Workshop on Graph colouring Problems Arising in Telecommunications: Final Report

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The channel assignment problem in radio or cellular phone networks is the following: we need to assign radio frequency bands to transmitters (each station gets one channel which corresponds to an integer). In order to avoid interference, if two stations are very close, then the separation of the channels assigned to them has to be large enough. Moreover, if two stations are close (but not very close), then they must also receive channels that are sufficiently apart.

Such problem may be modelled by $L(p,q)$-labellings of a graph G. The vertices of this graph correspond to the transmitters and two vertices are linked by an edge if they are very close. Two vertices are then considered close if they are at distance 2 in the graph. Let $dist(u,v)$ denote the distance between the two vertices u and v. An *$L(p,q)$-labelling of G* is an integer assignment f to the vertex set $V(G)$ such that:

- $|f(u) - f(v)| \geq p$, if $dist(u,v) = 1$, and
- $|f(u) - f(v)| \geq q$, if $dist(u,v) = 2$.

As the separation between channels assigned to vertices at distance 2 cannot be smaller than the separation between channels assigned to vertices at distance 1, it is often assumed that $p \geq q$.

The *span* of f is the difference between the largest and the smallest labels of f plus one. The *$\lambda_{p,q}$-number* of G, denoted by $\lambda_{p,q}(G)$, is the minimum span over all $L(p,q)$-labellings of G.

Moreover, very often, because of technical reasons or dynamicity, the set of channels available varies from transmitter to transmitter. Therefore one has to consider the list version of $L(p,q)$-labellings. A *k-list-assignment* L of a graph is a function which assigns to each vertex v of the graph a list $L(v)$ of k prescribed integers. Given a graph G, the *list $\lambda_{p,q}$-number*, denoted $\lambda^l_{p,q}(G)$ is the smallest integer k such that, for every k-list-assignment L of G, there exists an $L(p,q)$-labelling f such that $f(v) \in L(v)$ for every vertex v.

The problem of determining $\lambda_{p,q}(G)$ has been studied for some specific classes of graphs (see the survey of Yeh [22]). Generalizations of $L(p,q)$-labellings in which for each $i \geq 1$, a minimum gap of p_i is required for channels assigned to vertices at distance i, have also been studied (see for example [18] or [16]). Surprisingly, list $L(p,q)$-labellings have been studied only very litte explicitely and appear only very recently in the literature [14]. However, some of the proofs for $L(p,q)$-labellings also work for list $L(p,q)$-labellings.

Note that $L(1,0)$-labellings of G correspond to ordinary vertex colourings of G and $L(1,1)$-labelling of G to the vertex colourings of the square of G. Hence the $\lambda_{1,0}$-number of a graph G equals its *chromatic number* $\chi(G)$, and its $\lambda^l_{1,0}$-number eqauls its *choice number* $ch(G)$. The *square* of a graph G, denoted G^2, is the graph with vertex set $V(G)$ such that two vertices u,v are linked by an edge in G^2 if and only if u and v are at distance at most 2 in G. Formally, $E(G^2) = \{ uv \mid dist_G(u,v) \leq 2 \}$. Obviously, $L(1,1)$-labellings of G correspond to vertex colourings of G^2. So $\lambda_{1,1}(G) = \chi(G^2)$ and $\lambda^l_{1,1}(G) = ch(G^2)$

It is well known that $\omega(G) \leq \chi(G) \leq ch(G) \leq \Delta(G) + 1$, where $\omega(G)$ denotes the *clique number* of G, i.e., the size of a maximum clique in G, and $\Delta(G)$ denotes the *maximum degree* of G. Similar easy inequalities may be obtained for $L(p,q)$-labellings: $q\,\omega(G^2) - q + 1 \leq \lambda_{p,q}(G) \leq \lambda^l_{p,q}(G) \leq p\,\Delta(G^2) + 1$. As $\omega(G^2) \geq \Delta(G) + 1$, the previous inequality gives $\lambda_{p,q} \geq q\,\Delta + 1$. However, a straightforward argument shows that $\lambda_{p,q} \geq q\,\Delta + p - q + 1$. In the same way, $\Delta(G^2) \leq \Delta^2(G)$ so $\lambda^l_{p,q}(G) \leq p\,\Delta^2(G) + 1$ and the greedy algorithm shows $\lambda^l_{p,q}(G) \leq (2\,q - 1)\,\Delta^2(G) + (2\,p - 1)\,\Delta(G) + 1$. Taking a $L(\lceil p/k \rceil\,, \lceil q/k \rceil)$-labelling and multiplying each label by k, we obtain a $L(p,q)$-labelling. This proves the following easy observation.

Proposition 1 *For all graphs G and positive integers k, p, q we have*

$$\lambda_{p,q}(G) \;\leq\; k\,(\lambda_{\lceil p/k \rceil, \lceil q/k \rceil}(G) - 1) + 1.$$

In general, determining the $\lambda_{p,q}$-number of a graph is NP-hard [7]. In their seminal paper, Griggs and Yeh [9] observed that a greedy algorithm yields $\lambda_{2,1}(G) \leq \Delta^2 + 2\Delta + 1$, where Δ denotes the maximum degree of the graph G. Moreover, they conjectured that this upper bound can be decreased to $\Delta^2 + 1$.

Conjecture 2 ([9]) *For every $\Delta \geq 2$ and every graph G of maximum degree Δ,*

$$\lambda_{2,1}(G) \leq \Delta^2 + 1.$$

This upper bound would be tight: there are graphs with degree Δ, diameter 2 and $\Delta^2 + 1$ vertices, namely the 5-cycle, the Petersen graph and the Hoffman-Singleton graph. Thus, their square is a clique of order $\Delta^2 + 1$, so the span of every $L(2,1)$-labelling is at least $\Delta^2 + 1$.

However, such graphs exist only for $\Delta = 2, 3, 7$ and possibly 57, as shown by Hoffman and Singleton [11]. So one can ask how large may be the $\lambda_{2,1}$-number of a graph with large maximum degree. As it should be at least as large as the largest clique in its square, one can ask what is the largest clique number $\gamma(\Delta)$ of the square of a graph with maximum degree Δ. If Δ is a prime power plus 1, then $\gamma(\Delta) \geq \Delta^2 - \Delta + 1$. Indeed, in the projective plane of order $\Delta - 1$, each point is in Δ lines, each line contains Δ points, each pair of distinct points is in a line and each pair of distinct lines has a common point. Consider the *incidence graph* of the projective plane: it is the bipartite graph with vertices the set of points and lines of the projective plane, and every line is linked to all the points it contains. The properties of the projective plane implies that the set of points and the set of lines form two cliques in the square of this graph, and there are $\Delta^2 - \Delta + 1$ vertices in each.

Jonas [13] improved slightly on Griggs and Yeh's upper bound by showing that every graph of maximum degree Δ admits a $(2,1)$-labelling with span at most $\Delta^2 + 2\Delta - 3$. Subsequently, Chang and Kuo [5] provided the upper bound $\Delta^2 + \Delta + 1$ which remained the best general upper bound for about a decade. Král' and Škrekovski [17] brought this upper bound down by 1 as the corollary of a more general result. And, using the algorithm of Chang and Kuo [5], Gonçalves [8] decreased this bound by 1 again, thereby obtaining the upper bound $\Delta^2 + \Delta - 1$. Note that Conjecture 2 is true for planar graphs of maximum degree $\Delta \neq 3$. For $\Delta \geq 7$ it follows from a result of van den Heuvel and McGuiness [10], and Bella et al. [3] proved it for the remaining cases.

Combining results obtained at the workshop with earlier work, Havet, Reed, and Sereini have shown that Conjecture 2 holds for sufficiently large Δ. I.e. they prove:

Theorem 3 *There is a Δ_0 such that for every graph G of maximum degree $\Delta \geq \Delta_0$,*

$$\lambda_{2,1}(G) \leq \Delta^2 + 1.$$

This is one of the two main outcomes of the workshop, we now describe the second.

Because the transmitters are laid out on earth, $L(p,q)$-labellings of planar graphs are of particular interest. There are planar graphs for which $\lambda_{p,q} \geq \frac{3}{2}\,q\,\Delta + c(p,q)$, where $c(p,q)$ is a constant depending on p and q. For example, consider a graph consisting of three vertices x, y and z together with $3\,k - 1$ additonal vertices of degree two, such that z has k common neighbours with x and k common neighbours with y, x and y are connected and have $k - 1$ common neighbours (see Figure 1).

This graph has maximum degree $2\,k$ and yet its square contains a clique with $3\,k + 1$ vertices (all the vertices except z).

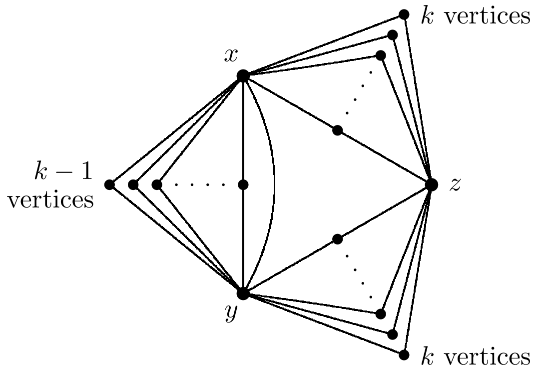

Figure 1: The planar graphs G_k.

A first upper bound on $\lambda_{p,q}(G)$, for planar graphs G and positive integers $p \geq$ has been proved by Van den Heuvel and McGuinness [10] : $\lambda_{p,q}(G) \leq 2\,(2\,q-1)\,\Delta + 10\,p + 38\,q - 24$. Molloy and Salavatipour [19] improved this bound by showing the following :

Theorem 4 (Molloy and Salavatipour [19]) *For a planar graph G and positive integers p, q,*

$$\lambda_{p,q}(G) \;\leq\; q \left\lceil \frac{5}{3}\,\Delta \right\rceil + 18\,p + 77\,q - 18.$$

Moreover, they described an $O(n^2)$ time algorithm for finding an $L(p,q)$-labelling whose span is at most the bound in their theorem.

The celebrated Four Colour Theorem by Appel and Haken [2] states that $\lambda_{1,0}(G) = \chi(G) \leq 4$ for planar graphs. Regarding the chromatic number of the square of a planar graph, Wegner [21] posed the following conjecture which is mentioned in Jensen and Toft [12, Section 2.18].

Conjecture 5 (Wegner [21]) *For a planar graph G of maximum degree Δ :*

$$\lambda_{1,1}(G) \;=\; \chi(G^2) \;\leq\; \begin{cases} 7, & \text{if } \Delta = 3,\\ \Delta + 5, & \text{if } 4 \leq \Delta \leq 7,\\ \left\lceil \frac{3}{2}\,\Delta \right\rceil + 1, & \text{if } \Delta \geq 8. \end{cases}$$

Wegner also gave examples showing that these bounds would be tight. For $\Delta \geq 8$, these are the same examples is in Figure 1.

Kotoschka and Woodall [15] conjectured that, for every square of a graph, the list-chromatic number equals the choose number. This conjecture and Wegner's one imply directly the following :

Conjecture 6 *For a planar graph G of maximum degree Δ :*

$$\lambda^{l}_{1,1}(G) \;=\; ch(G^2) \;\leq\; \begin{cases} 7, & \text{if } \Delta = 3,\\ \Delta + 5, & \text{if } 4 \leq \Delta \leq 7,\\ \left\lceil \frac{3}{2}\,\Delta \right\rceil + 1, & \text{if } \Delta \geq 8. \end{cases}$$

Wegner also showed that if G is a planar graph with $\Delta = 3$, then G^2 can be 8-coloured. Very recently, Thomassen [20] solved Wegner's conjecture for $\Delta = 3$ and Cranston and Kim [6] showed that the square of every connected graph (non necessarily planar) which is subcubic (i.e., with $\Delta \leq 3$) is 8-choosable, except for the Petersen graph. However, the 7-choosability of the square of subcubic planar graphs is still open. The first upper bound on $\chi(G^2)$ in terms of Δ was obtained by Jonas [13] who showed $\chi(G^2) \leq 8\,\Delta - 22$. This bound was later improved by Wong [**?**] to $\chi(G^2) \leq 3\,\Delta + 5$ and then by Van den

Heuvel and McGuinness [10] to $\chi(G^2) \leq 2\,\Delta + 25$. Better bounds were then obtained for large values of Δ. It was shown that $\chi(G^2) \leq \lceil \frac{9}{5}\,\Delta \rceil + 1$ for $\Delta \geq 749$ by Agnarsson and Halldórsson [1], and that $\chi(G^2) \leq \lceil \frac{9}{5}\,\Delta \rceil + 1$ for $\Delta \geq 47$ by Borodin et al. [4]. Finally, the best known upper bound before the workshop was obtained by Molloy and Salavatipour [19] as a special case of Theorem 4 :

Theorem 7 (Molloy and Salavatipour [19]) *For a planar graph G,*

$$\lambda_{1,1}(G) \;=\; \chi(G^2) \;\leq\; \left\lceil \frac{5}{3}\,\Delta \right\rceil + 78.$$

As mentioned in [19], the constant 78 can be reduced for sufficiently large Δ. For example, it was improved to 24 when $\Delta \geq 241$.

Havet, McDiarmid, Reed, and Van Den Heuvel, combining results obtained at the workshop with earlier results, managed to prove:

Theorem 8 *The square of every planar graph G of maximum degree Δ has list chromatic number at most $(1+o(1))\,\frac{3}{2}\,\Delta$. Moreover, given lists of this size, there is an acceptable colouring in which the colours on every pair of adjacent vertices of G differ by $\Delta^{1/4}$.*

As a corollary, for every planar graph G and any fixed p we get that $\lambda^l_{p,1}(G) \leq (1 + o(1))\,\frac{3}{2}\,\Delta(G)$.

Together with Proposition 1, this yields:

Corollary 9 *Let $p \geq q$ be two fixed integers. Then for any planar graph G we have $\lambda_{p,q}(G) \leq (1 + o(1))\,\frac{3}{2}\,q\,\Delta(G)$.*

Note that using exactly the same proof as for Theorem 8, one can

show that for any fixed $p \geq q$, for every planar graph G, $\lambda^l_{p,q}(G) \leq (1 + o(1))\,\frac{3}{2}\,(2\,q - 1)\,\Delta(G)$.

References

[1] G. Agnarsson and M.M. Halldórsson. Coloring powers of planar graphs. *SIAM J. Discrete Math.* 16:651–662, 2000.

[2] K. Appel and W. Haken. Every Planar Map is Four Colourable. *Contemporary Mathematics* 98. American Mathematical Society, Providence, RI, 1989.

[3] P. Bella, D. Král', B. Mohar and K. Quittnerova. Labeling planar graphs with a condition at distance two, *European J. of Combin.*, to appear.

[4] O.V. Borodin, H.J. Broersma, A. Glebov, and J. van den Heuvel. Minimal degrees and chromatic numbers of squares of planar graphs (in Russian). *Diskretn. Anal. Issled. Oper. Ser. 1* 8(4):9–33, 2001.

[5] G. J. Chang and D. Kuo. The L(2, 1)-labeling problem on graphs. *SIAM J. Discr. Math.* 9:309-316, 1996.

[6] D.W. Cranston and S.-J. Kim, *List-coloring the square of a subcubic graph.* Manuscript, 2006.

[7] J. P. Georges, D. W. Mauro and M. A. Whittlesey. Relating path coverings to vertex labelings with a condition at distance two. *Discrete Math.* 135:103-111, 1994.

[8] D. Gonçalves. On the L(p,1)-labelling of graphs. *Discrete Math.*, to appear. Earlier version in *European Conference on Combinatorics, Graph Theory, and Applications (EUROCOMB 2005).*

[9] J. R. Griggs and R. K. Yeh. Labeling graphs with a condition at distance two. *SIAM J. Discr. Math.* 11:585–595, 1992.

[10] J. van den Heuvel and S. McGuinness. Coloring the square of a planar graph. *J. Graph Theory* 42: 110–124, 2003.

[11] A. J. Hoffman and R. R. Singleton. On Moore graphs with diameter 2 and 3. *IBM J. Res. Develop.* 4:497–504, 1960.

[12] T. R. Jensen and B. Toft. *Graph coloring problems.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New-York, 1995.

[13] T.K. Jonas. *Graph coloring analogues with a condition at distance two : $L(2,1)$-labelings and list λ-labelings.* Ph.D. Thesis, University of South Carolina, 1993.

[14] A. Kohl, J. Schreyer, Z. Tuza, and M. Voigt. List version of $L(d,s)$-labelings. *Theoret. Comput. Sci.* 349:92–98, 2005.

[15] A.V. Kostochka and D.R. Woodall. Choosability conjectures and multicircuits. *Discrete Math* 240:123–143, 2001.

[16] D. Král'. Channel assignment problem with variable weights. *SIAM J. Discrete Math.* 20:690–704, 2006.

[17] D. Král' and R. Škrekovski. A theorem about the channel assignment. *SIAM J. Discr. Math.* 16:426–437, 2003.

[18] D.D.-F. Liu and X. Zhu. Multilevel distance labelings for paths and cycles. *SIAM J. Discrete Math.* 19:610–621, 2005.

In *Proc. 25th ACM Symposium on Theory of Computing*, pages 286-293, 1993.

[19] M. Molloy and M.R. Salavatipour. A bound on the chromatic number of the square of a planar graph. *J. Combin. Theory Ser. B* 94:189–213, 2005.

[20] C. Thomassen. The square of a planar cubic graph is 7-colorable. Manuscript, 2006.

[21] G. Wegner. Graphs with given diameter and a coloring problem. *Technical Report*, University of Dortmund, 1977.

[22] R.K. Yeh. A survey on labeling graphs with a condition at distance two. *Discrete Math.* 306:1217–1231, 2006.