Modular invariants and twisted equivariant K-theory (09rit146)

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1 Overview of CFT and twisted equivariant K-theory

Conformally invariant quantum field theory in 2 dimensions (CFT for short) is by now a well-established area of mathematical physics, with profound relations to several areas of pure mathematics. The two easiest classes of examples are associated to finite groups G (*holomorphic orbifolds*) and to the loop group $LG = \{f : S^1 \to G\}$ of compact Lie groups G (*Wess-Zumino-Witten models*), at some level $k \in \mathbb{Z}$. New examples can be constructed from old ones through the orbifold and GKO coset constructions.

A CFT consists of two halves, called *vertex operator algebras* (VOA), which are linked together by a *modular invariant*. Typically these two VOAs are isomorphic. For the nicest VOAs (called *rational*), e.g. those associated to finite groups or loop groups, the modules form a modular tensor category, and so among other things come with representations of braid groups and other mapping class groups such as the modular group SL(2,$\mathbb{Z}$). The Grothendieck ring of this category is called the *Verlinde ring*. In these rational theories — the only ones we consider — each Verlinde ring is finite-dimensional, associative, commutative and is perhaps the simplest algebraic structure associated to the CFT.

The modular invariant should be thought of as the glue linking together the two VOAs (or more specifically their modules) into the full CFT. The possible modular invariants for a finite group G are parametrized by pairs (H, ψ) for a subgroup H of $G \times G$ and $\psi \in H^2_H(\text{pt}; S^1)$. No such parametrisation is known for loop groups: the modular invariants at all levels k are known only for $LSU(2)$ (which have an A-D-E classification) and $LSU(3)$. More generally, we know that the generic loop group modular invariants are associated to affine Dynkin diagram symmetries. Those symmetries of the unextended Dynkin diagram are associated to outer automorphisms of G; the remaining symmetries are associated to subgroups Z of the centre of G, and yield the so-called *simple current modular invariants*. The remaining modular invariants — the *exceptional* ones — are primarily due to *conformal embeddings* (certain subgroups H of G) and *rank-level duality*. For example, in the A-D-E list of $LSU(2)$, the outer automorphisms of $SU(2)$ are trivial and give rise to the A-series of modular invariants, the only nontrivial subgroup of the centre $\mathbb{Z}_2$ of $SU(2)$ gives rise to the D-series, conformal embeddings give rise to the E_6 and E_8 modular invariants, while E_7 is due to rank-level duality.

The Verlinde ring is associated to each half. The analogous structure, associated to the full conformal field theory (or if you prefer, the modular invariant), is called the *full system* or *algebra of defect lines*. The *nimrep* or *boundary data* is a module for both the Verlinde ring and the full system. In some sense, every modular invariant of a pair of VOAs comes from a restriction of a larger VOA, twisted by an automorphism of the larger VOA. The problem in general is then to find such extensions. In practise (and in theory) the reverse procedure (inducing rather than restricting) is more valuable and is called *alpha induction*.

Much of this data is beautifully captured by *subfactors* (a subfactor is a containment $N \subset M$ of *factors*, which are simple von Neumann algebras), and *sectors* (equivalence classes) of endomorphisms. Here the Verlinde algebra is represented by sectors ${}_N\mathcal{X}_N$ on a III_1 factor N which are nondegenerately braided; multiplication is composition. In this picture, every modular invariant arises from a subfactor $N \subset M$ and an alpha-induction up to a closed (but unbraided) system ${}_M\mathcal{X}_M$ of sectors on M, and the nimrep to a system ${}_N\mathcal{X}_M$ of maps $N \to M$ closed under left compositions by ${}_N\mathcal{X}_N$ and right compositions by ${}_M\mathcal{X}_M$.

The other ingredient in our story is K*-theory*, which on a compact Hausdorff space X looks at the vector bundles over X, or equivalently the finitely generated projective modules over the C^*-algebra $C(X)$ of complex valued continuous functions on X. This gives the abelian group $K^0(X)$, as the Grothendieck group of vector bundles or modules. If a group G acts on our space X, we can define *equivariant* K-theory $K^0_G(X)$ for equivariant bundles, e.g. as the K-theory corresponding to the crossed product $C(X) \rtimes G$. For locally compact spaces, we need to be a bit careful, e.g. by inserting and removing one-point compactifications, but once we've done that we can define the group $K^1_G(X)$ as $K^0_G(\mathbb{R} \times X)$. These C^*-algebras (thought of as spaces of sections of the trivial bundle over X with fibres the compacts $\mathcal{K}$) can be twisted, by taking a non trivial bundle $\mathcal{K}_\tau$ over X. This results in *twisted* (equivariant) K-theory ${}^\tau K^*(X)$ (or ${}^\tau K^*_G(X)$ in the equivariant case). The possible twists τ are classified by a $\check{\mathrm{C}}\mathrm{ech}$ cohomology class of X, the Dixmier–Douady invariant $H^3(X;\mathbb{Z})$ (or $H^3_G(X;\mathbb{Z})$).

In a similar way, twisted equivariant K*-homology* ${}^\tau K^G_*(X)$ can be defined; these are related by Poincaré duality. The most important property of K-theory (or K-homology) is *Bott periodicity*, which says ${}^\tau K^{i+2}_G(X) \cong {}^\tau K^i_G(X)$ and ${}^\tau K^G_{i+2}(X) \cong {}^\tau K^G_i(X)$.

For example, let G be a compact connected simply connected Lie group. The equivalence classes of its finite-dimensional representations under direct sum and tensor product form the *representation ring* R_G. This ring can be realized as the equivariant K-group $K^0_G(\mathrm{pt})$ of G acting trivially on a point pt; the other K-group is $K^1_G(\mathrm{pt}) = 0$.

2 Recent Developments and Open Problems

The recent work of Freed-Hopkins-Teleman (see e.g. [5]) gives a K-theoretic interpretation for the Verlinde ring $\mathrm{Ver}_k(G)$ of a loop group LG at level k: $\mathrm{Ver}_k(G)$ is the twisted equivariant K-group ${}^{k+h^\vee}K^{\dim(G)}_G(G)$ where G here acts adjointly on itself, $h^\vee$ is the dual Coxeter number of G, and the twist $k+h^\vee$ lies in $H^3_G(G;\mathbb{Z}) \cong \mathbb{Z}$. The multiplication in $\mathrm{Ver}_k(G)$ is recovered from the push-forward of group multiplication. The other K-group, namely ${}^{k+h^\vee}K^{1+\dim(G)}_G(G)$, is 0.

A natural extension of Freed-Hopkins-Teleman would be to realise in a similar spirit (e.g. K-theoretically) the other data, such as the full system, nimreps, and alpha induction, for the modular invariants associated to loop groups. Freed-Hopkins-Teleman were helped to their loop group theory, through considering a toy model: the finite group G case, where it is much easier to see that the Verlinde ring is isomorphic to $K^0_G(G)$. But in [1], the finite group story is developed much more completely, guided by the braided subfactor approach. Consider a modular invariant associated to subgroup $H < G \times G$ and, for simplicity, trivial cocycle ψ in $H^2_H(\mathrm{pt}; S^1)$. Then the full system can be identified with $K^0_{H\times H}(G \times G)$, where $H \times H$ acts on $G \times G$ diagonally on the left and right, and $K^1_{H\times H}(G \times G) = 0$. The nimrep is $K^0_H(G)$, and again $K^1_H(G) = 0$.

We would expect something similar for loop groups. But one of the many ways in which finite groups G are easier than loop groups is that uniform parametrisation of modular invariants. For loop groups, we would expect a different description of the full system etc, for each class of modular invariants (namely those coming from outer automorphisms of G, from subgroups of the centre of G, from conformal embeddings, from rank-level duality,...).

Our recent paper [2] confronted these questions for the loop groups. It's long and technically complicated, and took us over 3 years to write, but will provide the foundation for all of our future work. In it we focussed primarily on what we thought would be the class closest to Freed-Hopkins-Teleman, namely conformal embeddings $H \subset G$; we expected the full system to be related to some twist of $K_H(G)$. This turned out to be far from straightforward, for reasons we only now understand, and we could only obtain partial matches. [2] also constructed the relevant Dixmier-Douady bundles realising the twists, and studied orbifold examples (again with only partial results).

3 Scientific Progress Made

We arrived in BIRS with several questions and some ideas. Our intention was to begin a sequel to our paper [2], which we had recently completed. A week later we left with 50+ pages of notes and the core of the sequel [3] worked out. Considering how hard [2] had been to write, we were both completely amazed at how much progress we made in so little time.

We have a new and much more promising approach to conformal embeddings, namely ${}^{\tau}K^0_{H\times H}(G\times G)$ where the action is diagonal: $(h_1,h_2).(g_1,g_2)=(h_1g_1h_2^{-1},h_1g_2h_2^{-1})$ (implicit here is the map $H\to G$). But we now realise that exact K-theoretic descriptions of conformal embeddings will require addressing the Clifford algebras implicit in [5].

But much more important, we obtained a complete understanding of the full system (including nimreps, alpha-induction,...) corresponding to the generic modular invariants, i.e. those coming from outer automorphisms and subgroups of the centre. For example the full system corresponding to a subgroup Z of the centre will be ${}^{\tau}K^0_{G\times G}(G/Z_0\times G/Z_0)$, again using the diagonal action, where Z_0 is a certain subgroup of Z, and τ some twist. The nimrep is ${}^{\tau}K^{\mathrm{dim}\,G}_{G/Z}(G)$. We accomplished this by first working out the complete picture for the special case of tori, which have a geometric description in terms of lattices. Furthermore, we obtained the K-theoretic description for the Verlinde ring of an infinite class of (non-holomorphic) orbifolds. We failed to do this for even one example in [2].

4 Outcome of the Meeting

Once we left BIRS we began fleshing out the details. We applied our K-theoretic descriptions to dramatically simplify nimrep formulas appearing in the CFT literature, and recovered K-theoretically formulas for D-brane charges which appeared in the CFT literature. The resulting paper [3] has been submitted it to Commun. Math. Phys. (We also began an unrelated paper, [4], which slowed somewhat our completion of [3].)

There are still some open questions left in [3] (e.g. we only have a partial understanding of rank-level duality and hence of the E_7 modular invariant of $LSU(2)$), but we both feel that the K-theoretic story is now close to complete, and the next step is to obtain direct KK-theoretic descriptions of the various maps here, namely the modular invariant, alpha inductions, the modular group representation, etc. These should be analysed via spectral triples, Fredholm modules and Dirac operators. Given the success of [3], developing this picture is the natural next step.

[2] took over 3 years to write. Partly this is because of its length (88 pages) and complexity, but partly it was because we work on opposite sides of the Atlantic and our visits together are diluted somewhat by other obligations (teaching, grad students, family etc). By contrast our week at BIRS was intense and distraction-free. It was a fabulous and invaluable experience, which pushed our desired extension of Freed-Hopkins-Teleman to new levels. [3] is a fine paper; it could not have been written in anything like this timeline without this Research-in-Teams at BIRS.

References

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