Multi-Parameter Nehari Theorems

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1 Nehari Theorems

The Nehari Theorem characterizes bounded Hankel operators. Let us state the result on $L^2(\mathbf{R})$. Let M_b be the operator of pointwise multiplication by b, $M_b\varphi = b \cdot \varphi$. Consider the standard decomposition $L^2(\mathbf{R}) = H^2(\mathbf{R}) \oplus H^2_-(\mathbf{R})$ of $L^2(\mathbf{R})$ into the Hardy spaces of analytic and anti analytic functions. Let $P_\pm$ be the orthogonal projections of L^2 onto H^2 and H^2_- respectively. A *Hankel operator with symbol* b maps H^2 into itself and is given by $H_b := P_+ M_b \overline{\varphi}$. This definition depends only on the analytic part of b. The Nehari Theorem [6] asserts that the bounded Hankel operators are exactly those which admit a bounded symbol.

Theorem 1 *H_b is bounded if and only if there is a bounded function β with $P_+\beta = P_+b$, and*

$$\|H_b\| = \inf\{\|\beta\|_\infty \mid P_+\beta = P_+b\}. \tag{1}$$

This Theorem is one of the foundations of modern operator theory. Coifman, Rochberg and Weiss [3] characterized real valued H^1 in several variables by way of a variant of Nehari's Theorem, this time stated in the language of commutators and the dual to H^1, BMO.

Theorem 2 (Coifman, Rochberg and Weiss) *Fix a dimension $d > 1$. Let R_j, $1 \le j \le d$, be the Riesz transforms on $\mathbf{R}^d$. We have the equivalence*

$$\sup_j \|[M_b, R_j]\|_{2\to 2} \simeq \|b\|_{\mathrm{BMO}}. \tag{2}$$

The latter space is real one parameter $\mathrm{BMO}(\mathbf{R}^d)$.

2 Multi-parameter Setting

We are interested in multi-parameter extensions of the results described above. Some results in this setting are already known, and for the purposes of this note, we restrict ourselves to the two parameter setting of Ferguson and Lacey [4], and stress that the higher parameter setting (which requires new ideas) is discussed in Lacey and Terwilleger [5].

The function theoretic setting for this Theorem is the Hardy space $H^2(\mathbf{R} \otimes \mathbf{R})$, consisting of functions f of two complex variables, analytic in each variable separately, with values on the boundary of $\mathbf{C}_+ \otimes \mathbf{C}_+$ that are square integrable. This is a closed subspace of $L^2(\mathbf{R}\otimes\mathbf{R})$, and we let $P_{+,+}$ be the orthogonal projection of L^2 onto this Hardy space. It is worth emphasizing that the complex domain is the product of two disks which is *not* pseudoconvex. It has boundary given by the product of two flat domains $\mathbf{R}\otimes\mathbf{R}$, hence the relevance of two parameter Harmonic Analysis.

The Hankel operators we consider are $H_b\varphi := P_{+,+}M_b\overline{\varphi}$, considered as an operator from H^2 to itself. This definition only depends upon the jointly analytic part of b, namely $P_{+,+}b$. These are the so called 'little Hankel operators' as the projection $P_{+,+}$ is the 'smallest' reasonable projection to use.

Theorem 3 (Ferguson and Lacey [4]) *A Hankel operator H_b is bounded if and only if it admits a bounded symbol. Namely, there is a bounded function β with $P_{+,+}\beta = P_{+,+}b$, and*

$$\|H_b\| = \inf_{\beta}\{\|\beta\|_\infty \mid P_{+,+}\beta = P_{+,+}b\}\,. \tag{3}$$

It is to be stressed that the relevant Hardy spaces here are on *product domains*, which do not fall in the scope of the elaborate theory built up around the classical Hardy spaces. In particular, the dual to $H^1(\mathbf{R}\otimes\mathbf{R})$ is a BMO$(\mathbf{R}\otimes\mathbf{R})$ space identified by S.-Y. Chang and R. Fefferman in a famous series of papers [1, 2].

3 Scientific Progress Made

Our focus has been to obtain a multi-parameter extension of the Coifman, Rochberg and Weiss result, and the Lacey Terwilleger result. This result, once established, would yield Nehari Theorems for certain Bergman spaces, and novel Div-Curl Lemmas. Namely, the principal result of our meeting is this Theorem.

We are concerned with product spaces $\mathbf{R}^{\vec{d}} = \mathbf{R}^{d_1}\otimes\cdots\otimes\mathbf{R}^{d_t}$ for vectors $\vec{d} = (d_1,\ldots,d_t)\in\mathbf{N}^t$. For Schwartz functions b, f on $\mathbf{R}^{\vec{d}}$, and for a vector $\vec{j} = (j_1,\ldots,j_t)$ with $1\le j_s\le d_s$ for $s = 1,\ldots,t$ we consider the family of commutators

$$C_{\vec{j}}(b,f)(x) =:= [\cdots[[M_b, R_{1,\,j_1}], R_{2,\,j_2}],\cdots](f)(x) \tag{4}$$

where $R_{s,\,j}$ denotes the jth Riesz transform acting on $\mathbf{R}^{d_s}$.

Theorem 4 *We have the estimates below, valid for $1<p<\infty$.*

$$\sup_{\vec{j}}\|C_{\vec{j}}(b,\varphi)\|_p \simeq \|b\|_{\mathrm{BMO}}. \tag{5}$$

By BMO*, we mean Chang–Fefferman* BMO*.*

Many of the techniques of proof used by Coifman Rochberg and Weiss are simply not available in the higher parameter setting. Many of the techniques of the Lacey Terwilleger approach apply, but they are not enough to conclude the proof of the Theorem. The argument of Lacey and Terwilleger relies at several points on the fact that the Hilbert transform is a difference of Fourier projections. And so several new methods must be brought to bear on the problem.

References

[1] Sun-Yung A. Chang, Robert Fefferman, Some recent developments in Fourier analysis and H^p-theory on product domains. *Bull. Amer. Math. Soc. (N.S.)* **12** (1985), 1—43

[2] Sun-Yung A. Chang, Robert Fefferman, A continuous version of duality of H^1 with BMO on the bidisc, *Ann. of Math. (2)* ***112*** *(1980), 179—201*

[3] R. R. Coifman, R. Rochberg, Guido Weiss, Factorization theorems for Hardy spaces in several variables. *Ann. of Math. (2)* **103** (1976), 611—635

[4] Sarah H. Ferguson and Michael T. Lacey, A characterization of product BMO by commutators. *Acta Math.* **189** (2002), 143—160

[5] Michael T Lacey and Erin Terwilleger, Hankel Operators in Several Complex Variables and Product BMO. Preprint.

[6] Zeev Nehari, On bounded bilinear forms. *Ann. of Math. (2)* **65** (1957), 153—162