Finiteness problems in arithmetic deformation theory 08rit135

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1 Overview of the project

Deformation theory pertains to the local behavior of moduli spaces. One example which has been very fruitful in solving various problems in number theory concerns deformations of Galois representations. Here one starts with a representation of a Galois group over a field of characteristic $p > 0$, and one tries to lift this representation over local rings having this field as their residue field [4].

We have been studying a generalization of this deformation theory in which one replaces a single representation by a complex of modules for an arbitrary profinite group [1]. Such complexes arise naturally in arithmetic geometry, e.g. from the action of Galois groups on the hypercohomology of etale sheaves.

Our main goal has been to prove finiteness theorems for universal deformations arising from arithmetic. We have shown in various interesting cases that the universal deformation of a complex of modules for a profinite group can be realized by a bounded complex of modules having terms which are finitely generated over the associated universal deformation ring. When this is true, one has much stronger methods for determining the deformation ring.

To study such finiteness problems, one would like to have an obstruction theory for deformations of complexes. We have been working on developing such an obstruction theory with Luc Illusie, following a suggestion of Ofer Gabber. Gabber's method is very concrete. Illusie proposed that it should be understood in the context of the low degree terms in a spectral sequence, as in his work on cotangent complexes in [3].

2 Outcome of the research in teams workshop

We spent most of the week attempting to reconcile Gabber's approach with the spectral sequence method of Illusie. For background on spectral sequences, see [2, §11]. We came to the following conclusions:

A. Usually when one speaks of the exact sequence of low degree terms associated to the spectral sequence of a bicomplex $L^{\bullet,\bullet}$, one assumes that the terms of $L^{\bullet,\bullet}$ are in the first quadrant. However, the $L^{\bullet,\bullet}$ which arises in our problem will not in general have all of its terms in the first quadrant.

B. We found that the correct object to focus on is the 0^{th} term $F_I^0 H^1(\mathrm{Tot}(L^{\bullet,\bullet}))$ of the first filtration of the first cohomology of the total complex of $L^{\bullet,\bullet}$ rather than the entire first cohomology group $H^1(\mathrm{Tot}(L^{\bullet,\bullet}))$. One does have a short exact sequence of low degree terms

$$0 \to E_\infty^{1,0} \to F_I^0 H^1(\mathrm{Tot}(L^{\bullet,\bullet})) \to E_\infty^{0,1} \to 0$$

arising from the first filtration spectral sequence associated to $L^{\bullet,\bullet}$.

C. We proved that in fact, $F_I^0 H^1(\mathrm{Tot}(L^{\bullet,\bullet}))$ is exactly the subgroup of $H^1(\mathrm{Tot}(L^{\bullet,\bullet}))$ which is defined by exact sequences of the kind considered by Gabber when defining lifting obstructions. This unified the approaches of Gabber and Illusie in a very satisfactory way. The tools we developed for showing this result should be very useful in carrying out computations pertaining to the finiteness problem discussed in §1.

References

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