Banff International Research Station
for Mathematical Innovation and Discovery

Schubert Calculus and Schubert Geometry
18–23 March 2007

MEALS

*Breakfast (Buffet): 7:00–9:00 am, Donald Cameron Hall, Monday–Friday
*Lunch (Buffet): 11:30 am–1:30 pm, Donald Cameron Hall, Monday–Friday
*Dinner (Buffet): 5:30–7:30 pm, Donald Cameron Hall, Sunday–Thursday
Coffee Breaks: As per daily schedule, 2nd floor lounge, Corbett Hall
***Please remember to scan your meal card at the host/hostess station in the dining room for each meal.**

MEETING ROOMS

All lectures will be held in Max Bell 159 (Max Bell Building accessible by bridge on 2nd floor of Corbett Hall). Hours: 6 am–12 midnight. LCD projector, overhead projectors and blackboards are available for presentations. *Please note that the meeting space designated for BIRS is the lower level of Max Bell, Rooms 155–159. Please respect that all other space has been contracted to other Banff Centre guests, including any Food and Beverage in those areas.*

SCHEDULE

Sunday

16:00	Check-in begins (Front Desk - Professional Development Centre - open 24 hours) Lecture rooms available after 16:00 (if desired)
17:30–19:30	Buffet Dinner, Donald Cameron Hall
20:00	Informal gathering in 2nd floor lounge, Corbett Hall

Monday

7:00–8:45	Breakfast
8:45–9:00	Introduction and Welcome to BIRS by BIRS Station Manager, Max Bell 159
9:00–10:00	Allen Knutson, *Matroids, shifting, and Schubert calculus*
10:00–10:30	Coffee Break, 2nd floor lounge, Corbett Hall
10:30–11:30	Izzet Coskun, *The geometry of flag degenerations*
11:30–13:00	Lunch
13:00–14:00	Guided Tour of The Banff Centre; meet in the 2nd floor lounge, Corbett Hall
14:00	Group Photo; meet on the front steps of Corbett Hall
14:30–15:30	Thomas Lam, *Affine Schubert calculus*
15:30–16:00	Coffee Break, 2nd floor lounge, Corbett Hall
16:00–17:00	Mark Shimozono, *Kac-Moody dual graded graphs and affine Schubert calculus*
17:30–19:00	Dinner
19:00–20:00	Elena Marchisotto, *Evaluating the research of Mario Pieri (1860-1913) in algebraic geometry*

Tuesday

7:00–9:00	Breakfast
9:00–10:00	Steve Mitchell, *Smooth and palindromic Schubert varieties in affine Grassmannians*
10:00–10:30	Coffee Break, 2nd floor lounge, Corbett Hall
10:30–11:30	Leonardo Mihalcea, *Chern-Schwartz-MacPherson classes for Schubert cells in the Grassmannian*
11:30–13:30	Lunch
14:00–15:00	Alex Yong, *Governing singularities of Schubert varieties*
15:00–15:30	Coffee Break, 2nd floor lounge, Corbett Hall
15:30–16:30	Alexander Varchenko, *The B. and M. Shapiro conjecture in real algebraic geometry and Bethe ansatz*
16:40–17:40	Evgeny Mukhin, *On generalizations of the B. and M. Shapiro conjecture*
17:40–19:30	Dinner

Wednesday

7:00–9:00	Breakfast
9:00–10:00	J. Tymoczko, *Divided difference operators for Grassmannians*
10:00–10:30	Coffee Break, 2nd floor lounge, Corbett Hall
10:30–11:30	Cristian Lenart, *K-theory and quantum K-theory of flag varieties*
11:30–13:30	Lunch
	Free Afternoon
17:30–19:30	Dinner

Thursday

7:00–9:00	Breakfast
9:00–10:00	Takeshi Ikeda, *Equivariant Schubert calculus for isotropic Grassmannians*
10:00–10:30	Coffee Break, 2nd floor lounge, Corbett Hall
10:30–11:30	Jim Ruffo, *A straightening law for the Drinfel'd Lagrangian Grassmannian*
11:30–13:30	Lunch
14:00–15:00	Hugh Thomas, *A combinatorial rule for (co)minuscule Schubert calculus*
15:00–15:30	Coffee Break, 2nd floor lounge, Corbett Hall
15:30–16:30	Nicolas Perrin, *Some combinatorial aspects of cominuscule geometry*
16:40–17:40	Kevin Purbhoo, *Geometric proofs of Horn and saturation conjectures*
17:40–19:30	Dinner

Friday

7:00–8:45	Breakfast
8:50–9:50	Jan Verschelde, *Numerical homotopy algorithms for enumerative geometry*
10:00–11:00	Anders Buch, *Equivariant Gromov-Witten invariants of Grassmannians*
11:30–13:30	Lunch

** 5-day workshops are welcome to use the BIRS facilities (2nd Floor Lounge, Max Bell Meeting Rooms, Reading Room) until 3 pm on Friday, although participants are still required to checkout of the guest rooms by 12 noon. **

Schubert Calculus and Schubert Geometry
18–23 March 2007

ABSTRACTS

Speaker: **Anders Buch** (Rutgers)
Title: *Equivariant Gromov-Witten invariants of Grassmannians*
Abstract: I will speak about joint work with L. Mihalcea, in which we prove that all equivariant (3-point, genus zero) Gromov-Witten invariants on Grassmannians are equal to equivariant triple-intersections on two-step flag varieties. This is a continuation of work with A. Kresch and H. Tamvakis, which established this result for the ordinary Gromov-Witten invariants. The non-equivariant case was obtained by showing that the curves counted by a Gromov-Witten invariant are in bijection with their kernel-span pairs, which consist exactly of the points in a triple-intersection of two-step Schubert varieties. Since the equivariant Gromov-Witten invariants have no enumerative interpretation (and also because they are defined relative to Schubert varieties that are not in general position), the proof in the equivariant case must be based on intersection theory. The main new construction is a blow-up of Kontsevich's moduli space that makes it possible to assign a kernel-span pair of the expected dimensions to every curve. By utilizing a construction of Chaput, Manivel, and Perrin, these results can be extended to all (co)minuscule homogeneous spaces.

Speaker: **Izzet Coskun** (MIT)
Title: *The geometry of flag degenerations*
Abstract: My goal is to explain the flat limits of certain subvarieties of partial flag varieties under one-parameter specializations. I will explain how this study leads to many new Littlewood-Richardson rules. I will try to get the audience to actively participate in the calculations, so we will have graph paper and colored pencils.

Speaker: **Takeshi Ikeda** (Okayama University of Science)
Title: *Equivariant Schubert calculus for isotropic Grassmannians*
Abstract: We describe the torus-equivariant cohomology ring of isotropic Grassmannians by using localization maps to any torus-fixed point. We present two formulas for equivariant Schubert classes of these homogeneous spaces. The first formula is given as a weighted sum over combinatorial objects which we call "excited Young diagrams." The second one is written in terms of factorial Schur Q- or P-functions. As an application, we give a Giambelli-type formula for the equivariant Schubert classes. This is a joint project with Hiroshi Naruse.

Speaker: **Allen Knutson** (UC San Diego)

Title: *Matroids, shifting, and Schubert calculus*

Abstract: In "A geometric Littlewood-Richardson rule", Vakil starts with an intersection of two Schubert varieties in a Grassmannian, and alternately degenerates, then decomposes into geometric components; the end result is a list of Schubert varieties. The primary geometric miracle is that each degeneration stays generically reduced, with the consequence that the original cohomology class (the product of two Schubert classes) is written as a multiplicity-free sum of Schubert classes.

There is an obvious generalization of this procedure to the flag manifold (where the geometric miracle is as yet unknown) but it doesn't give a combinatorial rule, just a geometric one.

In this talk I'll explain how to combinatorialize Ravi's procedure, replacing varieties with matroids, and degeneration with "shifting" (originally invented by Erdős-Ko-Rado for extremal combinatorics). One upshot will be that each of Ravi's varieties is defined by linear equations, allowing us to extend his proof to K-theory. Another is an easily stated conjecture for flag manifold Schubert calculus.

Speaker: **Thomas Lam** (Harvard)

Title: *Affine Schubert calculus*

Abstract: I will discuss some recent progress in understanding the (co)homology of the affine Grassmannian from the point of view of Schubert calculus. In particular, I will explain how to obtain polynomial representatives for Schubert classes and analogues of Pieri rules. If time permits, I hope also to explain some connections with calculations of the (co)homology ring of the affine Grassmannian due to Bott, to Ginzburg, to Bezrukavnikov, Finkelberg and Mirkovic and to Peterson. Part of this talk is based on joint work with Lapointe, Morse and Shimozono.

Speaker: **Cristian Lenart** (SUNY Albany)

Title: *K-theory and quantum K-theory of flag varieties*

Abstract: I previously presented Chevalley-type multiplication formulas in the T-equivariant K-theory of generalized flag varieties $K_T(G/P)$; these formulas were derived in joint work with A. Postnikov. In the first part of this talk, I will present a model for $K_T(G/P)$ in terms of a certain braided Hopf algebra called the Nichols-Woronowicz algebra. This model is based on the Chevalley-type formulas mentioned above, and has potential applications to deriving more general multiplication formulas. In the second part of the talk, I will show the way in which a presentation of the quantum K-theory of the classical flag variety $QK(Fl_n)$ leads to the construction of certain polynomials, called quantum Grothendieck polynomials, that are conjectured to represent Schubert classes in $QK(Fl_n)$. We present evidence for this conjecture; this includes the fact that the quantum Grothendieck polynomials satisfy a multiplication formula which is the natural generalization of the Chevalley-type formula mentioned above (in $K(Fl_n)$) and of the corresponding formula in quantum cohomology. This talk is based on joint work with T. Maeno.

Speaker: **Elena Marchisotto** (California State University, Northridge)

Title: *Evaluating the research of Mario Pieri (1860-1913) in algebraic geometry*

Abstract: Mario Pieri was an active member of the research groups surrounding Corrado Segre and Giuseppe Peano at the University of Turin around the turn of the nineteenth century. Pieri played a major role in foundations of mathematics. Can the same be said of his role in algebraic geometry? Was he the first to introduce the Schubert calculus to Italy? Did he accomplish all he set out to do with the appropriate rigor? Do his papers contain there valuable ideas for contemporary work? Are his contributions to enumerative geometry limited to the multiplication formula and theorem for correspondences on an n-dimensional projective space that are discussed in Fulton's book on intersection theory?

In my talk I will share my initial research in attempts to answer the above questions. I will provide an overview of Pieri the man and Pieri the algebraic geometer. I will briefly discuss Pieri's papers in algebraic geometry, in particular his results in enumerative geometry and his use of the Schubert calculus. I will report on existent letters of Castelnuovo, de Paolis, Enriques, Fouret, Schubert, Severi, and Zeuthen.

Speaker: **Leonardo Mihalcea** (Duke)

Title: *Chern-Schwartz-MacPherson classes for Schubert cells in the Grassmannian*

Abstract: A conjecture of Deligne and Grothendieck states that there is a functorial theory of Chern classes on possibly singular varieties, viewed as a natural transformation from the group of constructible functions to the homology (or Chow group) of a compact variety. This conjecture was solved in 1973 by R. MacPherson; the classes he defined (now known as Chern-Schwarz-MacPherson - or CSM) turned out to be the same as those defined by M.H. Schwartz, by different methods.

In joint work with Paolo Aluffi, we give explicit formulae for the CSM classes of the Schubert varieties in the Grassmannian. The main tools used in the computation are a new formula for the CSM classes recently discovered by P. Aluffi and a Bott-Samelson resolution of a Schubert variety. Given that Schubert varieties are singular, an unexpected feature is a certain effectivity satisfied by these classes; we have proved it in few cases, and it is conjectured to hold in general.

Speaker: **Steve Mitchell** (University of Washington)

Title: *Smooth and palindromic Schubert varieties in affine Grassmannians*

Abstract: The affine Grassmannian associated to a simple complex algebraic group G is an infinite-dimensional projective variety that is very much analogous to an ordinary Grassmannian. In particular, it comes equipped with a decomposition into Schubert cells whose closures are ordinary projective varieties. It turns out that for each fixed G, only finitely many of these Schubert varieties are smooth, and except in type A only finitely many are even palindromic.

In fact it is possible to determine the smooth and palindromic Schubert varieties explicitly. For example, in type C a Schubert variety is smooth if and only if it is a closed parabolic orbit, in which case it is a symplectic Grassmannian. It is palindromic if and only if it is either smooth or a subcomplex of the Schubert variety associated to the lowest nontrivial anti-dominant coroot lattice element; this variety is just the Thom space of a line bundle over projective space. The most eccentric type is type A, where in each rank there are two infinite families of singular palindromics. These latter varieties have interesting topological properties, which were studied in the '80's by myself and (independently) Graeme Segal.

Speaker: **Evgeny Mukhin** (IUPUI)

Title: *On generalizations of the B. and M. Shapiro conjecture*

Abstract: A proof of the B. and M. Shapiro conjecture is based on the consideration of the periodic quantum Gaudin model. We show what results are obtained in the similar way from quasiperiodic quantum Gaudin and from quasiperiodic XXX models.

Speaker: **Nicolas Perrin** (Paris)

Title: *Some combinatorial aspects of cominuscule geometry*

Abstract: In this talk we shall explain combinatorics tools (quivers) generalising Young diagrams that appear in the study of cominuscule homogeneous spaces.

We shall see how to read on these quivers some geometric properties of the Schubert varieties like their singularities, the fact that they are Gorenstein, locally factorial or even that they admit or not small resolutions.

These quivers are used by A. Yong and H. Thomas for classical Schubert calculus. We shall see how these quivers also help for computing some Gromov-Witten invariants.

Speaker: **Kevin Purbhoo** (UBC)

Title: *Geometric proofs of Horn and Saturation Conjecture*

Abstract:

Speaker: **James Ruffo** (TAMU)
Title: *A straightening law for the Drinfel'd Lagrangian Grassmannian*
Abstract: We present a structured set of defining equations for the Drinfel'd compactification of the space of rational maps of a given degree in the Lagrangian Grassmannian. These equations give a straightening law on a certain ordered set, which allows the use of combinatorial arguments to establish useful geometric properties of this compactification. For instance, its coordinate ring is Cohen-Macaulay and Koszul.

Speaker: **Mark Shimozono** (Virginia Tech)
Title: *Kac-Moody dual graded graphs and affine Schubert calculus*
Abstract: Motivated by the dual Hopf algebra structures on the homology and cohomology of the affine Grassmannians, particularly their Schubert structure constants, we exhibit families of dual graded graphs (in the sense of Fomin) associated to any Kac-Moody algebra, dominant weight and "positive" central element, yielding enumerative identities involving chains in the strong and weak Bruhat orders.

Speaker: **Hugh Thomas** (University of New Brunswick)
Title: *A combinatorial rule for (co)minuscule Schubert calculus*
Abstract: I will discuss a root system uniform, concise combinatorial rule for Schubert calculus of *minuscule* and *cominuscule* flag manifolds G/P. (The latter are also known as *compact Hermitian symmetric spaces.*) We connect this geometry to work of Proctor in poset combinatorics, thereby generalizing Schutzenberger's *jeu de taquin* formulation of the Littlewood-Richardson rule for computing intersection numbers of Grassmannian Schubert varieties.

I will explain the rule and, time permitting, discuss ideas used in its proof, including *cominuscule recursions*, a general technique relating the Schubert constants for different Lie types. I will also discuss *cominuscule dual equivalence*, a generalization of a concept due to Haiman. We use this to provide an independent proof of Proctor's jeu de taquin results in our context.

Speaker: **Julianna Tymoczko** (Michigan)
Title: *Divided difference operators for Grassmannians*
Abstract: We construct divided difference operators for Grassmannians. More precisely, we give an explicit combinatorial formula for divided difference operators on the equivariant cohomology of G/P, for any parabolic subgroup P and any complex reductive linear algebraic group G. One application generalizes a flag-variety result of Sara Billey's to compute the localizations of equivariant Schubert classes for G/P. Another provides equivariant Pieri rules for Grassmannians.

These divided difference operators are constructed using GKM (Goresky-Kottwitz-MacPherson) theory, which gives a combinatorial construction of equivariant cohomology for many suitably nice algebraic varieties. We will focus on how the theory works for Grassmannians of k-planes in complex n-dimensional space.

Speaker: **Alexander Varchenko** (North Carolina)
Title: *The B. and M. Shapiro conjecture in real algebraic geometry and Bethe ansatz.*
Abstract: I shall discuss the proof of Shapiro's conjecture by methods of math physics and representation theory. The Shapiro's conjecture says the following. If the Wronskian of a set of polynomials has real roots only, then the complex span of this set of polynomials has a basis consisting of polynomials with real coefficients.

Speaker: **Jan Verschelde** (University of Illinois-Chicago)

Title: *Numerical homotopy algorithms for enumerative geometry*

Abstract: In a 1998 paper on "Numerical Schubert Calculus", Birk Huber, Frank Sottile, and Bernd Sturmfels proposed numerical Pieri homotopy algorithms to solve problems in enumerative geometry. Implementations of these algorithms ran on specific examples of intersection problems whose solution set is entirely real. Jointly with Yusong Wang, the Pieri homotopies were applied to the output pole placement problem in linear systems control. Following Vakil's geometric proof of the Littlewood-Richardson rule, Sottile, Vakil, and I designed deformation methods to solve general Schubert problems. Current work is directed to implement these new Littlewood-Richardson homotopies.

Speaker: **Alex Yong** (University of Minnesota)

Title: *Governing singularities of Schubert varieties*

Abstract: We present a combinatorial and computational commutative algebra methodology for studying singularities of Schubert varieties of flag manifolds. We define the combinatorial notion of *interval pattern avoidance*. For "reasonable" invariants P of singularities, we geometrically prove that this governs (1) the P-locus of a Schubert variety, and (2) which Schubert varieties are globally not P.

The prototypical case is P="singular"; classical pattern avoidance applies admirably for this choice [Lakshmibai-Sandhya'90], but is insufficient in general. Our approach is analyzed for some common invariants, including Kazhdan-Lusztig polynomials, multiplicity, factoriality, and Gorensteinness, extending [Woo-Yong'05]; the description of the singular locus (which was independently proved by [Billey-Warrington '03], [Cortez '03], [Kassel-Lascoux-Reutenauer'03], [Manivel'01]) is also thus reinterpreted. Our methods are amenable to computer experimentation, based on computing with *Kazhdan-Lusztig ideals* (a class of generalized determinantal ideals) using Macaulay 2. This feature is supplemented by a collection of open problems and conjectures.