussed his recent work on embedding spaces of long $j$-dimensional planes in $\mathbb{R}^n$, showing in particular that the space of long knots is free over the space of prime knots with respect to the action of the 2-cubes operad, and also that it is free over the space of torus and hyperbolic knots with respect to the splicing operad. This structure allows for the computation of the homotopy type of a component in terms of simpler components; in particular, the homotopy types of the path components of hyperbolic and torus knots were previously computed by Hatcher. [1] [2]

**Hyam Rubinstein** presented his recent joint work with W. Jaco and S. Tillman, in which they utilize a norm on $H^1(M, \mathbb{Z}_2)$, analogous to Thurston's norm, to obtain a lower bound on the combinatorial complexity of a 3-manifold $M^3$. Assuming a type of (pseudo-)triangulation with one vertex called a 0-efficient triangulation, they analyze the normal surfaces dual to the elements of a rank-two subgroup of $H^1(M^3, \mathbb{Z}_2)$. Although it is generally not difficult to obtain an upper bound on complexity, such as by considering the triangulation associated to a Heegard decomposition of $M$, lower bounds had previously been elusive. [8]

The presenter also related joint work with B. Burton and Tillman, in which they employ related techniques to decide more efficiently if a normal surface identified by the Jaco-Oertel algorithm is incompressible. As a consequence, they answer a long-standing question of Thurston, showing that the Seifert-Weber hyperbolic dodecahedral space is not Haken (contains no incompressible surface). They analyze a 0-efficient triangulation that divides the dodecahedron along diagonals into 23 tetrahedra; it is conjecturally minimal. [3]

**Tara Brendle**, relating current joint work with Dan Margalit, described new relations in the symmetric Torelli group (which are by extension also new relations in the full Torelli group) and also discussed progress on an approach to Hain's conjecture that the symmetric Torelli group is generated by Dehn twists on separating curves [5]. Their strategy is to analyze the action of $\mathcal{SI}_g$ on the curve complex of the marked sphere via the branched covering projection given by the involution. They have proven that the Symmetric Birman Kernel - the kernel of the surjection from the symmetric Torelli group of the marked surface to that of the unmarked surface, obtained by "forgetting" the markings - is generated by twists around separating curves. (The paper on these results is in progress.)

**Allen Hatcher** presented his recent computation of the stable homology of MCG($H_g$), where $H_g$ is the handlebody of genus $g$, showing that for a given $i$ and sufficiently large $g$ and $N$, its degree-$i$ homology is the same as that of $\Omega^N S^N(BSO(3)_+)$, where $\Omega$ denotes looping and $S$ denotes suspension. With rational coefficients, this homology is classically known to be the polynomial ring $\mathbb{Q}[x_4, x_8, x_{12}, \ldots]$ (where $x_{4i}$ represents a class of degree $4i$). It is fitting that the Smale conjecture plays a (small) role in the proof, which proceeds by analyzing the the space of oriented submanifolds of $\mathbb{R}^\infty$ diffeomorphic to $H_g$ as the model for the classifying space BDiff$^+(H_g)$. (Hatcher and Wahl prove that the homology of MCG($H_g$) stabilizes in [7]; Hatcher's paper on calculation of this stable homology is in progress.)

This result fits neatly among other recent results on the stable homology of MCG($S_g$) and Aut($F_g$). (The group MCG($H_g$) maps injectively into the former and surjectively onto the latter.) Galatius proved, by analyzing the space of graphs in $\mathbb{R}^\infty$, that the homology of Aut($F_g$) is stably the same, in the sense just described, as that of $\Omega^N S^N$, which with rational coefficients is trivial. [4] Madsen and Weiss, in the work noted in the previous section, proved that the homology of MCG($S_g$) is stably the same as that of the $N^{th}$ iterated looping of the Thom space associated to the Grassmanian of oriented 2-planes in $R^N$. The rational homology of the latter was known to be the polynomial ring $\mathbb{Q}[x_2, x_4, x_6, \cdots]$, where the $x_{2i}$ represent certain classes of even degree. [10]

**Francois Laudenbach**, relating joint work with G. Meigniez, outlined a new geometric and self-contained proof in dimension three of Thurston's regularization theorem for co-orientable $\Gamma_1$-structures. Being free from reliance on either the perfectness of Diff($S^1$) or the homology equivalence between BDiff$_c(\mathbb{R})$ and $\Omega\mathrm{B}(\Gamma_1)_+$, their result applies more widely. Specifically, if $\xi$ is a tangentially $C^\infty$ and transversely $C^r$ $\Gamma_1$-structure on a closed, orientable 3-manifold, then $\xi$ is homotopic to a regular $\Gamma_1$ structure. Furthermore, this foliation may be taken to be an open book foliation modified by *suspension*, meaning a perturbation of the holonomy near a compact subsurface of a leaf. [9]

# 4 Directions for Future Research

As befits a workshop of this type, there was some informal discussion of future research directions in the field. The organizers thank Bob Edwards and Allen for catalyzing much of the discussion in this vein.

It was observed that many of the "holy grail" problems that framed and guided the field of low-dimensional topology during the twentieth century have been solved: the Poincaré Conjecture, the Smale Conjecture, the Geometrization Conjecture. (Open questions that might rate as exceptions include the generalized Smale Conjecture and the 4-dimensional smooth Schönflies problem.) Do we need new framing questions to motivate current researchers, and if so which ones? Or is it time to focus more on filling in the existing framework? (In particular, Wu-chung Hsiang suggested it would be desirable to develop an effective procedure to implement Ricci flow.)

Geometric topology seems to be arising in many other disciplines. Perhaps the focus in the foreseeable future should be on exploration of these applications and the topological questions that arise from them, with its potential to illuminate the sciences and social sciences, as well as other areas of mathematics.

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