Banff International Research Station

for Mathematical Innovation and Discovery

Positive Polynomials and Optimization
October 8-12, 2006

MEALS

Breakfast (Continental): 7:00–9:00 am, 2nd floor lounge, Corbett Hall, Sunday–Thursday
*Lunch (Buffet): 11:30 am–1:30 pm, Donald Cameron Hall, Sunday–Thursday
*Dinner (Buffet): 5:30–7:30 pm, Donald Cameron Hall, Saturday–Wednesday
Coffee Breaks: As per daily schedule, 2nd floor lounge, Corbett Hall
***Please remember to scan your meal card at the host/hostess station in the dining room for each lunch and dinner.**

MEETING ROOMS

All lectures will be held in Max Bell 159 (Max Bell Building accessible by bridge on 2nd floor of Corbett Hall). Hours: 6 am–12 midnight. LCD projector, overhead projectors and blackboards are available for presentations. *Please note that the meeting space designated for BIRS is the lower level of Max Bell, Rooms 155–159. Please respect that all other space has been contracted to other Banff Centre guests, including any Food and Beverage in those areas.*

SCHEDULE

Saturday

16:00	Check-in begins (Front Desk - Professional Development Centre - open 24 hours) Lecture rooms available after 16:00
17:30–19:30	Buffet Dinner, Donald Cameron Hall
20:00	Informal gathering in 2nd floor lounge, Corbett Hall Beverages and small assortment of snacks available on a cash honour-system.

Sunday

7:00–8:45	Breakfast
8:45–9:00	Introduction and Welcome to BIRS by BIRS Station Manager, Max Bell 159
9:00–10:00	Mihai Putinar, *The trigonometric moment problem in several variables and related SOS decompositions*
10:00–10:30	Coffee Break, 2nd floor lounge, Corbett Hall
10:30–11:30	Pablo Parillo, *Exact semidefinite representations for genus zero curves*
11:30–13:30	Lunch
14:00–15:00	Konrad Schmüdgen, *Positivity and Positivstellensätze for matrices of polynomials*
15:00–15:30	Coffee Break, 2nd floor lounge, Corbett Hall
15:30–16:30	Raul Curto, *Algebraic geometric techniques for the truncated moment problem*
16:30–17:00	Second Chances
17:30–19:30	Dinner

Monday

7:00–8:45	Breakfast
9:00–10:00	Claus Scheiderer, *Sums of squares and moment problems with symmetries*
10:00–10:30	Coffee Break, 2nd floor lounge, Corbett Hall
10:30–11:30	Tim Netzer, *An elementary proof of Schmüdgen's Theorem on the moment problem of closed semi-algebraic sets*
11:30–13:00	Lunch
13:00–14:15	Banff Centre tour
14:30–15:30	Luis Zuluaga, *Closed-form solutions to certain moment problems with applications to business*
15:30–16:00	Coffee Break, 2nd floor lounge, Corbett Hall
16:00–17:00	Murray Marshall, *Representations of non-negative polynomials, degree bounds and applications to optimization*
17:00–17:30	Second Chances
17:30–19:30	Dinner

Tuesday

7:00–8:45	Breakfast
9:00–10:00	M.-F. Roy, *Certificate of positivity in the Bernstein basis*
10:00–10:30	Coffee Break, 2nd floor lounge, Corbett Hall
10:30–11:30	Bill Helton, *Real algebraic geometry in a free *-algebra*
11:30–12:00	Second Chances
12:00–13:30	Lunch
	Free Afternoon
17:30–19:30	Dinner

Wednesday

7:00–8:45	Breakfast
9:00–10:00	Monique Laurent, *A numerical algorithm for the real radical ideal*
10:00–10:30	Coffee Break, 2nd floor lounge, Corbett Hall
10:30–11:30	Thorsten Theobald, *Symmetries in SDP-based relaxations for constrained polynomial optimization*
11:30	Group Photo
11:30–13:30	Lunch
14:00–15:00	Markus Schweighofer, *Global optimization of polynomials using gradient tentacles and sums of squares*
15:00–15:30	Coffee Break, 2nd floor lounge, Corbett Hall
15:30–16:30	Jiawang Nie, *Sparse SOS Relaxation and Applications*
16:30–17:00	Second Chances
17:30–19:30	Dinner

Thursday

7:00–8:45	Breakfast
9:00–10:00	Bruce Reznick, *Hilbert's construction of psd quartics and sextics that are not sos*
10:00–11:00	J.-B. Lasserre, *A Positivstellensatz which preserves the coupling of variables*
11:00–11:30	Last Chances
11:30–13:30	Lunch

** 5-day workshops are welcome to use the BIRS facilities (2nd Floor Lounge, Max Bell Meeting Rooms, Reading Room) until 3 pm on Thursday, although participants are still required to checkout of the guest rooms by 12 noon. **

Positive Polynomials and Optimization
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ABSTRACTS

Speaker: **Raúl Curto** (Iowa)
Title: *Algebraic Geometric techniques for the truncated moment problem*
Abstract: For a degree $2n$ real d-dimensional multisequence $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i \in Z_+^d, |i| \le 2n}$ to have a *representing measure* μ, it is necessary for the associated moment matrix $\mathcal{M}(n)(\beta)$ to be positive semidefinite, and for the algebraic variety associated to β, $\mathcal{V} \equiv \mathcal{V}_\beta$, to satisfy $\text{rank}\mathcal{M}(n) \le \text{card}\mathcal{V}$ as well as the following *consistency* condition: if a polynomial $p(x) \equiv \sum_{|i| \le 2n} a_i x^i$ vanishes on $\mathcal{V}$, then $p(\beta) := \sum_{|i| \le 2n} a_i \beta_i = 0$. In joint work with Lawrence Fialkow and Michael Möller, we employ tools and techniques from algebraic geometry (e.g., Hilbert polynomials, Gröbner and H-bases, representation of positive polynomials) to prove that for the *extremal* case ($\text{rank}\mathcal{M}(n) = \text{card}\mathcal{V}$), positivity of $\mathcal{M}(n)$ and consistency are sufficient for the existence of a (unique, $\text{rank}\mathcal{M}(n)$-atomic) representing measure.

Truncated moment problems (TMP) as above for which the support of a representing measure is required to lie inside a closed set K are called truncated K-moment problems (TKMP). In case K is a semi-algebraic set determined by polynomials $q_1 \ldots, q_m$, the study of TKMP is dual to determining whether a polynomial nonnegative on K belongs to the positive cone consisting of polynomials of degree at most $2n$ which can be expressed as sums of squares, and of squares multiplied by one or more distinct q_i's.

The extremal case, which we have now solved, is inherent in the TMP. A recent result of C. Bayer and J. Teichmann (extending a classical theorem of V. Tchakaloff and its successive generalizations given by I.P. Mysovskikh, M. Putinar, and L. Fialkow and the speaker) implies that if $\beta^{(2n)}$ has a representing measure, then it has a finitely atomic representing measure. Fialkow and the speaker had previously shown that $\beta^{(2n)}$ has a finitely atomic representing measure if and only if $\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta)$ admits an extension to a positive moment matrix $\mathcal{M}(n+k)$ (for some $k \ge 0$), which in turn admits a rank-preserving (i.e., *flat*) moment matrix extension $\mathcal{M}(n+k+1)$. Further, we proved that any flat extension $\mathcal{M}(n+k+1)$ is an extremal moment matrix for which there is a computable rank $\mathcal{M}(n+k)$-atomic representing measure μ. In this sense, the existence of a representing measure for $\beta^{(2n)}$ is intimately related to the solution of an extremal TMP.

Speaker: **Bill Helton** (UC San Diego)
Title: *Real Algebraic Geometry in a Free $*$-Algebra*
Abstract: The talk will describe recent results and focus on new directions.

Speaker: **J.-B. Lasserre** (Toulouse)
Title: *A Positivstellensatz which preserves the coupling of variables*
Abstract: We specialize Schmüdgen's Positivstellensatz and its Putinar and Jacobi–Prestel refinement, to the case of a polynomial $f^2\mathbb{R}[X,Y] + \mathbb{R}[Y,Z]$, positive on a compact basic semi-algebraic set K described by polynomials in $\mathbb{R}[X,Y]$ and $\mathbb{R}[Y,Z]$ only, or in $\mathbb{R}[X]$ and $\mathbb{R}[Y,Z]$ only (i.e. K is cartesian product). In particular, we show that the preordering $P(g,h)$ (resp. quadratic module $Q(g,h)$) generated by the polynomials $\{g_j\} \subset \mathbb{R}[X,Y]$ and and $\{h_k\} \subset \mathbb{R}[Y,Z]$ that describe K, is replaced with $P(g)+P(h)$ (resp. $Q(g)+Q(h)$), so that the absence of coupling between X and Z is also preserved in the representation. A similar result applies with Krivine's Positivstellensatz involving the cone generated by $\{g_j, h_k\}$.

Speaker: **Monique Laurent** (CWI, Amsterdam)
Title: *Semidefinite characterization and computation of real radical ideals*

Abstract: For an ideal $I \subseteq \mathbb{R}[x_1, \ldots, x_n]$ given by a set of generators $h_1, \ldots, h_m$, we propose a semidefinite characterization and a numerical method for finding the real radical ideal $\sqrt[\mathbb{R}]{I} = I(V_{\mathbb{R}}(I))$, provided it is zero-dimensional (even if I is not). Our method relies on expressing $I(V_{\mathbb{R}}(I))$ as the kernel of a suitable positive semidefinite moment matrix and uses semidefinite optimization for finding such a matrix.

One of our results can be sketched as follows. Let $M_t(y)$ be a maximum rank feasible solution to the system:

$$M_t(y) \succeq 0, \ M_{t-d_j}(h_j y) = 0 \ (j = 1, \ldots, m),$$

where $d_j := \lceil \deg(h_j)/2 \rceil$ and $t \geq \max_j d_j$. Then, $I(V_{\mathbb{R}}(I)) = \langle \mathrm{Ker} M_t(y) \rangle$ if the rank condition: $\mathrm{rank} M_t(y) = \mathrm{rank} M_{t-d}(y)$ holds. A maximum rank solution $M_t(y)$ can be found with a semidefinite programming solver; if the rank condition holds we have found $I(V_{\mathbb{R}}(I))$, otherwise iterate replacing t by $t+1$. The algorithm is +guaranteed to terminate when $V_{\mathbb{R}}(I)$ is finite.

With our method we can compute directly from the optimal matrix $M_t(y)$ the following objects: the set $V_{\mathbb{R}}(I)$ of real roots, a linear basis of the quotient vector space $\mathbb{R}[x]/I(V_{\mathbb{R}}(I))$, a border basis of $I(V_{\mathbb{R}}(I))$ as well as a Gröbner basis for a total-degree monomial ordering.

A feature of our method is that it exploits right from the beginning the real algebraic nature of the problem. In particular, it does not need the determination of a Gröbner basis of the ideal I and we do not compute (implicitly or +explicitly) the complex variety $V(I)$. The method also applies to finding $I(V_{\mathbb{R}}(I) \cap S)$ where S is basic closed semialgebraic set.

This is joint work with J.-B. Lasserre and P. Rostalski

Speaker: **Murray Marshall** (Saskatchewan)
Title: *Representations of non-negative polynomials, degree bounds and applications to optimization*
Abstract: Natural sufficient conditions for a polynomial to have a local minimum at a point are considered. These conditions tend to hold with probability 1. It is shown that polynomials satisfying these conditions at each minimum point have nice presentations in terms of sums of squares. Applications are given to optimization on a compact set and also to global optimization. In many cases, there are degree bounds for such presentations. These bounds are of theoretical interest, but they appear to be too large to be of much practical use at present. In the final section, other more concrete degree bounds are obtained which ensure at least that the feasible set of solutions is not empty.

Speaker: **Tim Netzer** (Konstanz)
Title: *An Elementary Proof of Schmüdgen's Theorem on the Moment Problem of Closed Semi-Algebraic Sets*
Abstract: We discuss a more elementary proof of the main result from Schmüdgen's 2003 article "On the moment problem of closed semi-algebraic sets". The result states, that the question whether a finitely generated preordering has the so called Strong Moment Property can be reduced to the same question for preorderings corresponding to fiber sets of bounded polynomials.

Speaker: **Jiawang Nie** (IMA, Minnesota)
Title: *Sparse SOS Relaxation and Applications*
Abstract: SOS relaxation provides very good approximation for finding global minimum and minimzer of polynomial functions. However, the size of the resulting SDP is often very large and makes it difficult to solve large scale problems. This talk will discuss the global optimiation of large polynomial functions that are given as the summation of small polynomials. The sparse SOS relaxations are proposed. We analyze the computational complexity and the quality of lower bounds. Some numerical implementations of randomly generated problems shows that this sparse SOS relaxation is very successful. This sparse SOS relaxation is very useful in solving large scale sparse polynomial systems, like the polynomial systems derived from nonlinear PDEs and distance geometry problems (e.g., sensor network localization).

Speaker: **Bruce Reznick** (Illinois)
Title: *Hilbert's construction of psd quartics and sextics that are not sos*
Abstract: We will discuss both of these constructions, which become almost intuitive when one "counts constants". New and simple examples will be derived.

Speaker: **Pablo Parrilo** (MIT)
Title: *Exact semidefinite representations for genus zero curves*
Abstract: The characterization of sets that admit an exact representation in terms of semidefinite programming constraints (perhaps with additional variables) is one of great interest in optimization. There have been a few recents results in this direction, based mainly on the work of Helton and Vinnikov and the related Lax conjecture, that point out to the existence of specific obstructions for (the interior of) a plane curve to be semidefinite representable. In this talk we discuss a procedure to explicitly construct exact representations for convex hulls of arbitrary segments of genus zero plane curves. In particular, it is shown that the new method enables the computation of representation for particular curves, for which a generic SOS-based construction fails.

Speaker: **Mihai Putinar** (UC Santa Barbara)
Title: *The multivariate trigonometric moment problem and related sums of squares decompositions*
Abstract: In the case of the one dimensional torus, Riesz-Fejer factorization of a non-negative trigonometric polynomial as the modulus square of another polynomial provides the basis of all positivity results related to the unit disk: Riesz-Herglotz parametrization of all non-negative harmonic functions, the solution to the trigonometric moment problem, as proposed by Schur, and separately by Caratheodory-Fejer, the spectral theorem for unitary operators.

In several variables, on the torus in $\mathbb{C}^n$, we reverse the flow, and start with Bochner's characterization of Fourier transforms of positive measures. This provides a sum of squares decomposition for positive trigonometric polynomials. And the result can easily be adapted to compact, semi-algebraic supports on the torus.

The most intriguing case is however the unit sphere in $\mathbb{R}^n$, where a decopmpsition into squares of spherical harmonics is available. For odd dimensional sphere, the complex structure induced from $\mathbb{C}^n$ provides a decomposition (of a positive polynomial) into squares of pluriharmonic polynomials. As a consequence I will indicate a novel proof of an old theorem of Quillen.

Finally, the non-commutative sphere, or torus, associated to the free-$*$ algebra reveals stronger SOS deecompositions.

Speaker: **M.-F. Roy** (Rennes)
Title: *Certificate of positiviy in the Bernstein basis*
Abstract: We prove the existence of a polynomial size (in the degree d and bitsize t of coefficients) certificate of positivity for a positive univariate polynomial on $[-1, 1]$ using Berstein basis of degree d on well-chosen subintervals. This improves by an exponential factor previsouly known results by Powers and Reznick.

Speaker: **Claus Scheiderer** (Konstanz)
Title: *Sums of squares and moment problems with symmetries*
Abstract: Let G be a real algebraic subgroup of $GL(V)$, the general linear group of a finite-dimensional real vector space V, and assume that G is (semi-algebraically) compact. We study the cones of sums of squares in $R[V]$ and in $R[V]^G$, the ring of G-invariants, and relate them through the operations of contraction and extension. More generally, we do the same for arbitrary quadratic modules. In doing this, we use (and partially re-prove, partially generalize) results of Procesi-Schwarz, Bröcker and Gatermann-Parrilo. We prove that the Reynolds operator maps the cone $\Sigma R[V]^2$ into itself, and that this property is characteristic of the case where G is compact.

Given a basic closed set K in V which is G-invariant, we ask for (finite) characterizations of the G-invariant K-moment functionals. We isolate two conditions under which such characterizations exist, and show by examples that this may happen at the same time when the usual (full) K-moment problem is not finitely solvable.

The talk will contain (plenty of) explicit examples and (a few) open problems. (Joint work with Salma Kuhlmann and Jaka Cimpric.)

Speaker: **Konrad Schmüdgen** (Leipzig)
Title: *Positivity and Positivstellensätze for Matrices of Polynomials*
Abstract: The notion of k-positivity ,$0 \leq k \leq n$, for (n,n)-matrices of polynomials is introduced and discussed. Generalizations of Stengle's Positivstellensatz to matrices are given.

Speaker: **Markus Schweighofer** (Konstanz)
Title: *Global optimization of polynomials using gradient tentacles and sums of squares*
Abstract: We combine the theory of generalized critical values with the theory of iterated rings of bounded elements (real holomorphy rings). We consider the problem of computing the global infimum of a real polynomial in several variables. Every global minimizer lies on the gradient variety. If the polynomial attains minimum, it is therefore equivalent to look for the greatest lower bound on its gradient variety. Nie, Demmel and Sturmfels proved recently a theorem about the existence of sums of squares certificates for such lower bounds. Based on these certificates, they find arbitrarily tight relaxations of the original problem that can be formulated as semidefinite programs and thus be solved efficiently. We deal here with the more general case when the polynomial is bounded from below but does not necessarily attain a minimum. In this case, the method of Nie, Demmel and Sturmfels might yield completely wrong results. In order to overcome this problem, we replace the gradient variety by larger semialgebraic sets which we call gradient tentacles. It now gets substantially harder to prove the existence of the necessary sums of squares certificates.

Speaker: **Thorsten Theobald** (Berlin)
Title: *Symmetries in SDP-based relaxations for constrained polynomial optimization*
Abstract: (joint work with L. Jansson, J.B. Lasserre and C. Riener)

We study methods for exploiting symmetries within semidefinite programming-based relaxation schemes for constrained polynomial optimization. Our main focus is on problems where the symmetric group or the cyclic group is acting on the variables. From the exact point of view, we extend the representation-theoretical methods of Gatermann and Parrilo for the unconstrained case to the constrained case (i.e., to Lasserre's relaxation scheme). In contrast to the viewpoint merely from the resulting semidefinite programs, the symmetries on the original variables induce much additional symmetry structure on the moment matrices of the relaxation scheme. We characterize the combinatorics of the resulting block decompositions in terms of Kostka numbers. Moreover, we present methods to efficiently compute lower and upper bounds for a subclass of problems where the objective function and the constraints are given by power sums.

Speaker: **Luis Zuluaga** (New Brunswick)
Title: *Closed-form solutions to certain moment problems with Applications to Business*
Co-authors: Donglei Du, Javier Pena, and Juan Vera.

Abstract: We present new closed-form solutions to certain moment problems with applications in mathematical finance, inventory theory, supply chain management, and Actuarial Science. In particular, we extend Lo's classical semiparametric closed-form bound for European call options by considering third-order moment information, and by finding a related semiparametric bound on the option's risk. Furthermore, we present a closed-form solution for a class of arbitrage bounds. We show how the latter result can be used to obtain a novel model for portfolio allocation with desirable properties for today's investors.