BIRS Workshop
Coarsely Quantized Redundant Representations of Signals
March 11 - March 16, 2006

MEALS

Breakfast (Continental): 7:00 - 9:00 am, 2nd floor lounge, Corbett Hall, Sunday - Thursday
*Lunch (Buffet): 11:30 am - 1:30 pm, Donald Cameron Hall, Sunday - Thursday
*Dinner (Buffet): 5:30 - 7:30 pm, Donald Cameron Hall, Saturday - Wednesday
Coffee Breaks: As per daily schedule, 2nd floor lounge, Corbett Hall
***Please remember to scan your meal card at the host/hostess station in the dining room for each lunch and dinner.**

MEETING ROOMS

All lectures are held in the main lecture hall, Max Bell 159. *Please note that the meeting space designated for BIRS is the lower level of Max Bell, Rooms 155-159. Please respect that all other space has been contracted to other Banff Centre guests, including any Food and Beverage in those areas.*

SCHEDULE

	Saturday	Sunday	Monday	Tuesday	Wednesday	Thursday
7:00-9:00	X	Continental Breakfast, 2nd floor lounge, Corbett Hall				
9:00	X	Opening	DeVore	Allebach	Li	Discussion and Open Problems
9:15	X	Daubechies				
9:45	X				Kutyniok	
10:00	X		Goyal	Nowicki		
10:30-11:00	X	Coffee Break, 2nd floor lounge, Corbett Hall				
11:00-11:45	X	Nguyen	Rauhut	Doerr	Boelcskei	
11:45-12:00	X	X	Group Photo[1]	X	X	X
11:30-13:30	X	Buffet Lunch, Donald Cameron Hall				
13:00-14:00	X	Guided Tour[2]	X	free afternoon	X	X
14:00-14:45	X	Benedetto	Wang	free afternoon	Keller	X
14:45-15:15	X	Lammers	Han	free afternoon	Jimenez	X
15:15-15:45	X	Coffee Break, 2nd floor lounge, Corbett Hall (except Tues.)				X
15:45-16:30	X	Casazza	Paulsen	free afternoon	Discussion	X
16:30-17:00	X	Yedlin	Bodmann	free afternoon	(organizers)	X
17:30-19:30	Buffet Dinner, Donald Cameron Hall					X

[1] A group photo will be taken on Monday at 11:45 am, directly after the last lecture of the morning. Please meet on the front steps of Corbett Hall.

[2] A free guided tour of The Banff Centre is offered to all participants and their guests on Sunday starting at 1:00 pm. The tour takes approximately 1 hour. Please meet in the 2nd floor lounge in Corbett Hall.

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PROGRAM

Sunday, March 12

9:00 am	Opening remarks
9:15 am	*Mathematical study of coarsely quantized representations of signals: what, why, how – an overview*
	Ingrid Daubechies, Princeton University
11:00 am	*The tiling phenomenon of Sigma-Delta modulators*
	Thao Nguyen, City College, CUNY
2:00 pm	*Sigma-Delta quantization and finite frames*
	John Benedetto, University of Maryland
2:45 pm	*Alternate Dual Frames and Sigma-Delta Quantization*
	Mark Lammers, University of North Carolina Wilmington
3:45 pm	*Frame Design for Sigma-Delta Quantization*
	Peter Casazza, University of Missouri
4:30 pm	*Industrial Applications of Sigma-delta Converters*
	Matt Yedlin, University of British Columbia

Monday, March 13

9:00 am	*Instance-optimal estimates in Compressed Sensing*
	Ronald DeVore, University of South Carolina
10:00 am	*Sparsity and Quantization: When Undersampling is Oversampling*
	Vivek Goyal, Massachusetts Institute of Technology
11:00 am	*Random Sampling of Sparse Trigonometric Polynomials*
	Holger Rauhut, University of Vienna
2:00 pm	*Sigma-Delta Quantization and the Traveling Salesman Problem*
	Yang Wang, Georgia Institute of Technology
2:45 pm	*Time Average MSE Analysis for the First Order Sigma-Delta Modulator*
	Bin Han, University of Alberta
3:45 pm	*Frame Paths and Sigma-Delta Quantization*
	Vern Paulsen, University of Houston
4:30 pm	*Zero-terminated frame paths and second order sigma-delta quantization*
	Bernhard Bodmann, University of Waterloo

Tuesday, March 14

9:00 am *Digital halftoning - a model-based perspective*
Jan Allebach, Purdue University

10:00 am *On the existence of a minimal attractor in Convex Dynamics*
Tomasz Nowicki, IBM Research

11:00 am *Discrepancy and Digital Halftoning*
Benjamin Doerr, Max-Planck-Institut fuer Informatik

Wednesday, March 15

9:00 am *A Dual Frame Formula and Pseudoframes with Applications*
Shidong Li, San Francisco State University

9:45 am *Fusion Frames: Redundant Representations under Distributed Processing Requirements*
Gitta Kutyniok, Justus-Liebig-University Giessen

11:00 am *Noise shaping and predictive quantizers of order $N > 1$ for arbitrary frame expansions*
Helmut Boelcskei, Swiss Federal Institute of Technology (ETHZ)

2:00 pm *The diffusion framework: a computational approach to data analysis and signal processing on data sets*
Yosi Keller, Yale University

2:45 pm *PCM Quantization Errors and the White Noise Hypothesis*
David Jimenez, Georgia Institute of Technology

3:45 pm Discussion
Moderators: **Alex Powell**, Vanderbilt University, and **Ozgur Yilmaz**, University of British Columbia

4:30 pm Discussion
Moderators: **Sinan Gunturk**, Courant Institute of Mathematical Sciences, and **Thao Nguyen**, City College, CUNY

Thursday, March 16

9:00 am Discussion and Open Problems

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ABSTRACTS
(in alphabetic order by speaker surname)

Speaker: **Allebach, Jan** (Purdue University)
Title: *Digital halftoning - a model-based perspective*
Abstract:

Speaker: **Benedetto, John** (University of Maryland)
Title: *Sigma-Delta quantization and finite frames*
Abstract: First order Sigma-Delta quantization schemes are constructed in the setting of finite frames; and optimal quantization searches are designed. Energy norm error estimates for various quantized frame expansions are derived, and, in particular, it is shown that first order Sigma-Delta quantizers outperform the standard pulse code modulation (PCM) schemes using linear reconstruction. Mean square error estimates are comparable when consistent reconstruction methods are used in conjunction with PCM.

This presentation emphasizes the refined error estimates requiring Gunturk's theorem and some inequalities from analytic number theory.

The theory and its extension to higher order schemes is a collaboration with Alex Powell and Ozgur Yilmaz. The analytic number theoretic portion also includes Aram Tangboondouangjit in the collaboration.

Speaker: **Bodmann, Bernhard** (University of Waterloo)
Title: *Zero-terminated frame paths and second order sigma-delta quantization*
Abstract: This is joint work with Vern Paulsen. We study the performance of finite frames for the encoding of vectors by applying a standard second order sigma-delta quantization to the frame coefficients. The frames under consideration are obtained from regular sampling of a path in a Hilbert space. In order to achieve error bounds that are comparable to the results on second-order sigma-delta quantization for oversampled bandlimited functions, we construct frame paths that terminate in the zero vector.

Speaker: **Boelcskei, Helmut** (Swiss Federal Institute of Technology (ETHZ))
Title: *Noise shaping and predictive quantizers of order $N > 1$ for arbitrary frame expansions*
Abstract: We briefly review the classical paper by Tewksbury and Hallock on "Oversampled, linear predictive and noise-shaping coders of order $N > 1$". An extension of the ideas of Tewksbury and Hallock to the case of oversampled filter banks, proposed by Boelcskei and Hlawatsch in 1997, is discussed and used to outline how noise shaping and linear predictive coders can be applied to arbitrary frame expansions. The last part of the talk describes such an extension and provides a proof of the reconstruction MSE decaying as r^{-2N+1} where r denotes the frame redundancy and N is the order of the noise shaping filter.

Speaker: **Casazza, Peter** (University of Missouri)
Title: *Frame Design for Sigma-Delta Quantization*

Abstract: We will give a number of frame design conjectures/problems concerning sigma-delta quantization. The idea is to stimulate discussion during the meeting.

Speaker: **Daubechies, Ingrid** (Princeton University)
Title: *Mathematical study of coarsely quantized representations of signals: what, why, how – an overview*
Abstract:

Speaker: **DeVore, Ronald** (University of South Carolina)
Title: *Instance-optimal estimates in Compressed Sensing*
Abstract: Discrete Compressed Sensing samples a discrete signal $x \in \mathbb{R}^N$ by n linear measurement each an inner product of x with a vector from $\mathbb{R}^N$. If n is the number of measurements allocated to the sensor then the whole process can be represented by an $n \times N$ matrix Φ. The vector $y = \Phi(x)$ represents the n samples we have about x. A decoder Δ is a mapping from $\mathbb{R}^n \to \mathbb{R}^N$. The vector $\Delta(\Phi(x))$ is the approximation we have to x from the information y. We will discuss how well such an encoding-decoding scheme can perform given n, N. In particular, we ask whether there is a value k such that $\|x - \Delta(\Phi(x))\|_{\ell_p} \leq C_0 \sigma_k(x)_{\ell_p}$ where σ_k is the error in k-term approximation. If so we are interested in the largest possible value of k given the vector length N and the information budget n.

Speaker: **Doerr, Benjamin** (Max-Planck-Institut fuer Informatik)
Title: *Discrepancy and Digital Halftoning*
Abstract: In the talk, I give a brief introduction to the field of discrepancy theory. I will then show how Asano et al. modelled the digital halftoning problem as a linear discrepancy problem. Finally, I will propose a solution for this problem based on dependent randomized rounding.

Speaker: **Goyal, Vivek** (Massachusetts Institute of Technology)
Title: *Sparsity and Quantization: When Undersampling is Oversampling*
Abstract: In a variety of applications, sparse solutions to underdetermined linear systems of equations are preferred, and for at least thirty years, practitioners have used 1-norm minimizations to promote sparsity. Recent research has turned this ad hoc procedure into the basis for certain deterministic and probabilistic guarantees, fueling further work in sparsity-based modeling. This talk will relate sparsity-based modeling to the author's earlier work on quantized overcomplete expansions, showing that large improvements are obtained by exploiting boundedness of quantization error. Universality of such source coding techniques will also be discussed.

Speaker: **Han, Bin** (University of Alberta)
Title: *Time Average MSE Analysis for the First Order Sigma-Delta Modulator*
Abstract: Based on experiment and numerical simulation, it has been widely believed that the time average mean square error (MSE) in the first order sigma-delta modulator with input of bandlimited signals decays likes $O(\lambda^{-3})$ as the sampling ratio λ goes to infinity. Building on several interesting works of Daubechies, DeVore and Gunturk, we will talk about the time Average MSE Analysis for the First Order Sigma-Delta Modulator.

Speaker: **Jimenez, David** (Georgia Institute of Technology)
Title: *PCM Quantization Errors and the White Noise Hypothesis*
Abstract: The White Noise Hypothesis (WNH), introduced by Bennett half century ago, assumes that in the pulse code modulation (PCM) quantization scheme the errors in individual channels behave like white noise, i.e. they are independent and identically distributed random variables. The WNH is key to estimating the means square quantization error (MSE). But is the WNH valid? In this paper we take a close look at the WNH. We show that in a redundant system the errors from individual channels can never be independent. Thus to an extend the WNH is invalid. Our numerical experiments also indicate that with coarse quantization the WNH is far from being valid. However, as the main result of this paper

we show that with fine quantizations the WNH is essentially valid, in which the errors from individual channels become asymptotically *pairwise* independent, each uniformly distributed in $[-\Delta/2, \Delta/2)$, where Δ denotes the stepsize of the quantization.

Speaker: **Keller, Yosi** (Yale University)
Title: *The diffusion framework: a computational approach to data analysis and signal processing on data sets*
Abstract: The diffusion framework is a computational approach to high dimensional data analysis and processing. Based on spectral graph theory, we define diffusion processes on data sets. These agglomerate local transitions reflecting the infinitesimal geometries of high-dimensional dataset, to obtain meaningful global embeddings. The eigenfunctions of the corresponding diffusion operator (Graph Laplacian) provide a natural embedding of the sets into a Euclidean s pace. In this talk, we show that the eigenfunctions of the Laplacian form manifold adaptive bases, which pave the way to the extension of signal processing concepts and algorithms from $\mathbb{R}^n$ spaces to general data sets.

We exemplify this approach by applying it to collaborative filtering and improving the resolution of coarsely quantized signals.

Speaker: **Kutyniok, Gitta** (Justus-Liebig-University Giessen)
Title: *Fusion Frames: Redundant Representations under Distributed Processing Requirements*
Abstract: Nowadays frame theory is indistinguishable from applications where redundant representations of data are required. Our focus is on applications under distributed processing requirements such as sensor networks or sensorineural systems. For these applications frames can be used locally, but the global structure cannot be handled by using classical frame theory.

In this talk we will introduce the new notion of a fusion frame, which is a sequence of subspaces satisfying a frame-like property. This property will be shown to ensure that local collections of frame elements linked together by using a fusion frame yield a global frame structure. In this sense the theory of fusion frames provides the link between distributed and centralized structures.

It will turn out that the theory of fusion frames has in fact many similarities with the classical frame theory for sequences of vectors. We will further show how our theory can be used to ease the construction of frames by considering local structures. Finally, we will discuss an application of our theory to noise reduction under distributed processing requirements.

This is joint work with P. G. Casazza (University of Missouri) and S. Li (San Francisco State University).

Speaker: **Lammers, Mark** (University of North Carolina Wilmington)
Title: *Alternate Dual Frames and Sigma-Delta Quantization*
Abstract: We investigate the use of alternate dual frames in sigma-delta quantization of finite frame expansions. We prove that reconstruction with alternate dual frames can substantially reduce quantization error in many settings, including pointwise and MSE error estimates for higher order sigma-delta schemes. In particular, this gives a method for dealing with "boundary term" issues, and allows k-th order schemes to achieve pointwise error of order $1/N^k$ in situations where this is not possible using the canonical dual frame. This is joint work with Alex Powell and Ozgur Yilmaz.

Speaker: **Li, Shidong** (San Francisco State University)
Title: *A Dual Frame Formula and Pseudoframes with Applications*
Abstract: Frames are known for their flexibility and great benefits so infused in many applications. One flexibility reflects in the existence of infinite many dual frames when a frame is non-exact. A dual frame formula has been obtained, with which optimal duals can be obtained through a parametric sequence of functions. On a closely related subject, there have been observations where more freedom than that of a frame is beneficial if not necessary. An extension of frames in the notion of *pseudoframes for subspaces* (PFFS) is thereby considered. PFFS functions in a manner just like a frame for a subspace $\mathcal{X}$ in $\mathcal{H}$. Yet none of the pair of sequences $\{x_n\}$ and $\{x_n^*\}$ is necessarily contained in $\mathcal{X}$. This gives rise to properties

exactly centering around the flexibility. Characterizations, constructions and properties of PFFS will be briefly discussed.

More focus of this talk will be on examples of dual frame and PFFS applications. We show how to construct compactly supported and/or fast decaying dual Gabor functions; We show how to use the dual frame formula to construct iteratively a tight frame that is (conjectured) closest to the given (starting) frame; We explain how to construct arbitrarily compactly supported (pseudoframe) biorthogonal duals of a B-spline Riesz sequence; We demonstrate the existence of tight pseudo-duals of frames of translates; We illustrate how PFFS can be applied in the construction of biorthogonal wavelets so as to obtain results that are more favorable; We discuss how PFFS is sufficient and even necessary in a quite general optimal noise suppression problem; Finally, we shall also exhibit examples where pseudo-dual or dual frame formula can be used to reduce perturbation and/or quantization noise in a general redundant frame or pseudoframe representation so as to provide a numerical approch and confirm the similar observations by other scholars.

Speaker: **Nguyen, Thao** (City College, CUNY)
Title: *The tiling phenomenon of Sigma-Delta modulators*
Abstract: Because Sigma-Delta modulators include a quantizer in a feedback loop, it is in general extremely difficult to determine their outputs explicitly in terms of their inputs. This has forced engineers to model the quantizer error signal as an independent ”white” noise. However, in the late 90’s, an outstanding phenomenon was discovered by S.Gunturk and I.Daubechies: on certain configurations of Sigma-Delta modulators under constant inputs, the state vectors appear to remain in a tile. Not only is this phenomenon interesting from an aesthetic point of view, but it also provides new theoretical tools for the explicit and deterministic analysis of Sigma-Delta outputs.

In this talk, we give a retrospective on this research from its origin to current investigations. We will cover the following questions: (i) when is tiling observed (single-bit, multi-bit, constant and time-varying inputs), (ii) why do have tiling (theorems available today), (iii) why is tiling useful (resolution of nonlinear dynamics, spectral analysis, limit cycle prediction, MSE predictions), (iv) how can the tiles be derived (recent results on fully parameterized tiles, Farey web, the ”frog problem”).

Speaker: **Nowicki, Tomasz** (IBM Research)
Title: *On the existence of a minimal attractor in Convex Dynamics*
Abstract: Convex Dynamics are derived from a family of the piecewise affine maps (translations) and were inspired by some application in printing and other analog-to-digital (or continuous-to-discrete) coding of sequences of signals which employ error diffusion. The pieces are Voronoi regions of the corners (sets of points for which such a corner is the closest one) of some polytope(s) and the translation vectors are the the vectors from the respective corners to arbitrary points of the polytope. The fundamental theorem is that such dynamics are bounded. We show that there exists a unique minimal bounded invariant set which contains the polytope itself and all the corners of the Voronoi regions.

Joint work with Charles Tresser.

Speaker: **Paulsen, Vern** (University of Houston)
Title: *Frame Paths and Sigma-Delta Quantization*
Abstract: Building on the ideas of Benedetto-Powell-Yilmaz, we study the performance of finite frames for the encoding of vectors by applying sigma-delta quantization algorithms to the sequence of frame coefficients. We focus on frame paths, which allow the user the luxury of increasing the redundancy of the frame to compensate for the errors created by quantization. We improve on the earlier bounds that had been obtained for the worst-case error and also obtain lower bounds on the worst case error. In addition, we introduce some new frame paths that have some interesting features. (Joint work with B. Bodmann.)

Speaker: **Rauhut, Holger** (University of Vienna)
Title: *Random Sampling of Sparse Trigonometric Polynomials*
Abstract: We study the problem of reconstructing a sparse multivariate trigonometric polynomial from

few sample values. By "sparse" we mean that only a small number of Fourier coefficients of the polynomial do not vanish. However, one does not know a priori which coefficients are non-zero.

Since it is very hard (or even impossible) to come up with deterministic results, we model the sampling points as random. We use two different probability models: (a) The sampling points are independently distributed according to a continuous uniform distribution on the unit cube. (b) They are independently distributed according to a discrete uniform distribution on a finite set of equidistant sampling nodes. This corresponds to the problem of recovering a sparse vector from few samples of its discrete Fourier transform.

We study two recovery methods: (1) Basis Pursuit (BP) (2) Orthogonal Matching Pursuit (OMP)

BP consists in minimizing the ℓ^1-norm of the Fourier coefficients subject to the condition that the corresponding trigonometric polynomial matches the original one on the sampling points. This scheme was studied recently in a series of papers by E. Candes, J. Romberg and T. Tao in the context of the discrete Fourier transform. They had surprisingly good theoretic and even better numerical results about the performance of this recovery method

OMP is a greedy algorithm, which was studied recently by Tropp et al. for the recovery problem in the context of Gaussian and Bernoulli measurements. It potentially is quite faster than Basis Pursuit (at least when the sparsity is small) and easier to implement.

We present a result that applies simultanuously to both of the probability models (a) and (b) and to both BP and OMP. It says that if the number of samples N is large enough compared to the sparsity (but possibly much smaller than the overall dimension of the underlying space of trigonometric polynomials) then with high probability the polynomial can be recovered both by BP and OMP. In particular, this includes a previous result of Candes, Romberg and Tao as a special case (with a different proof).

We illustrate our observation with numerical experiments. Indeed, the methods work quite well in practice.

Speaker: **Wang, Yang** (Georgia Institute of Technology)
Title: *Sigma-Delta Quantization and the Traveling Salesman Problem*
Abstract:

In this talk we consider the maximal and mean square errors as a result of quantization. We focus on the sigma-delta modulation quantization scheme in the finite frame expansion setting. We show that this problem is related to the classical Traveling Salesman Problem (TSP) in the Euclidean space. It is known [BPY-04] that the error bounds from the sigma-delta scheme depends on the ordering of the frame elements. By examining *a priori* bounds for the Euclidean TSP we show that error bounds in the sigma-delta scheme is superior to those from the pulse code mudulation (PCM) scheme in general. We also give a recursive algorithm for finding an ordering of the frame elements that will lead to good maximal error *and* mean square error.

Speaker: **Yedlin, Matt** (University of British Columbia)
Title: *Industrial Applications of Sigma-delta Converters*
Abstract:

Analog to digital (A/D) conversion plays a central role in modern technological applications which include digital audio receivers, sampling music synthesizers, biomedical data acquisition (EEG), seismometers and wireless communication systems. Central to this A/D technology is the sigma-delta analog to digital converter (ADC). The basis for this technology is delta modulation which was first proposed in 1946 and single-bit over-sampling including noise shaping implemented in solid state in 1962. Current CMOS technology is ideal for current implementations of the sigma-delta ADC.

In this seminar, the current industrial applications of sigma-delta ADCs will be presented, with a range from low to high frequency and corresponding resolution. A number of currently available commercial products will be quickly profiled, ranging from the 19 Hz 24 bit AD (Analog Devices) 7783 which is used for pressure and temperature sensing to a high-speed sigma-delta converter, AD 7725 with an input bandwidth of 350kHz, used for radar and sonar data acquisition.

Finally, a hardware presentation of the AD7725 sigma-delta converter, courtesy of the university program sponsored by Analog Devices, Inc., will illustrate some of the issues in using these devices.

The foregoing will be followed by a summary and a description of future possible applications.