Traceability of Graphs and Digraphs (08frg134)

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1 Overview of the Field, Recent Developments and Open Problems

A graph or digraph is *hamiltonian* if it contains a cycle that visits every vertex, and *traceable* if it contains a path that visits every vertex. A (di)graph D is *hypotraceable* if D is nontraceable but $D - v$ is traceable for every $v \in V(D)$. A (di)graph is k*-traceable* if each of its induced subdigraphs of order k is traceable. Clearly, a nontraceable digraph of order $k+1$ is k-traceable if and only if it is hypotraceable.

An *oriented* graph is a digraph without 2-cycles. Our interest in k-traceable oriented graphs stems from the following conjecture, which is stated in [1].

The Traceability Conjecture (TC): *For $k \geq 2$, every k-traceable oriented graph of order at least $2k-1$ is traceable.*

It has been proved that the TC holds for $k \in \{2, 3, 4, 5, 6\}$ (see [1] and [8]). The TC was motivated by the OPPC, an oriented version of the Path Partition Conjecture, which can be formulated as follows.

OPPC: *If D is an oriented graph with no path of order greater than λ and a is a positive integer such that $a < \lambda$, then $V(D)$ contains a set A such that the oriented graph induced by A has no path of order greater than a and $D - A$ has no path of order greater than $\lambda - a$.*

If the TC is true, it would imply that the OPPC is true for every oriented graph whose order is exactly one more than the order of its longest paths. Directed versions of the PPC are considered in [4], [5], [6], [7], [9] and [10]. Results supporting the TC are proved in [1], [2] and [8].

Clearly, 2-traceable oriented graphs are tournaments, i.e. their underlying graphs are complete. It is well-known that every strong tournament is hamiltonian. In

[1] we extended this result by showing that every strong k-traceable oriented graph of order greater than k is hamiltonian for every $k \in \{2, 3, 4\}$. However, when $k \geq 5$ the situation changes dramatically. We showed in [1] that for every $n \geq 5$ there exists a strong nonhamiltonian oriented graph of order n that is k-traceable for every $k \in \{5, \ldots, n\}$. Thus for $k \in \{2, 3, 4\}$ there are no strong nonhamiltonian k-traceable oriented graphs of order greater than k, while for each $k \geq 5$ there are infinitely many.

It is also well-known that every tournament is traceable, i.e., every 2-traceable oriented graph is traceable. It is therefore natural to ask: What is the largest value of k such that every k-traceable oriented graph of order at least k is traceable? And, are there nontraceable k-traceable oriented graphs of arbitrarily large order for some $k \geq 3$?

It is shown in [8] that for $k \geq 2$ every nontraceable k-traceable oriented graph of order $n > k$ contains a hypotraceable oriented graph of order h for some $h \in \{k+1, \ldots, n\}$ and also that there does not exist a hypotraceable oriented graph of order less than 8. In [3] it is shown that there exists a hypotraceable oriented graph of order n for every $n \geq 8$ except, possibly, for $n = 9$ or 11 and also that no hypotraceable oriented graph of order 8 is 5-traceable or 6-traceable. These results, together with the fact that the TC holds for $k \leq 6$, imply that for $k \in \{2, 3, 4, 5\}$, every k-traceable oriented graph of order at least k is traceable and every 6-traceable oriented graph of order n is traceable if $n = 6, 7$ or 8 or $n \geq 11$. Moreover, for every $k \geq 7$ except, possibly, for $k = 9$ or 11, there exists a nontraceable k-traceable oriented graph of order $k+1$. During the workshop we addressed the following two questions.

Question 1 Does there exist a nontraceable 6-traceable oriented graph of order 9 or 10?

Question 2 Do there exist nontraceable k-traceable oriented graphs of arbitrarily large order for some $k \geq 7$?

The underlying graph of a k-traceable oriented graph is, obviously, also k-traceable, so we also considered the following two questions during the workshop.

Question 4 What is the structure of k-traceable oriented graphs?

Question 5 Which k-traceable graphs have k-traceable orientations?

2 Outcome of the Focussed Research Workshop

We answered Question 1 by proving that there does not exist a 6-traceable oriented graph of order 9 or 10. Thus we conclude that for each $k \in \{2, 3, 4, 5, 6\}$ every k-traceable oriented graph of order at least k is traceable.

We answered Question 2 in the negative by establishing an upper bound in terms of k on the order of nontraceable k-traceable oriented graphs. We proved that the order of nontraceable k-traceable oriented graphs is at most $6k - 21$ for $k = 7, 8$ and at most $2k^2 - 20k + 58$ for every $k \geq 9$.

We made progress on Questions 4 and 5 by characterizing k-traceable graphs for $k = 3, 4, 5, 6$ and also characterizing k-traceable orientations of k-traceable graphs for $k = 3, 4$.

The results of our Workshop are written up in the two attached papers.

References

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