Topological graph theory and crossing numbers

Bojan Mohar (Simon Fraser University),
Janos Pach (Courant Institute and City College),
Bruce Richter (University of Waterloo),
Robin Thomas (Georgia Institute of Technology),
and
Carsten Thomassen (Technical University of Denmark)

October 21–26, 2006

1 Objectives

The main objective of this workshop is to bring together two groups of researchers, those working in topological graph theory and graph minors, and those working with crossing numbers. Both areas have developed methods, mathematical tools and powerful results that have great potential for being used in the other area. For instance, the most basic open problem about crossing numbers is the Turan's Brickyard problem. Would it be possible to use results about the genus of graphs and graph minors to get some new insight into this problem? On the other hand the study of crossing numbers of graphs on nonsimply connected surfaces may yield new results of interest for the topological graph theory.

We plan to organize some survey lectures where best mathematicians from both areas will present the current state of the art of the theory. Additionally, there will be corresponding problem sessions with intention to motivate the participants to apply their knowledge towards problems in the other area.

2 Overview of the Field

Roots of the Topological Graph Theory lie in the Heawood problem, one of very early discovered generalizations of the Four Color Problem. Heawood [4] proved in 1890 that every graph embedded in a closed surface of Euler characteristic $c \neq 2$ can be colored with $H(c) = \lfloor \frac{7+\sqrt{49-24c}}{2} \rfloor$ colors. However, it was left open for another 78 years if that many colors are really needed. G.A. Dirac proved in the 1950's that the answer to this question is equivalent to the fact that the genus (and the nonorientable genus) of the complete graph of order n is equal to $\lceil \frac{1}{12}(n-3)(n-4) \rceil$ (and $\lceil \frac{1}{6}(n-3)(n-4) \rceil$, respectively). Ringel and Youngs solved this problem completely in 1968, cf. [6]. Their solution and related works by other authors motivated further extensive research on embeddings of graphs in surfaces. The books by White [13] and later by Gross and Tucker [3] show the state of the art of the theory at the end of the 1980's.

In the late 1980's, two new directions of research brought additional insight and boosted the topological graph theory into even higher levels. The most important results came from Robertson and Seymour's theory of graph minors. Very influential was also Thomassen's work in which he first presented stimulating new proofs of the fundamentals of the theory, and later produced a number of deep results, most notably related to colorings of graphs on surfaces. These new developments are now accessible in a monograph by Mohar and Thomassen [5].

Today, topological graph theory and the related theory of graph minors are battling its way into the area of computer science. Applications have been discovered in areas like computational complexity, theory of algorithms, graph drawing, computer graphics, computer vision, etc. This is the direction for which we believe that important advances will be made in the future.

One particular branch of topological graph theory, the crossing number problems, has received particular attention in the last decade. Discoveries of F.T. Leighton in the early 1980's made this area of high importance in the theoretical computer science. Stunning discoveries made by Pach, Szekely, and many others in the 1990's have advanced the theory of crossing numbers into an independent subject with many applications in discrete geometry, combinatorics and computer science.

This workshop has brought together mathematicians working in these two areas with intention to meet together, exchange ideas, and use recent advances in both subjects to create new results on the extended grounds.

3 List of participants

Ackerman, Eyal, Simon Fraser University
Albertson, Mike, Smith College
Bokal, Drago, University of Waterloo
Bruhn, Henning, Universitt Hamburg
Cabello, Sergio, University of Ljubljana
Christian, Robin, University of Waterloo
Debowsky, Marisa, Courant Institute
DeVos, Matt, Simon Fraser University
Ebrahimi B., Javad, Simon Fraser University
Ellingham, Mark, Vanderbilt University
Fox, Jacob, Princeton University
Goddyn, Luis, Simon Fraser University
Hajiaghayi, MohammadTaghi, Carnegie Mellon University
Hlineny, Petr, Masaryk University
Hutchinson, Joan, Macalester College
Kawarabayashi, Ken-ichi, National Institute of Informatics
Kral, Daniel, Charles University
Mohar, Bojan, Simon Fraser University
Norine, Serguei, Georgia Institute of Technology
Oum, Sang-il, Georgia Institute of Technology
Pach, Janos, Courant Institute and City College
Pelsmajer, Michael, Illinois Institute of Technology
Pikhurko, Oleg, Carnegie Mellon University
Pinchasi, Rom, Technion - Israel Institute of Technology
Richter, Bruce, University of Waterloo
Robertson, G. Neil, Ohio State University
Salazar, Gelasio, Universidad Autonoma de San Luis Potosi
Schaefer, Marcus, DePaul University
Shahrokhi, Farhad, University of North Texas
Solymosi, Jozsef, University of British Columbia
Song, Zixia, University of Central Florida
Szekely, Laszlo, University of South Carolina
Tardos, Gabor, Simon Fraser University
Thomas, Robin, Georgia Institute of Technology
Thomassen, Carsten, Technical University of Denmark
Vodopivec, Andrej, University of Ljubljana

Yerger, Carl, Georgia Institute of Technology

4 Scientific program

The program has been composed of long lectures (mostly surveys), short lectures (important new results), open problems sessions, and five-minute lectures. Each participant (except some of those giving another talk) have been asked to present a short summary of their work of no more than five minutes in duration. Although this is hard to obey, the time limit was strictly enforced (with relatively good success). The purpose was to get to know each other's research programs and interests so that like-minded individuals can pursue informal interactions. This applied to all participants from senior researchers to graduate students.

Saturday, October 21, 2006

17:30–19:30 Dinner
19:30–24:00 Informal gathering in Corbett Hall lounge

Sunday, October 22, 2006

09:00–09:15 Introduction and Welcome by the BIRS Station Manager
09:15–10:00 Carsten Thomassen, Planar representations of finite and infinite graphs
10:00–10:30 Coffee
10:30–12:15 Five-minute lectures
12:15–14:00 Lunch
14:00–14:40 Gelasio Salazar, A biased survey on crossing numbers, plus two doable important open problems
14:40–14:45 Short break
14:45–15:30 Five-minute lectures
15:30–16:00 Coffee
16:00–17:30 Five-minute lectures
17:30–19:30 Dinner
19:30–24:00 Informal gathering in Corbett Hall lounge

Monday, October 23, 2006

09:00–10:00 MohammadTaghi Hajiaghayi, Algorithmic graph minor theory
10:00–10:30 Coffee
10:30–11:30 Kenichi Kawarabayashi, Linear-time algorithm for computing crossing number
11:30–11:40 Short break
11:40–12:00 Matt DeVos, Describing Fullerenes
12:00–14:00 Lunch
14:00–17:30 Free
17:30–19:30 Dinner
19:30–24:00 Informal gathering in Corbett Hall lounge

Tuesday, October 24, 2006

09:00–10:00 Janos Pach, Extremal Graph Theory and Geometric Graphs
10:00–10:30 Coffee
10:30–11:15 Marcus Schaefer, Graphs with rotation
11:20–11:40 Laszlo Szekely, On lower bounds for the minor crossing number
11:45–12:05 Michael Albertson, Distinguishing labelings of geometric graphs
12:00–14:00 Lunch
14:00–15:00 Gabor Tardos, Extremal Theory of Topological Graphs

15:00–15:30 Coffee
15:30–16:15 Jacob Fox, Ramsey-type results for intersection graphs
16:20–16:40 Petr Hlineny, Crossing number of almost planar graphs
16:50–17:10 Drago Bokal, Crossing-critical graphs with presecribed average degree and crossing number
17:30–19:30 Dinner
19:30–24:00 Informal gathering in Corbett Hall lounge

Wednesday, October 25, 2006

09:00–09:45 Neil Robertson, My favorite open problems in Topological Graph theory
09:50–10:20 Henning Bruhn, MacLane's criterion for higher surfaces
10:20–10:50 Coffee
10:50–11:10 Bruce Richter, Cycle spaces of infinite graphs
11:15–11:35 Mark Ellingham, The orientable genus of some joins of complete graphs with large edgeless graphs
Afternoon: Work in groups

Thursday, October 26, 2006

No scheduled talks.

5 Abstracts of talks

On the Crossing Number of Almost Planar Graphs

Petr Hlineny and Gelasio Salazar

Crossing minimization is one of the most challenging algorithmic problems in topological graph theory, with strong ties to graph drawing applications. Despite a long history of intensive research, no practical good algorithm for crossing minimization is known (that is hardly surprising, since the problem itself is NP-complete). Even more surprising is how little we know about a seemingly simple particular problem: to minimize the number of crossings in an almost planar graph, that is, a graph with an edge whose removal leaves a planar graph. This problem is in turn a building block in an "edge insertion" heuristic for crossing minimization. In this paper we prove a constant factor approximation algorithm for the crossing number of almost planar graphs with bounded degree. On the other hand, we demonstrate nontriviality of the crossing minimization problem on almost planar graphs by exhibiting several examples, among them new families of crossing critical graphs which are almost planar and projective.

MacLane's criterion for higher surfaces

Henning Bruhn and Reinhard Diestel

MacLane's planarity criterion can be seen as listing a number of properties of the facial cycles of a plane graph which, together, are strong enough to imply the following: that whenever we have any collection of cycles with these properties and attach a 2-cell to each of them, the 2-complex obtained is homeomorphic to the sphere. We shall provide such a list for higher surfaces. The difficulty here does not lie in the proof but rather in finding properties that are as simple and natural as possible.

Ramsey-type results for intersection graphs

Jacob Fox

The intersection graph of a collection C of sets has vertex set C and two elements of C are adjacent if and only if they have nonempty intersection. J. Pach, Cs. Toth, and I recently proved several Ramsey-type results for intersection graphs of geometric objects that are outlined below.

(1) There is a positive constant c such that for every intersection graph G of $n > 1$ convex bodies in the plane, G or its complement contains a complete bipartite graph with at least cn vertices in each of its vertex classes.

(2) An arrangement of pseudosegments is a collection of continuous arcs in the plane such that no pair cross more than once. There is a positive constant c such that the intersection graph of any arrangement of n pseudosegments in the plane contains a clique or independent of size at least n^c.

(3) An x-monotone curve is a continuous arc in the plane such that no vertical line intersects it in more than one point. There is a positive constant c such that for every intersection graph G of $n > 1$ x-monotone curves in the plane, G contains a complete bipartite graph on at least $cn/\log n$ vertices in each of its vertex classes or the complement of G contains a complete bipartite graph with at least cn vertices in each of its vertex classes. In the other direction, Pach and G. Tóth showed that for each $\epsilon > 0$ and n sufficiently large, there is an intersection graph G of a collection of n x-monotone curves in the plane that does not contain a complete bipartite graph with at least $\frac{14n}{\epsilon \log n}$ vertices in each of its vertex classes and every vertex of G is adjacent to all but at most n^ϵ other vertices.

(4) For each positive integer k, there is a positive constant c_k such that for every intersection graph G of $n > 1$ x-monotone curves in the plane with no pair intersecting in more than k points, G or its complement contains a complete bipartite graph with at least $c_k n$ vertices in each of its vertex classes.

The above results are proved using structural theorems that demonstrate close relationships between certain families of intersection graphs of geometric objects and cocomparability graphs.

Turán-Type Results for Arrangements of Curves

J. Pach and M. Sharir

Let C be a family of n compact connected sets in the plane, whose intersection graph $G(C)$ has no complete bipartite subgraph with k vertices in each of its classes. Then $G(C)$ has at most n times a polylogarithmic number of edges, where the exponent of the logarithmic factor depends on k. In the case where C consists of convex sets, we improve this bound to $O(n \log n)$. If in addition $k = 2$, the bound can be further improved to $O(n)$.

Linear time algorithm for computing crossing number

Ken-ichi Kawarabayashi and Bruce Reed

We show that for every fixed k, there is a linear time algorithm that decides whether or not a given graph has crossing number at most k, and if this is the case, then the algorithm computes a drawing of the graph into the plane with at most k crossings. This answers the question posed by Grohe (STOC01 and JCSS 2004). Our algorithm can be viewed as a generalization of the seminal result by Hopcroft and Tarjan, which says that planarity of graphs can be decided in linear time. Our algorithm can also be compared with the algorithm by Mohar (STOC96 and Siam J. Disrete Math 2001), which says that there is a linear time algorithm for a given graph G to give either an embedding of G into a fixed surface S, i.e, Euler genus k for fixed k, or a minimal forbidden subgraph for embeddability in S. Our algorithm has several appealing features. First, unlike the algorithm by Grohe, our algorithm does not involve any huge hidden constant. In fact, the time complexity can be written as $O(2^{O(k^4)}n)$. This is because our proof does not depend on the excluded grid minors theorem, which is the case in Grohe. Second, our algorithm consists of several interesting ingredients. It uses a deep result of Mohar. It also uses techique by Reed, Robertson, Schrijver and Seymour, which improves the time complexity of the seminal result of Robertson and Seymour for planar graphs and graph on a fixed graph. The algorithm also needs a deep result in discrete geometry. Finally, it uses the algorithm by Bodlaender which shows how to give linear time algorithms for many NP-hard problems in graphs of bounded tree-width. Third, we can apply our algorithm to other problem. For instance, given a graph G and fixed k. Can we make G planar after deleting at most k edges? Our algorithm does give a linear time algorithm for this problem.

Describing Fullerenes

Matt DeVos and Bojan Mohar

With a notion of a polyhedral surface, one can use a fundamental result of Alexandrov to represent various families of planar graphs in a particularly simple geometric way. In particular, we show how this can be done for all triangulations of the sphere with maximum vertex degree 6. Dually, this applies to cubic planar graphs of maximum face size 6. In particular, this representation can be used for the class of fullerenes, cubic planar graphs whose face sizes are only 5 or 6.

An independent discovery of essentially the same result was obtained earlier by Thurston.

My interests in Topological graph theory

Serguei Norine

I am interested in topological description of Pfaffian and k-Pfaffian graphs along the lines of the following conjecture of mine. Conjecture 1. For a graph G and a non-negative integer g the following are equivalent

(1) There exists a drawing of G on an orientable surface of genus g such that the number of pairwise crossings of edges of M is even for every perfect matching M of G.

(2) G is $4g$-Pfaffian.

(3) G is $(4g + 1 - 1)$-Pfaffian.

I have been able to prove that the above conjecture holds for $g = 0$, that (1) implies (2) in general, and that (1) and (2) are equivalent for $g = 1$. However, for other implications the method I used breaks down for $g > 1$ and new ideas are required to settle the conjecture.

I am also interested in applying methods used in proofs of the subcases of the above conjecture to the theory of crossing numbers and in unifying various existing approaches to studying the parity of crossing numbers. In particular, I want to study the hypergraph problem described below, which is related to Turans brickyard problem.

Cycle spaces of infinite graphs

Bruce Richter

Recent work by several authors have developed the theory of cycle spaces of infinite graphs to usefully allow infinite circuits. A single locally finite graph can have many different compactifications. Each of these has its own, typically different, cycle space. If compactification C_1 is the continuous image of compactification of C_2, preserving the graph itself, then the C_2-cycle space contains the C_1-cycle space. What is a basis for the quotient space? There are many other questions one could ask.

2-crossing-critical graphs

Bruce Richter

Bogdan Oporowski has made substantial progress on trying to determine all 2-crossing-critical graphs. I have gotten sucked into this project. But I am also interested in determining all large 3-crossing-critical graphs. (A graph is k-crossing-critical if its crossing number is at least k and all proper subgraphs have crossing number ¡ k. The graph $C_3 \times C_3$ has crossing number 3, but all its proper subgraphs have crossing number at most 1, so it is both 2-crossing-critical and 3-crossing-critical.)

Some recent work on Topological Graph Theory

Robin Thomas

With M. DeVos, R. Hegde, K. Kawarabayashi, S. Norine and P. Wollan we have shown:

THEOREM. There exists an integer N such that every 6-connected graph G with no K_6 minor has a vertex v such that $G - v$ is planar.

Jorgensen conjectured that the above statement holds for all 6-connected graphs regardless of size. That remains open. The theorem suggests the following conjecture.

CONJECTURE. For every integer t there exists an integer N such that every t-connected graph G with no K_t minor has a set X of at most $t - 5$ vertices such that $G - X$ is planar.

Here the assumption that G be big is necessary. Notice that if G has a set X as above, then it has no K_t minor. The excluded clique theorem of Robertson and Seymour gives structural information about graphs with no K_t minor, but not enough to easily deduce the above conjecture.

With Daniel Kral we have obtained a polynomial-time algorithm to compute the chromatic number of a graph G embedded on a fixed surface S with all faces even. This is a warm-up for the more general and still open problem whether the same can be done for triangle-free graphs, regardless of face sizes. An exact statement of our theorem requires a number of definitions, and so let me state it informally. We need a definition and an observation. Let C be a cycle with vertex-set $\{v_1, \ldots, v_k\}$ in order, let $v_0 = v_k$, and let $c : V(C) \to \{1, 2, 3\}$ be a (proper) 3-coloring of C. We define $w(C)$, the winding number of C, as the number of indices $i \in \{1, \ldots, k\}$ such that $c(v_i) = 1$ and $c(v_{i-1}) = 2$ minus the number of indices i such that $c(v_i) = 2$ and $c(v_{i-1}) = 1$. Now let G be a graph embedded in an orientable surface S, let $C_1, \ldots, C_k$ be facial cycles, and assume that every other face is bounded by a cycle of length four. Then if c is a 3-coloring of G, then $\sum_{i=1}^{k} w(C_i) = 0$. In other words, if some faces are precolored and all other faces are bounded by cycles of length four, then a necessary condition for the precoloring to extend to a 3-coloring of G is that the sum of the winding numbers of the precolored faces be zero. We will call this the winding number condition. Our theorem says that for every surface S (orientable or not) there exists an integer N such that for every graph G embedded in S with all faces even there exists a subgraph H of G on at most N vertices such that for every face f of H:

(i) if f includes a face of G with boundary length exceeding four, then every 3-coloring of the boundary of f extends to a 3-coloring of the subgraph G_f of G contained in f, and

(ii) otherwise a 3-coloring of the boundary of f extends to a 3-coloring of G_f if and only if it satisfies the winding number condition.

A 3-coloring algorithm follows: for every 3-coloring of H test if it extends into every face of H.

Summary of my recent research activities related to the topic of the workshop

Daniel Kral'

I am working in the area of graph colorings and I have recently started being interested a lot in colorings of graphs embedded in surfaces. As examples of my recent work, let me name the following results:

Theorem 1 (K., Mohar, Nakamoto, Pangrac, Suzuki) An Eulerian triangulation of the Klein bottle is 5-colorable unless it contains a complete graph of order six as a subgraph.

Theorem 2 (K., Stehlik) Every triangle-free graph on the double-torus is 4-colorable.

Theorem 3 (K., Thomas) A triangle-free quadrangulation of the torus is 3-colorable if and only if it does not contain a Cayley graph for the group Z_{13} with generators 1 and 5 as a subgraph.

Theorem 4 (K., Thomas) For every fixed $g \geq 0$, the chromatic number of an even-faced graph embedded in the orientable surface of genus g can be determined in polynomial time.

Planar representations of finite and infinite graphs

Carsten Thomassen

Results on representations of finite planar graphs may be non-trivial to extend to the infinite case. We have recently found an extension method which applies to rectangular representations, bar representations and visibility graphs.

A biased survey on crossing numbers, plus two doable important open problems

Gelasio Salazar

We will try to cover most of the mainstream avenues of research in Crossing Numbers, including the work that has steadily led to the study of crossing numbers in nonplanar surfaces.

Regarding open problems, I'll take Donald Knuth's position when a Computer Science student asked him, during a lecture at Munchen: What are the 5 most important open problems in Computer Science? Knuth replied: "I don't like this "top ten" business. It's the bottom ten that I like. You've got to go for the little things, the stones that make up the wall". I have two serious candidates for the "bottom ten".

Graphs with Rotation

Michael Pelsmajer, Marcus Schaefer, Daniel Štefankovič

A *rotation system* for a graph specifies a clockwise ordering of incident edges at each vertex. Rotation systems have traditionally been used to describe embeddings of graphs in surfaces. We have recently applied them in various graph drawing problems concerned with *imbeddings* of graphs: odd crossing number, minor-monotone crossing number, the Hanani-Tutte theorem, and generalized thrackles.

The crossing number, $\texttt{cr}(G)$, of a graph G is the smallest number of crossings in any drawing of the graph.[1] The *odd crossing number*, $\texttt{cr}_{odd}(G)$, is the smallest number of pairs of edges that cross an odd number of times in any drawing. Obviously, $\texttt{cr}_{odd}(G) \leq \texttt{cr}(G)$. We were able to show that for every $\epsilon > 0$, there is a graph G such that $\texttt{cr}_{odd}(G) < (\sqrt{3}/2 + \epsilon)\texttt{cr}(G)$ (2005).

The proof uses a multigraph on two vertices with rotation. Combining contractions with rotations, we can also show that

$$\texttt{cr}(G) \leq \texttt{cr}_{odd}(G)\binom{n+4}{4}/5$$

for a multigraph on n vertices.

Call an edge in a drawing *even* if it intersects every other edge an even number of times.

Theorem 1 (Hanani-Tutte) *If a non-planar graph is drawn in the plane, then the drawing contains two non-adjacent edges that intersect an odd number of times.*

We give a new and direct geometric proof of this result which, in turn, is based on a strengthening of a result by Pach and Tóth:

Theorem 2 (Pelsmajer, Schaefer, Štefankovič) *If D is a drawing of G in the plane, and E_0 is the set of even edges in D, then G can be drawn in the plane so that no edge in E_0 is involved in an intersection and there are no new pairs of edges that intersect an odd number of times.*

The proof of this result relies on a geometric redrawing idea, using rotations in an essential way. This result leads to an easy proof of another result of Pach and Tóth: $\texttt{cr}(G) \leq 2\texttt{cr}_{odd}(G)^2$. It also allows us to prove that odd crossing number and crossing number are the same for small values: $\texttt{cr}(G) = \texttt{cr}_{odd}(G)$ for $\texttt{cr}_{odd}(G) \leq 3$. The case analysis in the proof again uses rotation systems.

A graph is a *thrackle* if it can be drawn such that any pair of edges intersects exactly once, where a common endpoint of two edges counts as an intersection of these two edges. A *generalized thrackle* is a graph that can be drawn such that any pair of edges intersects an odd number of times (again counting endpoints).

As it turns out, our proof techniques allow us another simple proof of the following well-known result.

Theorem 3 (Cairns, Nikolayevsky (2000)) *If G is a bipartite, generalized thrackle on a surface of genus g, then G can be embedded on that surface.*

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[1] No more than two edges are allowed to cross in a point, and edges cannot pass through vertices.

The special case $g = 0$ of the theorem was first proved by Lovász, Pach, and Szegedy, 1997: if a bipartite graph is a generalized thrackle, then it is planar.

We can naturally ask about the complexity of problems—such as the crossing number—for graphs with rotation system. We can show that computing the crossing number (or odd-crossing number or pair-crossing number) of a graph with rotation system is **NP**-hard. As a corollary we obtain Hliněný's result (2004) that computing the crossing number of a cubic graph (without rotation system) and computing the minor-monotone crossing number is **NP**-complete.

If we restrict the number of vertices to 1 or 2, the crossing number problem (with rotation) lies in **P**. The case of three vertices is open. For pair crossing number even the case $k = 2$ is open.

Crossing-critical graphs with prescribed average degree and crossing number

Drago Bokal

Crossing-critical graphs were introduced by Širáň, who proved existence of infinite families of 3-connected k-crossing-critical graphs for every $k \geq 3$. Kochol proved existence of infinite families of simple 3-connected k-crossing-critical graphs, $k \geq 2$. Richter and Thomassen started the research on degrees in crossing-critical graphs by proving that there are only finitely many simple k-crossing-critical graphs with minimum degree r for every two integers $r \geq 6$ and $k \geq 1$. Salazar observed that their argument implies the same conclusion for every rational $r > 6$, integer $k \geq 1$, and simple k-crossing-critical graphs with average degree r. For every rational $r \in [4, 6)$ he proved existence of an infinite sequence $\{k_{r,i}\}_{i=0}^{\infty}$ such that for every $i \in \mathbb{N}$ there exists an infinite family of simple 4-connected $k_{r,i}$-crossing-critical graphs with average degree r and asked about existence of such families for rational $r \in (3, 4)$. The question was partially resolved by Pinontoan and Richter, who answered it positively for $r \in (3\frac{1}{2}, 4)$.

In the talk, we extend the theory of tiles, developed by Pinontoan and Richter, to encompass a generalization of the crossing-critical graphs constructed by Kochol. Combining tiles with a new graph operation, the zip product, which preserves the crossing number of the involved graphs, we settle the question of Salazar and combine the answer with the results of Širáň and Kochol into the following theorem: there exists a convex continuous function $f : (3, 6) \to \mathbb{R}^+$, such that, for every rational number $r \in (3, 6)$ and every integer $k \geq f(r)$, there exists an infinite family of simple 3-connected crossing-critical graphs with average degree r and crossing number k.

Recent and current work on geometric and topological graphs

Michael O. Albertson

My most recent work (with Debra Boutin) concerns distinguishing labelings of geometric graphs. A labeling of a graph $f : V(G) \to \{1, 2, \ldots, d\}$ is said to be d-*distinguishing* if no nontrivial automorphism of G preserves the labels. The *distinguishing number* of a graph G, denoted by $\mathrm{Dist}(G)$, is the minimum d such that G has a d-distinguishing labeling.

An automorphism of a geometric graph that preserves both crossings and noncrossings of edges is called a *geometric automorphism*. We prove two theorems constraining the action of a geometric automorphism on the boundary of the convex hull of a geometric clique. First, any geometric automorphism that fixes the boundary of the convex hull fixes the entire clique. Second, if the boundary of the convex hull contains at least four vertices, then it is invariant under every geometric automorphism.

We use the above results, and the theory of determining sets, to prove that certain geometric cliques are 2-distinguishable. A subset of vertices is a *determining set* for a graph if every automorphism is uniquely determined by its action on this subset. The main theorem connecting these concepts says that a graph is d-distinguishable if and only if it has a determining set that can be $(d-1)$-distinguished. These ideas readily extend to geometric graphs. Using these results, we prove that if $(n \geq 7)$ and $\overline{K}_n$ is a geometric clique in which the boundary of the convex hull contains at least 4 vertices, then $\mathrm{Dist}(\overline{K}_n) \leq 2$. In prior work we had shown that for $n \geq 6$ there does exist a rigid $\overline{K}_n$. We have conjectured that $\mathrm{Dist}(\overline{K}_n) \leq 2$ when $n \geq 7$ and the boundary of the convex hull of $\overline{K}_n$ is a triangle. We know from prior work that $\mathrm{Dist}(\overline{K}_n) \leq 3$ in these circumstances.

Current projects include

1. Geometric automorphisms of other families of geometric graphs;
2. Graphs (geometric graphs) with large distinguishing number;
3. Edge coloring embedded graphs; and
4. Induced acyclic subgraphs of embedded graphs.

Distances in embedded graphs

Sergio Cabello (joint work with Erin W. Chambers)

We give an $O(g^2 n \log n)$ algorithm to represent the shortest path tree from all the vertices on a single specified face f in a genus g graph. From this representation, any query distance from a vertex in f can be obtained in $O(\log n)$ time. This generalizes a result of Klein (SODA'05) for plane graphs. We also show how to use these shortest path trees to find a shortest non-contractible cycle and a shortest non-separating cycle in a graph embedded in an orientable surface in $O(g^3 n \log n)$ time.

The orientable genus of some joins of complete graphs with large edgeless graphs

Mark Ellingham and Chris Stephens

In an earlier paper the authors showed that with one exception the nonorientable genus of the graph $\overline{K_m} + K_n$ with $m \geq n-1$, the join of a complete graph with a large edgeless graph, is the same as the nonorientable genus of the spanning subgraph $\overline{K_m} + \overline{K_n} = K_{m,n}$. The orientable genus problem for $\overline{K_m} + K_n$ with $m \geq n-1$ seems to be more difficult, but here we find the orientable genus of some of these graphs. In particular, we determine the genus of $\overline{K_m} + K_n$ when n is even and $m \geq n$, the genus of $\overline{K_m} + K_n$ when $n = 2^p + 2$ for $p \geq 3$ and $m \geq n-1$, and the genus of $\overline{K_m} + K_n$ when $n = 2^p + 1$ for $p \geq 3$ and $m \geq n+1$. In all of these cases the genus is the same as the genus of $K_{m,n}$, namely $\lceil (m-2)(n-2)/4 \rceil$.

Embedding metrics into constant-dimensional geometric spaces

MohammadTaghi Hajiaghayi

Embedding metrics into constant-dimensional geometric spaces, such as the Euclidean plane, is relatively poorly understood. Motivated by applications in visualization, ad-hoc networks, and molecular reconstruction, we consider the natural problem of embedding shortest-path metrics of unweighted planar graphs (planar graph metrics) into the Euclidean plane. It is known that, in the special case of shortest-path metrics of trees, embedding into the plane requires $T(vn)$ distortion in the worst case, and surprisingly, this worst-case upper bound provides the best known approximation algorithm for minimizing distortion. We answer an open question posed in this work and highlighted by Matousek by proving that some planar graph metrics require $O(n^{2/3})$ distortion in any embedding into the plane, proving the first separation between these two types of graph metrics. We also prove that some planar graph metrics require $O(n)$ distortion in any crossing-free straight-line embedding into the plane, suggesting a separation between low-distortion plane embedding and the well-studied notion of crossing-free straight-line planar drawings. Finally, on the upper-bound side, we prove that all outerplanar graph metrics can be embedded into the plane with $O(vn)$ distortion, generalizing the previous results on trees (both the worst-case bound and the approximation algorithm) and building techniques for handling cycles in plane embeddings of graph metrics.

List-coloring classes of planar graphs when the lists vary in size

Joan P. Hutchinson

We prove the following theorem. If a graph is a 2-connected outerplanar near-triangulation and a list assignment L satisfies $|L(v)| \geq \min\{\deg(v), 5\}$ for every vertex v, then the graph is L-list-colorable except for K_3 with three identical 2-lists.

Connectivity cannot be reduced and the result does not hold for 2-connected K_4-minor-free graphs. The bound of five is best possible due to an example of A. Kostochka.

Another theorem is: If a 2-connected bipartite outerplanar graph has $|L(v)| \geq \min\{\deg(v), 4\}$ for every vertex v, then the graph is L-list-colorable.

This result does not hold for non-bipartite graphs, for 1-connected graphs, nor for K_4-minor-free graphs, and the bound of four cannot be decreased to three. We have conjectures about other variations in which this sort of list-coloring might be possible.

Pivot-vertex-minors of graphs

Sang-il Oum

Our main research interest is on graph structure theory related to pivot-vertex-minors of graphs. Local complementation at a vertex v of a graph is an operation to replace the subgraph of G induced by neighbor of v by its complementary graph. A vertex-minor of a graph is a graph obtained by local complementation and vertex deletion. Pivoting an edge uv is an operation to toggle adjacency of two vertices in different sets among three sets of vertices defined by adjacency to u and v (common neighbors of u, v, neighbor of u but nonneighbor of v, or neighbor of v but nonneighbor of u). A pivot-minor is a graph obtained by pivoting and vertex deletion.

The Kuratowski-Wagner theorem states that a graph is planar if and only if it has no minor isomorphic to K_5 or $K_{3,3}$. A circle graph is an intersection graph of chords of a circle. Bouchet proved in 1994 that a graph is a circle graph if and only if it has no “vertex-minor” isomorphic to one of three graphs.

With Jim Geelen, we proved that there are finitely many excluded “pivot-minors” for circle graphs, and obtained 15 excluded pivot-minors by computer search. In particular, this implies Kuratowski-Wagner theorem.

Circle graphs have a quite topological feature. De Fraysseix (1984) showed that a bipartite graph is a circle graph if and only if it is a fundamental graph of a cycle matroid of a planar graph. In this sense, planar graphs are strongly related to circle graphs.

We would be very interested to know any other graph classes that are pivot-minor-closed or vertex-minor-closed. Are there any examples arising from surfaces other than sphere? I know that bipartite graphs, circle graphs, distance-hereditary graphs, PU-orientable graphs, graphs of rank-width at most k are closed under pivot-minors.

On lower bounds for the minor crossing number

Drago Bokal, Éva Czabarka, László A. Székely, and Imrich Vrťo

The *minor crossing number* of a graph G is defined as the minimum crossing number of all graphs that contain G as a minor:

$$mcr(G) := \min\{cr(H) \;:\; G \leq_m H\}.$$

This concept was introduced by D. Bokal, G. Fijavž, and B. Mohar in “The minor crossing number”, to appear in *SIAM J. Discrete Mathematics*. The point of this definition is that the family of graphs with minor crossing number at most k is minor closed, unlike the family of graphs with ordinary crossing number at most k. Bokal et al. showed for the complete graph $(1 + o(1))n^2/4 \leq mcr(K_n) \leq (1 + o(1))n^2/2$, and for the hypercube $4^n \frac{1+o(1)}{5n^2} \leq mcr(Q_n) \leq 2(1 + o(1)) \cdot 4^{n-2}$.

We improve on the bounds for the hypercube by showing $4^n \frac{1+o(1)}{3263n} \leq mcr(Q_n) \leq 4^n \frac{1+o(1)}{\sqrt{\pi n}}$. The key tool for establishing the lower bound is an adaptation of the bisection width lower bound to the minor crossing number.

As the cited result for K_n shows, one cannot expect the Leighton Lemma lower bound for $mcr(G)$, as $mcr(K_n) = O(n^2)$. We present an adaptation of the Leighton Lemma that still might be useful, and also discuss the adaptation of the embedding method to set lower bound for $mcr(G)$.

Ramsey-Type Results for Arrangements of Curves

Janos Pach (joint with J. Fox and Cs. Tóth)

An arrangement of *pseudosegments* is a family of continuous curves such that any pair of them cross at most once. We prove that

1. there is a positive constant c such that the intersection graph of any arrangement of n pseudosegments contains a clique or independent set of size at least n^c.
2. there is a positive constant c such that if G is the intersection graph of $n > 1$ convex compact sets in the plane, then G or its complement contains a complete bipartite graph with cn vertices in each of its vertex classes.

An x*-monotone* curve is a continuous curve in the plane that is intersected by any vertical line in at most one point. We prove that there is a positive constant c such that for every intersection graph G of n x-monotone curves in the plane, G contains a complete bipartite graph with at least $\frac{cn}{\log n}$ vertices in each of its vertex classes or the complement of G contains a complete bipartite graph with at least cn vertices in each of its vertex classes.

Degenerate Crossing Numbers

J. Pach and G. Tóth

Let G be a graph with n vertices and $e \geq 4n$ edges, drawn in the plane in such a way that if two or more edges (arcs) share an interior point p, then they must properly cross one another at p. It is shown that the number of crossing points, counted without multiplicity, is at least constant times e and that the order of magnitude of this bound cannot be improved. If, in addition, two edges are allowed to cross only at most once, then the number of crossing points must exceed constant times $(e/n)^4$.

6 Carsten Thomassen's exposition about Planar representations of finite and infinite graphs

Many results and problems on finite graphs have natural counterparts for infinite graphs. One of the most useful tools for going from finite graphs to infinite graphs is the following which is called König's infinity lemma.

Theorem 4 *Let v be a vertex in an infinite connected graph K, and let $D_1, D_2, \ldots$ be the distance classes from v. Then K has a one-way infinite path $vv_1v_2\ldots$ such that v_i belongs to D_1 for $i = 1, 2, \ldots$.*

One of the first applications (if not the first) is the extension of Kuratowski's theorem which characterizes the finite planar graphs in terms of forbidden subgraphs. This was generalized to infinite graphs by P. Erdős, see [2], as follows.

Theorem 5 *Let G be a countably infinite graph such that every finite subgraph is planar. Then G can be drawn in the plane such that no two edges intersect except at a common end.*

To prove this, Erdos constructed an auxiliary graph K as follows. First the vertices of G are enumerated $x_1, x_2, \ldots$. Then G_n denotes the subgraph induced by $v_1, v_2, \ldots, v_n$. Two planar embeddings of G_n are said to be *equivalent* if there is a homeomorphism of the plane taking one to the other. Then there are only finitely many non-equivalent embeddings. Each of them will be a vertex in a set D_n. The vertex set of the graph K is the union of all the sets D_n, $n = 1, 2, \ldots$. If we delete x_n from an embedding of G_n, then we obtain an embedding of G_{n-1}, and we add an edge between these embeddings in K. Then we apply König's infinity lemma to K, and we use the resulting path in K to draw successively $x_1, x_2 \ldots$ such that each drawing is equivalent with the corresponding vertex in the path of K.

Numerous results on finite graphs can be extended to infinite graphs by the same argument. However, the limitations of the method are perhaps more interesting. For example, it is a well-known result on finite graphs (also attributed to Erdős) that every finite graph has a so-called *unfriendly partition*, that is, a vertex partition such that each vertex has at least as many neighbors in the opposite part as in its own part. Using König's infinity lemma, it is easy to prove that the same holds for locally finite graphs. However, it is not known whether every countable graph has an unfriendly partition.

The notation and terminology below are the same as in [5]. In addition, we say that an infinite graph is *locally finite* if every vertex has finite degree. Following [8] we say that an embedding of a graph in the plane is *rectangular* if every edge is a vertical or horizontal straight line segment. The planar graphs having rectangular embeddings have not been characterized, but Ungar [12] proved that every finite planar, cubic, cyclically 4-edge-connected graph has a rectangular embedding after four edges on the outer cycle are subdivided. In such an embedding every face is bounded by a rectangle. Over 20 years ago I conjectured in [9] that Ungar's theorem extends to infinite graphs. One may try to repeat the above argument by Erdős. But, the infinite path in the auxiliary graph K cannot always be used to find the embedding because it is not always possible to extend a rectangular representation to a bigger one even if the homeomorphism properties are satisfied. In [11] we overcome this obstacle by introducing what we call a *grid representation* which is defined as follows.

Let $L_1, L_2, \ldots, L_p$ be a collection of pairwise parallel horizontal lines in the plane, and let $Q_1, Q_2, \ldots, Q_q$ be a collection of pairwise parallel vertical lines. Let G be a finite graph such that each vertex of G can be represented by an intersection point of some L_i and Q_j, and such that each edge of G is contained in one of $L_1, L_2, \ldots, L_p, Q_1, Q_2, \ldots, Q_q$, and such that no two edges cross. If each of the lines $L_1, L_2, \ldots, L_p, Q_1, Q_2, \ldots, Q_q$ contains a vertex of G, then these lines together with the representation of G is called a *grid representation* of G. We say that the intersection point of L_i and Q_j has the *coordinates* i, j. Two grid representations of G are *equivalent* if they have the same number of vertical lines and the same number of horizontal lines, and if every vertex of G has the same coordinates in the two representations. Clearly, a finite graph has only finitely many non-equivalent grid representations. The following was proved in [11].

Theorem 6 *Let G be a countably infinite graph. Let E be a collection of edges of G. (E may equal the edge set of G. E may be empty.) Let E be colored with the two colors $0, 1$. Assume that every finite subgraph of G has a rectangular representation such that each edge of E in the subgraph is vertical if it has color 1 and horizontal if it has color 0. Then G has a rectangular representation such that each edge of E is vertical if it has color 1 and horizontal if it has color 0.*

The proof ensures that all vertices and edges can be kept inside a prescribed square of the plane if we wish so. In the proof of the above theorem we get accumulation points. (A *vertex accumulation point* is a point each neighborhood of which contains infinitely many vertices of the graph. An *edge accumulation point* is a point each neighborhood of which intersects infinitely many edges of the graph.) The representation can be chosen such that no point on an edge of G is an accumulation point. It is easy to give examples of a cubic plane graph which has a grid representation and also has a planar drawing without accumulation points but with the property that every grid representation has accumulation points if we allow E to be the whole edge set. It is also easy to give such examples where E is empty and G is 4-regular. In fact, the only 4-regular graph which has a grid representation without accumulation points is the 2-dimensional grid. (For, if some facial walk is not a 4-cycle, then it is a two-way infinite path forming a spiral, and one of the one-way infinite subspirals is bounded.)

A cubic graph G is *cyclically 4-edge connected* if it is 3-connected and any edge-cut of G consists of the three edges incident with a vertex. Ungar [12] proved that every finite, planar, cubic, cyclically 4-edge-connected graph has a rectangular embedding provided some four edges on the outer cycle are subdivided. It follows from [9] that any facial cycle can play the role of the outer cycle

One noteworthy feature of Ungar's theorem is that it emphasizes that the difficulties in the 4-color problem (and many other problems on planar graphs) are of purely combinatorial nature and not of geometric or topological nature, as the countries can be chosen to be rectangles.

In [9] it was conjectured that Ungar's theorem extends to the infinite case. In [11] we prove that conjecture by combining Ungar's result with Theorem 6.

Theorem 7 *Every infinite, planar, cubic, cyclically 4-edge-connected graph G has a rectangular representation in the plane.*

This proof of this result applies in other contexts as well. A *bar representation* of a graph G is a representation such that the vertices of G are pairwise disjoint horizontal straight line segments and each edge is a vertical straight line segments joining its two ends and intersecting no other vertex. If all possible edges between vertices are present, then G is a *visibility graph*. If G is a graph, then we define the graph G^* as follows: If each component of G is 2-connected, then $G^* = G$. If some component of G is not 2-connected, then G^* is obtained from G by adding a new vertex and joining it to all cutvertices of G.

Tamassia and Tollis [7] proved that a finite graph G is a visibility graph if and only if G^* is planar. In other words, G is planar and has a plane representation such that all cutvertices are on the outer face boundary. In particular, every 2-connected finite planar graph is a visibility graph, and every finite planar graph has a bar representation. In [11] the following infinite counterpart is proved.

Theorem 8 *A countably infinite graph has a bar representation if and only if it is planar.*

Consider now a countably infinite graph G with the property that each finite subgraph is planar. The proof of Theorem 8 shows that G can be representated such that every vertex is a rectangle and every edge is a vertical straight line segment joining its ends and intersecting no other rectangle. (Instead of successively adding one horizontal line in the proof of Theorem 8 we add two horizontal lines close to one another.) By representing a vertex by a point inside a rectangle it is now easy to obtain a planar drawing of the graph. This argument may not be as natural as the one by Erdős but it avoids the geometric details in the transition from the finite case to the infinite case.

Perhaps the rectangles in the previous paragraph can even be chosen to be squares, see Conjecture 4 below.

Theorem 9 *If G is a countably infinite, locally finite graph G such that G^* is planar, then G is a visibility graph. In particular, every countably infinite, locally finite, 2-connected graph is a visibility graph.*

The converse of Theorem 9 is not true. To see this, consider any finite or countably infinite visibility graph. Every edge xy can be represented by two vertical straight line segments which, together with parts of the line segments representing x, y form a rectangle whose interior intersects no vertex or edge. We divide any such rectangle into two rectangles by adding a horizontal straight line segment inside the rectangle, and then we add in each of the two smaller rectangles a one-way infinite path starting at x or y. The resulting graph is a visibility graph, and every vertex is a cutvertex. More elaborate examples are possible so a complete characterization of the infinite visibility graphs seems complicated, even in the locally finite case.

An additional complication occurs if we allow vertices of infinite degree. The proof of Theorem 9 shows that any graph G such that G^* is planar has a bar representation such that the only edges that can be added to the representation are some which join two vertices of infinite degree.

All previous results are 2-dimensional. However the method can be extended to higher dimensions although there may not be many examples demonstrating that. We shall here mention one nontrivial example. A *box graph* is a graph such that every vertex is a box in 3-space, that is, the cartesian product of three closed bounded intervals in the real line. No two boxes have an interior point in common. Two vertices are neighbors if they have a rectangle (with positive area) in common. The box graphs have not been characterized in terms of forbidden subgraphs. In [10] I proved that every finite, planar graph is a box graph. In [11] this is extended to the infinite case as follows.

Theorem 10 *A countably infinite graph is a box graph if and only if every finite subgraph is a box graph. In particular, every countably infinite, planar graph is a box graph.*

The proof of Theorem 10 is analogous to that of Theorem 6.

We conclude with some open problems which are related to the results above but seem to require more elaborate methods.

In [9] I conjectured the following:

Conjecture 1 *Every infinite, cubic, cyclically 4-edge-connected graph which has a planar representation with no vertex-accumulation point and no edge-accumulation point has a rectangular representation in the plane with no vertex-accumulation point and no edge-accumulation point.*

There is an analogous problem for bar representations:

Conjecture 2 *Every infinite, locally finite graph which has a planar representation with no vertex-accumulation point and no edge-accumulation point has a bar representation with no vertex-accumulation point and no edge-accumulation point.*

In Theorem 9 the graphs are locally finite. Perhaps this condition can be omitted.

Conjecture 3 *Every countably infinite, 2-connected graph is a visibility graph.*

Conjecture 4 *Every countably infinite, planar graph has a representation such that every vertex is a square such that no two squares intersect and every edge is a vertical straight line segment joining the squares representing its ends.*

Conjecture 5 *Every countably infinite, planar graph has a representation such that every vertex is a closed disc such that no two discs intersect and every edge is a vertical straight line segment joining the discs representing its ends.*

Note that Conjecture 4 implies Conjecture 5 by taking the largest discs inside the squares. So the remark following Conjecture 4 shows that Conjecture 5 is true for finite graphs. This also follows from the theorem of Koebe (see e.g. [5], page 51) that every finite planar graph is a *coin graph*, that is, a graph whose vertices are closed discs no two of which have an interior point in common and such that two discs are neighbors if and only they have a point in common. To obtain Conjecture 5 in the finite case, just shrink the discs a little (and possibly rotate the collection of discs slightly).

Conjecture 6 *Every countably infinite, locally finite, planar graph is a coin graph.*

Conjecture 6 cannot be extended to all countable graphs. The graph obtained from a two-way infinite path by adding two new vertices each of which is joined to all other vertices in the graph is a countable, planar graph, but it cannot be represented as a coin graph.

7 Some open problems presented at the problem sessions

In problem sessions, many old and new open problems have been presented. Some of them are recorded below.

Problem 1 *Let G be a planar graph and $x, y \in V(G)$. How difficult is to compute the crossing number of $G + xy$?*

Note: Salazar and Hlineny have conjectured that this problem is solvable in polynomial time. On the other hand, Cabello and Mohar believe that this problem is NP-hard at least when the planar graph has weighted edges and a crossing of edges with weights a and b counts as ab.

Problem 2 (Bruce Richter) *A graph G is said to be k-*crossing-critical *if $\texttt{cr}(G) \geq k$ and $\texttt{cr}(G-e) < k$ for every edge e of G. It is known that 2-crossing-critical graphs have "cyclic" structure apart from a finite number of sporadic cases. Is there a similar result for large 3-crossing-critical graphs?*

Problem 3 (Petr Hlineny) *Is it true that for every positive integer k there exists an integer D such that every k-crossing-critical graph has maximum degree less than D?*

Petr Hlineny thinks that this may not be true.

Problem 4 *Is it true that for every positive integer k there exists an integer B such that every k-crossing-critical graph has bandwidth at most B?*

Problem 5 (Robin Thomas) *Let $\mathcal{I}$ be a lower ideal for minor ordering, i.e., a minor-closed class of graphs which does not contain all graphs. Is there a polynomial time algorithm which for every $G \in \mathcal{I}$ computes an integer k such that $|\chi(G) - k| \leq 10$?*

The constant 10 is fictious. The problem makes sense if it is replaced by any positive integer. Robin Thomas conjectured that the answer to the above problem is affirmative.

Problem 6 (Drago Bokal) *Is there an integer k for which there exist infinitely many k-crossing-critical (simple) graphs of average degree 6.*

It is known that infinitely many k-crossing-critical graphs of average degree d exist if d is any rational number in the interval $(3, 6)$ and that this is no longer the case if $d > 6$.

Problem 7 (Sergio Cabello) *Find a polynomial fixed-parameter-tractable (FPT) algorithm for deciding isomorphism of graphs of genus at most g, i.e.,the time complexity should be $O(f(g)p(n))$, where p is a polynomial and n is the size of the input.*

Note: Known algorithms have polynomial complexity $O(n^{O(g)})$, which is not FPT.

Problem 8 *Is there a graph which is a minimal forbidden minor for two distinct surfaces?*

If yes, then the two surfaces would both be nonorientable and their nonorientable genera would differ by one.

Problem 9 (Gelasio Salazar) *For a graph G, let $\mathtt{cr}_g(G)$ be the crossing number of G with respect to the drawings of G in the orientable surface of genus g. Is it possible that a graph satisfies $\mathtt{cr}_0(G) = N$, $\mathtt{cr}_1(G) = N - 1$, and $\mathtt{cr}_2(G) = 0$, where N is a very large integer?*

This problem has been recently solved by Matt DeVos, Bojan Mohar, and Robert Šamal [1].

Let G and H be geometric graphs, represented by straight-line segments in the plane. We say that G and H are *isomorphic* if there is an isomorphism f of (abstract) graphs G and H such that for every pair of edges ab and cd of G, the line segments corresponding to these two edges intersect if and only if the line segments of edges $f(a)f(b)$ and $f(c)f(d) \in E(H)$ intersect in the representation of H. The isomorphism f is *strong* if each line segment representing an edge e of G intersects other edges of G in the same order as the line segment of $f(e)$ intersects their f-images in the representation of H. The following problem was motivated by the lecture of Mike Albertson on distinguishing geometric graphs.

Problem 10 (Bojan Mohar) *How hard it is to decide if two geometric graphs are (strongly) isomorphic.*

Further open problems have been presented by Matt DeVos, Mark Ellingham, Luis Goddyn, and others.

8 Scientific progress and some outcomes of the meeting

Some of the spirit of the workshop can be seen from the exposition of Carsten Thomassen in Section 6. Old results and open problems presented in a new light, many open problems and conjectures.

The exchange between the various research groups at the workshop proved to be very fruitful. From the point of view of crossing numbers, there were two main outcomes.

The presentation by Salazar on the problem of determining the crossing number of a graph G such that G has an edge e so that $G - e$ is planar led to much discussion. The question is:

Is there an efficient (that is, polynomial time) algorithm to determine the crossing number of such a graph G?

This is such a "simple" problem and yet there is no known solution. Examples show that the crossing number of G can be arbitrarily large. Earlier work by Mutzel et al. showed that there is an efficient algorithm to find the drawing of G having the fewest crossings subject to the additional requirement that all the crossings involve e. These are quite natural drawings of such a graph, but Hlineny and Salazar show by example that they are not necessarily optimal in the sense of giving a drawing of G having fewest crossings. (The same result was discovered independently by Mutzel et al. and published in the journal version of their original presentation.) They further show that they can efficiently find a drawing that is within a constant factor of being optimal.

Petr Hlineny gave a short presentation about the important class of k-crossing-critical graphs. These are the graphs G having crossing number at least k, but every proper subgraph has crossing number less than k. There are several standard questions about such graphs. Two standard ones have the same form:

Is there a number $f(k)$ so that any k-crossing-critical graph has maximum degree or bandwidth at most $f(k)$?

The best result along these lines is Hlineny's theorem that they have bounded path-width. The result is known for $k = 2$ but open for $k \geq 3$. The discussion during the workshop led to further discoveries of k-crossing-critical graphs with larger maximum degrees than were previously known, but still did not resolve the question.

The presentation by Richter on a problem, deriving from the Graph Minors Project, concerning how often a set of disjoint arcs, joining specified pairs of points in the boundary of a bordered surface must meet the different homeomorphism types of non-separating curves led quite directly to its complete solution shortly after the conference.

János Pach reported of a new joint paper that he has just finished with Jacob Fox, that grew out of work started in Banff. This improves on the results of his earlier joint work with Micha Sharir. The paper is entitled "Separator theorems and Turán-type results for planar intersection graphs" (by Jacob Fox and János Pach) and its contents can be summarized as follows:

> "We establish several geometric extensions of the Lipton-Tarjan separator theorem for planar graphs. For instance, we show that any collection C of Jordan curves in the plane with a total of m crossings has a partition into three parts $C = S \cup C_1 \cup C_2$ such that $|S| = O(\sqrt{m})$, $\max\{|C_1|, |C_2|\} \leq \frac{2}{3}|C|$, and no element of C_1 has a point in common with any element of C_2. These results are used to obtain various properties of intersection patterns of geometric objects in the plane. In particular, we prove that if a graph G can be obtained as the intersection graph of n convex sets in the plane and it contains no complete bipartite graph $K_{k,k}$ as a subgraph, then the number of edges of G cannot exceed $c_k n$, for a suitable constant c_k."

Bojan Mohar also reported of a couple of new results about crossing numbers that have resulted from discussions at this meeting. First, he has extended results related to János Pach's exposition on Degenerate Crossing Numbers. He proved that this version of a crossing number is almost equivalent to the notion of the non-orientable genus of graphs – an unexpected relation whose importance lies in the fact that the non-orientable genus is monotone under graph minors relation. Another achievement is a recent proof by Matt DeVos, Bojan Mohar, and Robert Šamal [1] which, in particular, completely solves Problem 9.

Drago Bokal reports on an ongoing research jointly with Éva Czabaraka, Lszl A. Szkely, and Imrich Vrťo on General lower bounds for minor crossing number, on which they also made important progress during the workshop:

> "There are three general techniques for bounding crossing numbers of graphs: the Crossing lemma, the Bisection method, and the Embedding method. Recently we established similar results in the context of the minor crossing number. As a result, we were able to tighten the bounds for the minor crossing number of hypercubes. We are looking for further applications of the new tools."

Additionally, Petr Hlineny informed us that he has finished and submitted a paper with a crossing-critical construction with high even degrees, and he acknowledged great influence of this workshop in obtaining this result.

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