Giant vortex and the breakdown of pinning in a rotating Bose–Einstein condensate

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Abstract:

This talk presents work in collaboration with A. AFTALION (Paris-6) and LIA BRONSARD (McMaster).

We consider the following variational problem arising from a two-dimensional model for rotating Bose–Einstein Condensates (BEC.) Let $a = a(r)$ be a real-analytic radially symmetric function in the plane, with the property that

$$\mathcal{A} = \{x \in \mathbf{R}^2 : \ a(|x|) > 0\}$$

is an *annulus,* and such that a vanishes linearly at each edge of the annulus $\mathcal{A}$. Examples include $a(r) = -b_0 + b_1 r^2 - b_2 r^4$ with appropriately chosen coefficients. Let $\Omega \in \mathbf{R}$, $x = (x_1, x_2) \in \mathbf{R}^2$, $x^\perp = (-x_2, x_1)$, and $\epsilon > 0$. We study minimizers $u \in H_0^1(\mathcal{A}; \mathbf{C})$ of the energy functional

$$E_\epsilon(u) = \int_{\mathcal{A}} \left\{ \frac{1}{2}|\nabla u|^2 - \Omega \mathbf{x}^\perp \cdot (iu, \nabla u) + \frac{1}{4\epsilon^2}(|u|^2 - a(x))^2 \right\} \, dx,$$

in the singular limit as $\epsilon \to 0$. In the context of BEC, u is the quantum wave-function, Ω is the angular speed of rotation, and $-a(r)$ gives a potential well imposed to "trap" the condensate (by means of lasers) in a bounded region of space. The choice of an annular trap here is meant to simulate certain current experiments in BEC.

The aim of this talk is to show how the annular topology of the condensate domain affects the presence and location of vortices as a function of the angular speed Ω. Our results concern both fixed rotation Ω (independent of ϵ and rotations which grow with ϵ. When Ω is fixed, we prove that minimizers converge to a non-zero (radially equivariant) solution away from the hole, while the hole itself plays the role of a "Giant Vortex" with degree increasing with Ω. We also consider angular velocities of the form

$$\Omega = \omega_0 |\ln \epsilon| + \omega_1 \ln |\ln \epsilon|,$$

with ω_0, ω_1 constant. We show that there is a critical value of the coefficient $\omega_0 = \omega_0^*$ so that whenever $\omega_0 < \omega_0^*$ minimizers have no vorticity in the interior of the annulus $\mathcal{A}$, but when $\omega_0 = \omega_0^*$ and ω_1 is large enough then vortices begin to appear inside $\mathcal{A}$. The location of these vortices is completely determined by the coefficient a: they lie on one or several concentric circles in $\mathcal{A}$ whose radii attain a given minimization problem involving $a(r)$.

The method involves deriving sharp upper and lower bounds on the energy of minimizers via a vortex-ball construction as in the work of Sandier–Serfaty. In order to determine the number and location of the vortices for supercritical rotations we must take into account the effect of the Giant Vortex in constructing the upper and lower bounds.

Nodal solutions of elliptic equations

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We report on some joint work with Tobias Weth (Giessen) and Michel Willem (Louvain-La-Neuve) about the existence and the properties of nodal solutions of

$$(1) \qquad -\Delta u = f(u) \qquad \text{or} \qquad (2) \qquad -\varepsilon^2 \Delta u = f(u), \quad \varepsilon \to 0$$

on a bounded domain $\Omega \subset \mathbb{R}^N$ with Dirichlet boundary conditions. The nonlinearity is assumed to be superlinear and subcritical in u as $|u| \to \infty$. The following results are obtained with variational methods applied to the energy functional $J : H_0^1(\Omega) \to \mathbb{R}$ associated to (1) or (2).

If Ω contains a large ball then (1) has at least three nodal solutions, two of which have precisely two nodal domains, the third one has either two or three nodal domains. Generically the Morse indices of the first two solutions are 2 and $N+1$, respectively, the Morse index of the third solution is $N+2$. Ω may be topologically trivial. This result is a consequence of a more general theorem which gives a lower bound for the number of nodal solutions of the singularly perturbed equation (2) with precisely two nodal domains in terms of the topology of the configuration space of Ω.

If Ω is radially symmetric we can say more about the minimal nodal solutions. These are obtained as minimizers of J on the set (not manifold!)

$$\mathcal{M} = \{u \in H_0^1(\Omega) : \ u^+ \neq 0 \neq u^-, \ J'(u)u^+ = 0 = J'(u)u^-\}.$$

If $\inf J|\mathcal{M}$ is achieved then a minimizer is in fact a nodal solution of (1). On a radially symmetric domain such a minimal nodal solution is "foliated Schwarz symmetric" but not radially symmetric in general. This kind of symmetry appears also for minimal nodal solutions of weighted asymmetric eigenvalue problems, or for Fucik eigenfunctions of $-\Delta$ corresponding to the first Fucik curve.

HETEROCLINIC SOLUTIONS IN A PHASE TRANSITION MODEL WITH INDEFINITE NONLOCAL INTERACTIONS

PETER W. BATES, XINFU CHEN, AND ADAM J.J. CHMAJ

We construct a global minimizer of the van der Waals' free energy functional

$$\mathcal{E}(u) := \tfrac{1}{4} \iint_{\mathbb{R}^2} J(x-y)(u(x)-u(y))^2 dx dy + \int_{\mathbb{R}} F(u(x))\, dx, \tag{1}$$

in the space $u_0 + L^1 \cap L^\infty$, where $u_0 := \pm 1$ if $\pm x > 0$. F is a double-well potential with equal local minima at ± 1, e.g., $F(s) = \frac{1}{4}(s^2-1)^2$. The double integral term in (1), with an even kernel J, such that $\int_{\mathbb{R}} J > 0$, replaces the more common $\frac{1}{2}\int_{\mathbb{R}} u'(x)^2 dx$ used, e.g., in the Ginzburg-Landau functional. One can derive (1) as the Helmholtz free energy of a continuous spin system as was suggested by van der Waals in the 1890's. In his, and subsequent analyses, the double integral term is expanded in a formal series of higher order derivatives, and truncated at the first gradient term.

In the derivation, J, sometimes called the intermolecular potential appears as $J = J^{AA} + J^{BB} - 2J^{AB}$, where the positive J^{ij}'s are Ising energies of interaction between spins i and j. Thus in general, J clearly can change sign, which is the case studied in this work. We note that short-range repulsive and long-range attractive interactions appear in, e.g., the van der Waals' forces (often modeled by the Lennard-Jones potential).

Note that since there is no gradient term in (1), the underlying space is not restricted to differentiable functions and critical points are possibly discontinuous functions. Indeed, monotone discontinuous heteroclinic critical points of (1) were constructed in previous work by the authors and others. For instance, we discovered families of critical points, discontinuous along arbitrarily prescribed interfaces, which are seemingly stable, since the formal second variation is positive. However, there is no variational sufficiency condition for minimizers of (1), since if it is defined on the natural space L^2 or $u_0 + L^2$, it is only $C^{1,1}$ and these discontinuous solutions which usually form nonsmooth continua in L^2 are in general not local minimizers.

Without loss of generality, let $\int_{\mathbb{R}} J = 1$. Here we study the case when the Fourier transform, $\hat{J} \leq 1$, which assures that (1) is bounded below by 0. If $J \geq 0$, a monotone global minimizer of (1) can be constructed using monotone rearrangements. However, if J changes sign, monotonicity methods are not applicable, and in general the global minimizer will not be monotone. Instead we use ideas from concentration and convexification techniques.

Shadowing of collision chains for the elliptic 3 body problem

S. Bolotin

Suppose Sun of mass 1, Jupiter of mass ε and Asteroid of negligible mass move in $\mathbb{R}^2$. Let $u(t)$ be the elliptic T-periodic orbit of the Jupiter. The motion of the Asteroid is described by a Lagrangian system (L_ε) with

$$L_\varepsilon(q,\dot q,t)=|\dot q|^2/2+|q+\varepsilon u(t)|^{-1}+\varepsilon|q-u(t)|^{-1}.$$

System (L_ε) is a singular perturbation of the Kepler problem (L_0).

Fix $m,n\in\mathbb{N}$. Let Π be the set of chains $c=(c_i)_{i=1}^n$ of collision curves $c_i:[t_{i-1},t_i]\to\mathbb{R}^2\setminus\{0\}$ such that $c_i(t_{i-1})=u(t_{i-1})$, $c_i(t_i)=u(t_i)$. The time moments $t_0<\ldots<t_{n-1}$ are independent variables and $t_n=t_0+mT$. Thus Π is an open set in $W_0^{1,2}([0,1],\mathbb{R}^{2n})\times\mathbb{R}^n$. Critical points of the action functional

$$I(c)=\sum I(c_i),\qquad I(c_i)=\int L_0(c_i(t),\dot c_i(t),t)\,dt$$

are chains of collision orbits of system (L_0) such that the relative Hamiltonian $h=H_0-\dot q\cdot\dot u(t)$ does not change at collisions: $h_i^+=h_i^-=h_i$. We say that $c=(c_i)_{i=1}^n$ is a *nondegenerate collision chain* if it is a nondegenerate critical point of I and at each collision the direction of relative velocity $v=\dot q-\dot u(t)$ changes: $v_i^+\nparallel v_i^-$.

Theorem 1. *For any nondegenerate periodic collision chain $c=(c_i)_{i=1}^n$, there exists $\varepsilon_0>0$ such that for any $\varepsilon\in(0,\varepsilon_0)$, there exists a unique mT-periodic solution of system (L_ε) which is $O(\varepsilon)$-close to $c_i(t)$ for $t_{i-1}\le t\le t_i$.*

Such shadowing periodic orbits were called by Poincaré the periodic solutions of the second kind. However, Poincaré didn't prove their existence.

A similar result holds for infinite collision chains. Take N open bounded sets $U_k\subset\mathbb{R}^2$ such that for each $(t_1,t_2)\in U_k$ there exists a collision orbit $c:[t_1,t_2]\to\mathbb{R}^2$ of (L_0) with $c(t_1)=u(t_1)$, $c(t_2)=u(t_2)$, smoothly depending on (t_1,t_2). In particular $t_1<t_2$ are not conjugate along c. Then $I(c)=S_k(t_1,t_2)$ is a smooth function on U_k. Sequences $\varkappa=(k_i)_{i\in\mathbb{Z}}$ and $\tau=(t_i)_{i\in\mathbb{Z}}$ such that $(t_{i-1}-Tm_i,t_i-Tm_i)\in U_{k_i}$, $m_i\in\mathbb{Z}$, define a collision chain $c=(c_i)_{i\in\mathbb{Z}}$. Set

$$A_\varkappa(\tau)=\sum I(c_i)=\sum S_{k_i}(t_{i-1},t_i).$$

The functional is formal but its derivative $A'_\varkappa(\tau)\in l_\infty$ is well defined. A collision chain $c=(c_i)_{i\in\mathbb{Z}}$ corresponding to the critical point τ is called nondegenerate if the second derivative $A''_\varkappa(\tau):l_\infty\to l_\infty$ has a bounded inverse and the changing direction condition is uniform in i. Then for small $\varepsilon\in(0,\varepsilon_0)$ there exists an orbit of (L_ε) shadowing the chain c.

If S_k satisfies the twist condition $D^2_{t_1t_2}S_k\ne 0$, critical points of $A_\varkappa$ correspond to orbits of compositions $f_{k_n}\circ\ldots\circ f_{k_0}$ of symplectic maps $f_{k_i}:(t_{i-1},h_{i-1})\to(t_i,h_i)\in(\mathbb{R}/T\mathbb{Z})\times\mathbb{R}$ with generating functions S_{k_i}. Such random dynamical systems have rich hyperbolic dynamics even if every map f_k is integrable. This makes it possible to construct many nondegenerate collision chains and hence periodic and chaotic shadowing orbits for system (L_ε).

Minimization methods for quasi-linear problems and stability of solitary water waves

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Minimization for quasi-linear elliptic problems arises when studying the stability of capillary-gravity water waves. The energy functional is of the form $U\backslash\{0\} \ni w \to K(w) + \dfrac{\mu^2}{L(w)}$, where U is a ball included is some Hilbert space, $K, L \in C^2(U)$, L is a (possibly not coercive) positive-definite quadratic form and $\mu > 0$ is a parameter.

Solitary water waves are found as limits of periodic waves obtained by minimization, and their stability is then a consequence of the method. This gives a new proof of the existence result of Amick and Kirchgässner (1989) and of the stability result of Mielke (2002), and this leads to new results on the stability of solitary water waves in the presence of weak surface tension.[1] [2]

The abstract method being quite general, it is of independent interest. Consider a real Hilbert space $(X_0, \langle\cdot,\cdot\rangle_0)$ with norm $\|\cdot\|_0$ and a (possibly unbounded) positive definite self-adjoint operator A in X_0 such that A^{-1} exists as a continuous operator on X_0. Denote by X_n the domain of $A^{n/2}$ for $n \in \mathbb{N}$, which is dense in X_0. The inner-product and norm on X_n are defined by $\langle u, w\rangle_n = \langle A^{n/2}u, A^{n/2}w\rangle_0$ and $\|w\|_n = \|A^{n/2}w\|_0$.

Let $R_2 > 0$ and $U \subset X_2$ be the open ball $\{w \in X_2 : \|w\|_2 < R_2\}$. The functional K is assumed to satisfy $K(0) = 0$, $K \geq 0$ on U and $K'(w)Aw \geq C_1\|w\|_2^2$ for all $w \in X_4 \cap U$ and for some constant $C_1 > 0$. Assume also that $L'(w)Aw \leq C_2 K(w)^q \|w\|_2^{2(1-q)}$ for all $w \in X_4 \cap U$ and for some constants $C_2 > 0$ and $q \in (0, 1]$. Observe that $K'(0) = L'(0) = 0$. Suppose that $\inf_{w\in X_2\backslash\{0\}}\{K''(0)(w,w)\}/\{2L(w)\} = 1$. The next hypothesis is about gain of regularity: for all $\gamma > 0$, $\epsilon > 0$ and $w \in U$,

$$K'(w) - \gamma L'(w) + \epsilon A^2 w = 0 \;\Rightarrow\; w \in X_4,$$

where A^2w denotes the functional in X_2^* taking the value $\langle Aw, Au\rangle_0$ at any $u \in X_2$.

Finally suppose that if $\{w_n\} \subset U$ converges weakly in X_2 to $w \in U$, then $\lim_{n\to\infty} K(w_n) = K(w)$ and $\lim_{n\to\infty} L(w_n) = L(w)$.

Theorem. *Let $\mu > 0$ satisfy $\dfrac{4C_2(2\mu)^q}{C_1} < R_2^{2q}$. For $u \in U\backslash\{0\}$, set $J(u) := K(u) + \dfrac{\mu^2}{L(u)}$ and assume that there exists $u_* \in U\backslash\{0\}$ such that $J(u_*) < 2\mu$. Then there exists $w \in U\backslash\{0\}$ such that $K'(w) = \dfrac{\mu^2}{L(w)^2}L'(w)$ and $J(w) = \min\{J(u) : u \in U\backslash\{0\}\} < 2\mu$.*

Its proof is close to the one of an analogous theorem in a paper by Buffoni-Séré-Toland. [3]

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[1] B. Buffoni, *Existence and conditional energetic stability of capillary-gravity solitary water waves by minimization*, to appear in the Arch. Rat. Mech. Anal.

[2] B. Buffoni, *Conditional energetic stability of gravity solitary waves in the presence of weak surface tension*, to appear in Topological Methods in Nonlinear Analysis, Journal of the Juliusz Schauder Center.

[3] B. Buffoni, É. Séré, and J. F. Toland, *Minimization methods for quasi-linear problems, with an application to periodic water waves*, to appear in SIAM JMA.

Almut Burchard (University of Virginia)

Compactness via Symmetrization

Lack of compactness is a principal analytical difficulty in the study of functionals on unbounded domains. For symmetric functionals, the existence of minimizers can often be established by first restricting the problem to radially symmetric functions with the help of a rearrangement inequality, and then using the additional compactness properties of symmetric functions, as captured by the Strauss radial lemma [1977], to find a convergent minimizing sequence. This strategy was used in the determination of the sharp Sobolev constants by Talenti [1976], in the analysis of the sharp Hardy-Littlewood-Sobolev inequalities by Lieb [1983], and for the study of ground states for many functionals of Mathematical Physics.

Certain dynamical stability problems can also be reduced to the study of related variational problems. Here, it is the compactness of arbitrary minimizing sequences, not just the existence of minimizers, that plays the key role. In a series of famous papers, Lions [1984] introduced a general abstract *concentration compactness* principle which has lead to many applications. In order to apply this principle to a specific problem, some additional analysis is usually needed.

In recent joint work with Y. Guo [2004], we closely examine the role of translations for minimizing sequences of two classes of functionals that appear in many applications of the concentration compactness principle: convolution integrals of the form

$$\mathcal{I}(f) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) K(|x-y|) f(y)\, dxdy$$

with some strictly decreasing, positive definite kernel K, and gradient integrals of the form

$$\mathcal{J}(g) = \int_{\mathbb{R}^n} \Phi(|\nabla g(x)|)\, dx$$

with some strictly convex, increasing integrand Φ. Special cases are the Coulomb kernel in three dimensions, and the p-norm of the gradient. We show that the difference between a minimizing sequence and the corresponding sequence of symmetrized functions is characterized by appropriate translations. Besides the interest of our results in classical analysis, this characterization suggests a practical two-step procedure for establishing compactness on an unbounded domain. *Step 1.* Show convergence of all symmetric minimizing sequences. *Step 2.* Show convergence up to translations for general minimizing sequences, assuming that their symmetrizations converge. The first step implies the existence of minimizers; it is also a necessary ingredient in the proof that these minimizers are dynamically stable under symmetric perturbations. We focus on the second step, which implies dynamical stability under more general perturbations. We discuss applications to symmetric galaxy configurations appearing in recent work of Guo and Rein [1999-2001], and to functionals with additional scaling symmetries.

Technically, our results are inspired by *asymmetry* inequalities, which estimate the difference between a function or a body and a symmetric one by a related geometric quantity. The most powerful result in that direction, due to Hall [1992], states that a body whose surface area is close to the surface area of a ball of the same volume is in fact close (in symmetric difference) to a suitable translate of the ball. We expect that asymmetry inequalities should hold for large classes of symmetric functionals, including the Coulomb electrostatic energy. We hope that our approach can give another perspective on concentration compactness for symmetric functionals.

Ivar Ekeland (University of British Columbia and PIMS)

Existence and regularity of solutions for a new type of variational problems

Abstract: When computing conditional expectations by Monte-Carlo methods, one tries to minimize the mean variance of the error. Applying Malliavin calculus to the problem, one is led to a novel type of Sobolev space, consisting of all functions on the positive orthant of R^n, such that every derivative not containing terms in $(d^p)/(dx_i)^p$ with $p = 2$ or more is square integrable. The last derivative with this property is $d^n/(dx_1)...(dx_n)$. We show that this is a bona fide Sobolev space, and we consider the problem of minimizing a quadratic form on that space under boundary conditions. We show existence, uniqueness and regularity

Maria J. Esteban

CNRS and University Paris-Dauphine

Title : About a physical notion of ground-state solutions for a highly indefinite variational problem.

Abstract :

The Dirac-Fock equations are the Euler-Lagrange equations corresponding to the Dirac-Fock energy functional in a "sphere" of $L^2(R^3, C^4)^N$, N being a positive integer. This model corresponds, in an approximated way, to the search of stationary states for relativistic atoms and molecules. The Dirac operator being unbounded, both from above and from below, the corresponding energy functional is higly indefinite. However, this models "should contain" a notion of ground state if it is to describe a physical situation in which "minimal energy" solutions should exist and correspond to "most probable" configurations for the physical system. Moreover, the nonrelativistic limit of these equations (in the high light speed limit) can be shown to be the Hartree-Fock equations, for which ground state solutions exist under reasonable conditions (the Hartree-Fock energy is bounded from below).

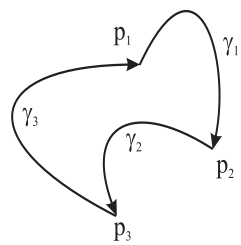
In this talk I will first describe the Dirac-Fock equations, and how taking the nonrelativistic limit leads us to the Hartree-Fock equations. Then, I will concentrate in showing how for high light speeds, various different variational problems are equivalent to the one that we use to show existence of solutions. From this, we will be able to obtain a physically relevant notion of ground state solutions for this model. By doing so, what we really show is that the critical points that are physically relevant all lie in a subset of the "sphere" defined by a nonlinear constraint. What we indirectly show is that the Dirac-Fock energy is bounded from below in that set (it is not in the whole "sphere"), that its minimum is reached there and that the minimizers are critical points of the energy which correspond to the solutions that we had previously found by using an unconstrained variational argument.

Note : All the results presented in this talk have been obtained in collaboration with Eric Séré.

HOMOCLINIC CYCLES, CLOSED 1-FORMS, AND HOMOTOPY INVARIANTS

M. FARBER

In the talk I shall describe a new *focusing effect* developed in [1]. It is an approach which uses *homotopy* information and predicts existence of homoclinic cycles in certain dynamical systems. A *homoclinic cycle* is determined by a sequence of fixed points $p_1, \dots, p_k$ of the flow and by a sequence of connecting orbits $\gamma_1, \dots, \gamma_k$ such that $\alpha(\gamma_i) = \{p_i\}$ for $i = 1, \dots, k$, $\omega(\gamma_i) = \{p_{i+1}\}$ for $i = 1, \dots, k-1$ and $\omega(\gamma_n) = \{p_1\}$.



A new numerical invariant $\text{cat}(X, \xi)$ was defined in [1]. Here X is a finite polyhedron and $\xi \in H^1(X; \mathbf{R})$ is a real cohomology class. In the special case $\xi = 0$ the number $\text{cat}(X, \xi)$ coincides with the Lusternik-Schnirelman category $\text{cat}(X)$. One of important properties of $\text{cat}(X, \xi)$ is its *homotopy invariance*. Also, $\text{cat}(X, \xi)$ allows cohomological lower bounds which use cup-products of cohomology classes of certain local coefficient systems.

Theorem 1. *If the number of zeros of a smooth closed 1-form ω is less than $\text{cat}(M, \xi)$, where $\xi = [\omega] \in H^1(M; \mathbf{R})$ denotes the cohomology class of ω, then the flow of any gradient-like vector field for ω has a homoclinic cycle.*

Theorem 1 guaranties existence of homoclinic cycles and shows their surprising stability. The conditions of Theorem 1 are satisfied for any nonzero cohomology class $\xi \in H^1(M; \mathbf{R})$ with $\text{cat}(M, \xi) > 1$: it is proven in [1] that any nonzero cohomology class ξ can be realized by a closed 1-form ω having at most one zero.

References

[1] M. Farber, *Topology of closed one-forms*, Mathematical Surveys and Monographs, Vol 108, American Mathematical Society, 2004.

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Nassif Ghoussoub (UBC and PIMS)

Anti-self dual Lagrangians and new variational formulations of boundary value problems and evolution equations

Abstract: We develop a theory of anti-self dual Lagrangians that allows for variational formulations and resolutions for many boundary value and initial value problems which normally cannot be obtained as Euler-Lagrange equations of action functionals. Examples include non-potential operator equations (like nonlinear transport and others involving first order differential operators), as well as certain dissipative evolution equations (like the heat equation, porous media, other gradient flows and the Navier-Stokes equations).

Vortices and current in the three-dimensional thin-film Ginzburg-Landau model of superconductivity

Glotov, Purdue University

We study the variable thickness Ginzburg-Landau equations describing type-II superconducting thin films. While the convergence of the order parameter is discussed in a paper by Chapman, Du, and Gunzburger, we turn our attention to the equation for the magnetic potential and obtain results on convergence of various quantities involved in the latter equation. We also show that the limiting order parameter is a minimizer of the two-dimensional thin-film energy. The limiting problem, among other properties, has an advantage, from a computational point of view, of being restricted to a bounded domain. The regularity of the solutions to the three-dimensional problem presents another interest for us. Using regularity, we obtain uniform convergence of the three-dimensional minimizers. This in turn allows us to conclude, thanks to the description of the vortex structure for minimizers of the two-dimensional thin-filmn energy available from the work of Ding and Du, that the three-dimensional minimizers exhibit vortices and their degree is preserved as the thickness of the film tends to zero.

Hichem, Hajaiej(University of Virginia)

Existence and non-existence of Schwarz symmetric ground states for elliptic eigenvalue problems

ABSTRACT

We determine a class of Carathéodory functions G for which the minimum of the following constrained minimization problem.

$$M(c) = \inf\{J(u) : \breve{i}nH^1(R^N) \text{ and } \int_{\mathbb{R}^N} u(x)^2 dx = c^2\}\text{where}c \quad > 0 \tag{1}$$

is prescribed and$(u) = \int_{\mathbb{R}^N} \frac{1}{2} |\nabla u(x)|^2 - G(|x|, u(x))dx.$

is achieved by a Schwarz symmetric function.We also discuss the optimality of our results,proving that,in some cases,the minimum is not attained if one of our assumptions is ruled out.

Periodic Solutions of Second Order Superquadratic Hamiltonian Systems with Potential Changing Sign

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Let h be continuous, 2π-periodic, sign changing and satisfy

$$\overline{\{t|h(t)>0\}} \cap \overline{\{t|h(t)<0\}} = \emptyset. \qquad (h_0)$$

Set $h_- = min\{0, h\}$ and $h_+ = max\{0, h\}$, we consider the periodic solutions of the second order Hamiltonian system

$$-\ddot{x} - \lambda x = h_-(t)V_1'(x) + h_+(t)V_2'(x). \qquad (1)$$

Let $S_0 = S^1 \backslash (\overline{\{t|h(t)>0\}} \cup \overline{\{t|h(t)<0\}})$, $\sigma(S_0)$ be the eigenvalues of $-\ddot{x} = \lambda x$ with the Dirichlet boundary value on S_0 and $\sigma(S^1) = \{k^2 | k \quad \text{is integer}\}$.

Theorem 1. *Let V_1 and V_2 be C^2 functions satisfying*

(V1) there are constants $\theta > 2$ and $r > 0$ such that

$$V'(x) \cdot x \geq \theta V(x) > 0 \quad |x| \geq r;$$

(V2) $\lambda \notin \sigma(S^1)$ and $|V_1'(x)| = |V_2'(x)| = o(|x|)$.

If $\lambda \notin \sigma(S_0)$, then (1) has a nonzero 2π-periodic solution, and has a sequence of unbounded 2π-periodic solutions if V_1 and V_2 are even in x.

Similar result hold if $h \in C^1(S^1)$, sign changing and all zeros of h are simple.

On the Nirenberg Problem

Ji, Min

Let (S^2, g_0) be the standard 2-sphere. We consider the Nirenberg problem: which positive function R can be the scalar curvature of some metric g which is pointwise conformal to g_0? Writing $g = e^u g_0$, the problem is equivalent to the solvability of the following PDE:

$$-\triangle_{g_0} u + 2 - R\, e^u = 0, \qquad \text{on} \quad S^2. \tag{1}$$

Moser first proved the solvability for R being an even function. Later there were much researches about it. In fact, it can be reduced to a variational problem. The corresponding functional is bounded from below, however has no minimum if R is not a constant. Then most of works made efforts to look for minimax type of solutions. The following result is due to A.Chang–P.Yang:

Chang–Yang Theorem Suppose R has only isolated non-degenerate critical points and in addition satisfies

$$|\bigtriangledown R| + |\triangle R| \neq 0. \tag{2}$$

If $\Sigma_{x\in S_-}(-1)^{\text{ind}(x)} \neq 1$, where $S_- := \{x \in S^2 : \bigtriangledown R(x) = 0, \triangle R(x) < 0\}$, then (1) has a solution.

This result was proved first by using a delicate minimax procedure, and reproved later by other authors in different way, such as, Morse theory, degree theory.

Below is our consideration. For $R \in C^3$ define a simple map $G : S^2 \to R^3$ by

$$G(x) = \bigtriangledown R(x) \cdot \bigtriangledown x \; - \triangle R(x) x \qquad x \in S^2$$

We denote by B the unit ball in R^3.

Theorem 1 Let R satisfy (2). If $\deg(G,\ B,\ 0) \neq 0$, then (1) has a solution.

Since our map G has a simple expression, it is easy to calculate their degree, and some previous important results can be derived and improved immediately.

Corollary 1 If

$$\triangle R(x) \triangle R(-x) - \bigtriangledown R(x) \cdot \bigtriangledown R(-x) \geq 0 \quad \forall x \in S^2,$$

then the equation (1) has a solution provided that (2) holds.

Corollary 2 If R is non-degenerate in $S_+ = \{x \in S^2 : \nabla R(x) = 0, \triangle R(x) > 0\}$ (or in $S_- = \{x : \nabla R(x) = 0, \triangle R(x) < 0\}$) and

$$\sum_{x\in S_+} (-1)^{\text{ind}(x)} \neq 1 \quad (\text{ or } \sum_{x\in S_-} (-1)^{\text{ind}(x)} \neq 1),$$

then (1) has a solution provided that (2) holds.

Remark Corollary 1 may be viewed as an extension of Moser's result. Corollary 2 extends Chang–Yang's theorem which assumes all critical points of R to be non-degenerate. And some other results with completely form can be also derived from our Theorem 1.

Classification of Solutions for a System of Integral Equations

Wenxiong Chen, Congming Li, Biao Ou

Abstract

I will present the joint work of Wenxiong Chen, Biao Ou and myself related to the well-known Hardy-Littlewood-Sobolev inequality:

$$\int_{R^n}\int_{R^n} f(x)|x-y|^{\alpha-n}g(y)dxdy \\ \leq C(n,s,\alpha)||f||_r||g||_s.$$

Here $f \in L^r(R^n)$, $g \in L^s(R^n)$, $0 < \alpha < n$ and $\frac{1}{r}+\frac{1}{s}=\frac{n+\alpha}{n}$.

We are mainly interested in the study of non-negative solutions to its Euler-Lagrange equations which can be transformed to the following system of integral equations in R^n:

$$\begin{cases} u(x) = \int_{R^n} |x-y|^{\alpha-n} v(y)^q dy \\ v(x) = \int_{R^n} |x-y|^{\alpha-n} u(y)^p dy \end{cases} \tag{0.1}$$

with $\frac{1}{q+1}+\frac{1}{p+1}=\frac{n-\alpha}{n}$. First, under the natural integrability conditions $u \in L^{p+1}(R^n)$ and $v \in L^{q+1}(R^n)$, we prove that all the solutions are radially symmetric and monotone decreasing about some point. In the special case $p=q$, we classified all the solutions which solved a open problem posted by Professor E. Lieb.

I will also present some of our work on regularity, radial symmetry, and monotonicity of solutions to this and some related systems which include subcritical cases, super critical cases, and singular solutions in all cases; and obtain qualitative properties for these solutions.

Some related systems of partial differential equations are also studied.

On the Yamabe problem and a fully nonlinear version of it

YanYan Li, Rutgers University

Abstract

Let (M, g) be a compact smooth Riemannian manifold of dimension $n \geq 3$, and let

$$A_g := \frac{1}{n-2}(Ric_g - \frac{R_g}{2(n-1)}g)$$

denote the Schouten tensor of g, where Ric_g and R_g denote respectively the Ricci tensor and the scalar curvature of g. We use $\lambda(A_g) = (\lambda_1(A_g), \cdots, \lambda_n(A_g))$ to denote the eigenvalues of A_g with respect to g.

Let V be an open convex subset of $\mathbb{R}^n$ which is symmetric with respect to the coordinates. We assume that $\emptyset \neq \partial V$ is smooth and satisfies

$$\nu(\lambda) \in \{\mu \in R^n | \mu_i > 0, \forall 1 \leq i \leq n\}, \qquad \forall\ \lambda \in \partial V,$$

and

$$\nu(\lambda) \cdot \lambda > 0, \qquad \forall\ \lambda \in \partial V.$$

Let

$$\Gamma(V) := \{s\lambda \mid \lambda \in V,\ 0 < s < \infty\}$$

be the cone with vertex at the origin generated by V.

Conjecture. *Let (M^n, g), V and $\Gamma(V)$ be as above. We assume that*

$$\lambda(A_g) \in \Gamma(V), \qquad \text{on } M^n.$$

Then there exists a smooth positive function $u \in C^\infty(M^n)$ such that the conformal metric $\hat{g} = u^{\frac{4}{n-2}} g$ satisfies

$$\lambda(A_{\hat{g}}) \in \partial V, \qquad \text{on } M^n.$$

For $V = \{\lambda \in \mathbb{R}^n \mid \sum_{i=1}^n \lambda_i > 1\}$, the conjecture is the Yamabe Conjecture in the positive case.

We present some recent results concerning the conjecture, which include some results on the existence and compactness of solutions as well as some Liouville type theorems. These are joint works with Aobing Li and with Lei Zhang.

Multiple Brake Orbits in Bounded Convex Symmetric Domains

Yiming Long, Duanzhi Zhang, Chaofeng Zhu

Nankai University, Tianjin, China

Abstract Let $V \in C^2(\mathbf{R}^n, \mathbf{R})$ and $h > 0$ such that $\Omega \equiv \{q \in \mathbf{R}^n | V(q) < h\}$ is bounded, open and connected. Consider the following given energy problem of the second order Hamiltonian system:

$$\ddot{q}(t) + V'(q(t)) = 0, \quad \text{for } q(t) \in \Omega, \tag{1}$$

$$\frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) = h, \qquad \forall t \in \mathbf{R}, \tag{2}$$

$$\dot{q}(0) = \dot{q}(\frac{\tau}{2}) = 0, \tag{3}$$

$$q(\frac{\tau}{2} + t) = q(\frac{\tau}{2} - t), \qquad q(t+\tau) = q(t), \quad \forall t \in \mathbf{R}. \tag{4}$$

A solution (τ, q) of (1)-(4) is called a *brake orbit* on Ω. We call two orbits q and $p : \mathbf{R} \to \mathbf{R}^m$ *geometrically distinct*, if $q(\mathbf{R}) \neq p(\mathbf{R})$. Denote by $\mathcal{J}(\Omega)$ and $\tilde{\mathcal{J}}(\Omega)$ the sets of all brake orbits and geometrically distinct brake orbits in Ω respectively.

In 1948, H. Seifert proved ${}^{\#}\mathcal{J}(\Omega) \geq 1$ provided V is analytic, Ω is homeomorphic to the unit ball in $\mathbf{R}^n$, and $V'(q) \neq 0$ for $q \in \partial\Omega$. Then he conjectured that ${}^{\#}\tilde{\mathcal{J}}(\Omega) \geq n$ holds under the same conditions. Since then many studies have been carried out for brake orbits. Specially in 1983-1984, K. Hayashi, H. Gluck-W. Ziller, and V. Benci proved indeoendently ${}^{\#}\mathcal{J}(\Omega) \geq 1$, if V is C^1, $\bar{\Omega} = \{V \leq h\}$ is compact, and $V'(q) \neq 0$ for all $q \in \partial\Omega$. In 1987, P. Rabinowitz proved the corresponding result for first order Hamiltonian systems. For multiplicity results concerning Seifert's this conjecture, we are only aware of the papers of E. van Groesen in 1985, A. Szulkin in 1989, and A. Ambrosetti-V. Benci-Y. Long in 1993, in which ${}^{\#}\tilde{\mathcal{J}}(\Omega) \geq n$ was proved under various pinching conditions on the hypersurface $\partial\Omega$.

In this paper we study the multiplicity of brake orbits without any pinching conditions. Our main result in this paper is the following:

Theorem. *For $n \geq 2$ and $V \in C^2(\mathbf{R}^n, \mathbf{R})$, suppose $V(0) = 0$, $V(q) \geq 0$, $V(-q) = V(q)$, and $V''(q)$ is positive definite for all $q \in \mathbf{R}^n \backslash \{0\}$. Then for any given $h > 0$ and $\Omega \equiv \{q \in \mathbf{R}^n | V(q) < h\}$, there holds*

$$^{\#}\tilde{\mathcal{J}}(\Omega) \geq 2. \tag{5}$$

Symmetry properties of positive solutions to nonlinear second order finite difference boundary value problems

P.J. McKenna & W. Reichel

Over the past quarter century, one field of intense research activity has been the study of what symmetry properties the solution of a nonlinear elliptic boundary value problem can inherit from the domain on which it is being solved.

A classic paper is that of Gidas-Ni-Nirenberg, in which a typical result of the type we have in mind was: a positive solution of the boundary value problem

$$\Delta u = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1}$$

must be radially symmetric if Ω is a ball.

More recently, a related area has been attracting growing attention, namely how does one approximate solutions of this type of nonlinear boundary value problem? Typically, the work in this area relies on a suitable discretization of (1), (most commonly by finite-differences), and then uses theoretical ideas from nonlinear analysis such as monotonicity methods, mountain pass algorithms, or linking methods, to develop an approximate or exact solution to the discretized problem.

The purpose of this paper is to begin to address the so-far-neglected question: if the partial differential equation (1) has inherited certain symmetry properties from the domain, to what extent does the discretized problem also inherit these symmetry properties?

Consequently, we are led to study the most natural discretization of (1), namely,

$$\begin{aligned} &u_{i+1} - 2u_i + u_{i-1} = h^2 f(u_i), \quad u_i > 0, \quad i = -(N-1)\dots N-1, \quad (2)\\ &u_{-N} = u_N = 0, \end{aligned}$$

where $h = L/N > 0$ is the mesh-size of an equidistant mesh on $[-L, L]$. We suppose that $f : [0, \infty) \to R$ is a given function. A solution of (2) is represented as a vector $u = (u_{-N}, \dots, u_N) \in^{2N+1}$. With $\|u\|_\infty = \max_{i=-N\dots N} |u_i|$ we denote its maximum norm. The first natural conjecture would be that the discrete approximate solution u_i would have a maximum at $j = 0$, and be symmetric about 0 in the sense that $u_{-j} = u_j$. This would exactly reflect the symmetry properties of the analogous continuous problem. This is false.

Roughly speaking our result states *as* $h \to 0$*, the solution becomes more and more symmetric about the origin and the maximum* $\to$ *towards the origin.* Thus, the correct result is that for a sufficiently small space step, the solution will be "approximately" symmetric about the origin. We shall prove an analogous result in the partial differential equation setting.

1 Stable Vortex Solutions to the Ginzburg-Landau Equations with and without Magnetic Field

Montero, McMaster University

Superconductivity is a phenomenon in which certain metals, such as mercury, lead and tin, lose electrical resistance below a critical temperature. It was first observed by H. Kamerlingh-Onnes in 1908. In 1950, Ginzburg and Landau presented a model for this phenomenon that today is accepted as a valid macroscopic model.

The theory of Ginzburg and Landau is based on minimizing the Helmhotz free energy. In a non-dimensional form this energy can be expressed as

$$G_\varepsilon(u, A) = \int_\Omega \big(\frac{1}{2}|(\nabla - iA)u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2\big)\, dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla \times A - h_{ap}|^2\, dx. \tag{1}$$

Here $\Omega \subseteq \mathbb{R}^n$ ($n = 2$ or 3) represents the superconducting sample. The case $n = 2$ pertains to the situation where either Ω is a cross section of an infinite sample, or represents a thin film. The quantity h_{ap} is an applied magnetic field, $A : \mathbb{R}^n \to \mathbb{R}^n$ is the vector potential of the induced magnetic field, and $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{C}$ is a kind of wave function for the superconducting electrons. A most relevant quantity is $|u|^2$: it represents the density of superconducting electrons at a point in the sample. A value $|u|^2 = 1$ indicates complete superconductivity, while $|u|^2 = 0$ means that the material is in the completely normal state. These portions that remain in the normal state are usually referred to as vortices. Finally, the parameter $\varepsilon > 0$ depends on the material and is known as the Ginzburg-Landau parameter.

From a mathematical point of view, the study of (1) is an arduous task. To make the problem more tractable we consider initially a simplified version of G_ε in which we ignore h_{ap} and A. This leads to what we will refer to as the reduced Ginzburg-Landau energy

$$E_\varepsilon(u) = \int_\Omega \left\{ \frac{|\nabla u|^2}{2} + \frac{(1-|u|^2)^2}{4\varepsilon^2} \right\} dx, \tag{2}$$

defined for $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{C}$.

Although the simplification that leads from G_ε to E_ε may seem arbitrary, it has been the case in the past that results for E_ε have led to a better understanding of G_ε. Let us consider the following decomposition of G_ε due to

Bethuel and Riviere:

$$G_\varepsilon(u,A) = \quad E_\varepsilon(u) - \int_\Omega j(u)\cdot A\,dx + \frac{1}{2}\int_\Omega |u|^2\,|A|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\times A - h_{ap}|^2\,dx,$$

where $j(u) = \frac{1}{2i}(\bar{u}\nabla u - u\nabla\bar{u})$ and $\bar{u}$ means complex conjugation. This decomposition underscores the importance of E_ε in the study of G_ε, and suggests that local minimizers of the former could indicate the presence of local minimizers of the latter, at least when h_{ap} is small.

A crucial tool for us is the theory of weak Jacobians, as developed by R. L. Jerrard and M. H. Soner. These authors show that, for a family of functions $\{u_\varepsilon\} \in W^{1,2}(\Omega;\mathbb{C})$ with $E_\varepsilon(u_\varepsilon) \le C\,|\ln\varepsilon|$, the energies $E_\varepsilon(u_\varepsilon)$ asymptotically bound from above the length of the zero sets (or vortex sets) of u_ε. Note that for functions mapping $\Omega \subseteq \mathbb{R}^n$ to $\mathbb{C}$, these zero sets should generically be of dimension $n-2$. This suggests that the presence or absence of a locally length minimizing curve lying in $\Omega \subseteq \mathbb{R}^3$ may have some bearing on the presence or absence of local minimizers of E_ε in $W^{1,2}(\Omega;\mathbb{C})$ when $\varepsilon > 0$ is small.

In order to make this approach work we appeal to tools from geometric measure theory. In fact, the weak Jacobians of Jerrard and Soner can be viewed as linear functionals that act on Hölder continuous, compactly supported vector fields in Ω. We use the limiting behavior of E_ε to identify a geometric condition on Ω that guarantees the existence of local minimizers of E_ε. This condition essentially amounts to the existence of a line segment in Ω, with endpoints in $\partial\Omega$, that locally minimizes length. This is joint work with P. Sternberg and W. Ziemer.

We also present an existence result for G_ε in 3-d simply connected domains when the applied field h_{ap} is not too big. In particular, for the case $h_{ap} = 0$, this provides what is perhaps the first existence result via Ginburg-Landau theory of permanent currents in the presence of vortices. This also is joint work, in this case with R. Jerrard and P. Sternberg.

Stability of Spot and Ring Solutions of the Diblock Copolymer Equation

Xiaofeng Ren
Department of Mathematics and Statistics
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May, 2004

Abstract. A molecule in a diblock copolymer is a linear sub-chain of A monomers grafted covalently to another sub-chain of B monomers. The different type sub-chains tend to segregate locally, resulting in micro-domains rich in A and B monomers. These micro-domains form morphology patterns/phases in a larger scale.

The Ohta-Kawasaki free energy of a diblock copolymer melt is a functional of the A monomer density field $u(x)$. When there is high A monomer concentration at x, $u(x)$ is close to 1; when there is high concentration of B monomers at x, $u(x)$ is close to 0. A value of $u(x)$ between 0 and 1 means that a mixture of A and B monomers occupies x. The re-scaled, dimensionless free energy of the system is

$$I(u) = \int_D \{\frac{\epsilon^2}{2}|\nabla u|^2 + \frac{\epsilon\gamma}{2}|(-\Delta)^{-1/2}(u-a)|^2 + W(u)\}\, dx,$$

which is defined in the admissible set

$$X_a = \{u \in W^{1,2}(D) : \overline{u} = a\}$$

where $\overline{u} = \frac{1}{|D|}\int_D u\, dx$ is the average of u in D. a is a fixed constant in $(0,1)$. It is the ratio of the number of the A monomers to the number of all the monomers in a chain molecule. One can take

$$W(u) = \frac{1}{4}(u^2 - u)^2.$$

The two parameters ϵ and γ characterize the system. We consider the parameter range

$$\epsilon \to 0,\ \gamma \sim 1.$$

We study two solutions: the spot solution and the ring solution of K interfaces, both in a unit disc. The spot solution models a cell in a cylindrical phase of the diblock copolymer and the ring solution models a defective lamellar phase.

Using the Γ-convergence theory we show that the spot solution exists for all $\gamma > 0$ and there exists $\gamma_1 > 0$ such that the ring solution exists for $\gamma > \gamma_1$.

Next we consider the stability of these solutions by analyzing their critical eigenvalues. We will show that there exists $\gamma_0 > 0$ such that the spot solution is stable if $\gamma < \gamma_0$ and unstable if $\gamma > \gamma_0$. For the ring solution there exists $\gamma_2 > \gamma_1$ such that the ring solution is stable if $\gamma \in (\gamma_1, \gamma_2)$ and unstable if $\gamma > \gamma_2$.

Finally we make a comparison between the diblock copolymer problem and the Cahn-Hilliard problem, which is obtained by setting $\gamma = 0$ in the definition of I.

Eric Sere

CNRS and University Paris-Dauphine

Title : TITLE: Existence of a stable polarized vacuum in the Bogoliubov-Dirac-Fock approximation.

This is joint work with Christian HAINZL and Mathieu LEWIN.

Abstract :

According to Dirac's ideas, the vacuum consists of infinitely many virtual electrons which completely fill up the negative part of the spectrum of the free Dirac operator D^0 (this model is called the "Dirac sea"). In the presence of an external field, these virtual particles react and the vacuum becomes polarized.

In this work, we consider a nonlinear model of the vacuum derived from QED, called the Bogoliubov-Dirac-Fock model (BDF). In this model, the vacuum is represented by a bounded self-adjoint operator Γ on $L^2(R^3)$. An energy of this vacuum is defined. A stable vacuum is a minimizer of this BDF energy functional, under some convex constraints.

We show the existence of a minimizer of the BDF energy in the presence of an external electrostatic field. Then we prove that this minimizer is a projector, which solves a self-consistent equation of Hartree-Fock type. This minimizer is interpreted as the polarized Dirac sea.

Gamma-convergence of gradient flows with applications to Ginzburg-Landau

Sylvia Serfaty, Courant Institute

This is partly joint work with Etienne Sandier from Paris-XII (Creteil).

We present a method to prove convergence of gradient-flows of families of energies which Gamma-converge to a limiting energy. It provides lower bound criteria to obtain the convergence, which correspond to a sort of C^1-order Gamma-convergence of functionals. More specifically, assuming that a family E_ε of functionals Γ-converges to F (in particular for $u_\varepsilon \rightharpoonup^S u$ in a sense S to be specified, u_ε and u not belonging necessarily to the same space, we have $\underline{\lim}_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \geq F(u)$), we ask the question of whether solutions of the gradient-flow

$$\partial_t u_\varepsilon = -\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)$$

(gradient-flow with respect to a certain Hilbert structure X_ε) with appropriate initial data converge to solutions of

$$\partial_t u = -\nabla_Y F(u),$$

for some structure Y to be specified. It turns out that this is not in general true without further assumptions, but that sufficient conditions are (with suitable hypotheses)

$$(1)\ \text{for } u_\varepsilon(x,t) \rightharpoonup^S u(x,t) \qquad \underline{\lim}_{\varepsilon\to 0} \int_0^s \|\partial_t u_\varepsilon\|^2_{X_\varepsilon}(t)\,dt \geq \int_0^s \|\partial_t u\|^2_Y(t)\,ds$$

$$(2)\qquad \text{for } u_\varepsilon(x) \rightharpoonup^S u(x) \qquad \underline{\lim}_{\varepsilon\to 0} \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon} \geq \|\nabla_Y F(u)\|_Y.$$

We then apply this method to establish the limiting dynamical law of a finite number of vortices for the heat-flow of the Ginzburg-Landau energy in dimension 2,

$$\frac{\partial_t u}{|\log \varepsilon|} = \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2).$$

In this case, the limiting objects whose dynamics we study are the limiting vortices of the maps u_ε, and the limiting energy is a "renormalized energy", defined on the finite-dimensional space of possible vortex-locations. We prove that the conditions above are satisfied and thus re-obtain with a different method the result of Lin and Jerrard-Soner, that the limiting vortices follow the gradient-flow of the renormalized enery. We also obtain the analogue new result for the full Ginzburg-Landau model with magnetic field.

One extension of this method is to push it to "second order" to compare the C^2 structures of the energy-landscapes of E_ε and F near critical points. This gives necessary conditions for stable/unstable critical points of E_ε to converge to stable/unstable critical points of F. This is again applied in the case of Ginzburg-Landau to obtain stability results on the limiting vortex-configurations, and a nonexistence result of nontrivial stable critical points with Neumann boundary condition with no magnetic field.

Another extension is to apply it to Ginzburg-Landau vortex-dynamics with suitable space-time rescalings, which allow to continue studying dynamics at times of collisions of vortices. This, coupled with a new estimate relating $\|\nabla E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon}$ (in the case of Ginzburg-Landau) to the vortex-distances, allows to give energy-dissipation rates at collision time and optimal estimates on those collision-times, and under certain assumptions, to extend the limiting dynamics after collision.

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available at `http://www.math.nyu.edu/faculty/serfaty`

Things I don't know (but wish I did) about local minimizers to Ginzburg-Landau, Allen-Cahn and Cahn-Hilliard

Peter Sternberg
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Abstract

Over the past twenty-some years there has been enormous activity surrounding the analysis of critical points to the Cahn-Hilliard energy and Ginzburg-Landau energy. These models have critical points characterized by low dimensional structures (domain walls, vortices) and the stability properties of minimizers, both local and global, in general follow from the stability of these objects.

Much is known about existence of these local minimizers exhibiting a variety of structures as well as about which kinds of structures cannot exist in a stable setting and I will touch on some of these results in my talk. However, the main goal is to point out various basic questions along these lines that remain, to my knowledge, unanswered. This talk in essence represents a laundry list of easy-to-state open questions that I wish someone would settle so that I can sleep better.

Most of the talk will focus on questions related to possible vortex configurations for the Ginzburg-Landau energy without magnetic field

$$E(u) = \int_\Omega V(u) + \frac{1}{2}|\nabla u|^2 \, dx,$$

for $\Omega \subset \mathbb{R}^n$, $V : \mathbb{R}^1 \to \mathbb{R}^1$ given by $V(t) = \frac{1}{4}(t^2-1)^2$ and $u : \Omega \to \mathbb{C}$, as well as for its counterpart with (induced) magnetic field

$$G(u, A) = \int_\Omega V(|u|) + \frac{1}{2}|(\nabla - iA)u|^2 \, dx + \int_{\mathbb{R}^n \setminus \Omega} \frac{1}{2}|\nabla \times A|^2 \, dx,$$

where $A : \mathbb{R}^n \to \mathbb{R}^n$. Topics to be raised will include the possible presence of stable vortex solutions in two dimensions as well as questions about classifying the co-dimension of vortices. (Note: by 'vortex' I simply mean the set $\{u(x) = 0\}$.)

I will also mention some lingering (but again basic) questions about the structure of transition layers to local minimizers of the Cahn-Hilliard energy, given by $E(u)$ above if one instead takes $u : \Omega \to \mathbb{R}^1$, perhaps subject to a mass constraint

$$\int_\Omega u \, dx = m.$$

On Some Weighted Hardy-Sobolev Inequalities

Zhi-Qiang Wang, Utah State University

Consider a family of weighted Hardy-Sobolev type inequalities due to Caffarelli, Kohn and Nirenberg: There is $S(a,b) > 0$ such that for all $u \in C_0^\infty(\mathbf{R}^N)$, it holds

$$\int_{\mathbf{R}^N} |x|^{-2a}|\nabla u|^2\, dx \geq S(a,b) \left(\int_{\mathbf{R}^N} |x|^{-bq}|u|^q\, dx \right)^{2/q} \tag{1}$$

for $N \geq 3$: $-\infty < a < \frac{N-2}{2}, 0 \leq b-a \leq 1$ and $q = \frac{2N}{N-2+2(b-a)}$. These inequalities extend to $D_0^{1,2}(\mathbf{R}^N) := \overline{C_0^\infty(\mathbf{R}^N)}^{||\cdot||}$ with respect to the norm $||u||_a^2 = \int_{\mathbf{R}^N} |x|^{-2a}|\nabla u|^2 dx$, and have the associated Lagrange equation $-div(|x|^{-2a}\nabla u) = |x|^{-bq}u^{q-1}$, which is a prototype of more general anisotropic type nonlinear elliptic PDEs with multiple singularities and degeneracies. Topics to be discussed include symmetry property of extremal functions (i.e., ground state solutions of the PDEs) and Hardy-Sobolev inequalities with remainder terms.

• **Symmetry and symmetry breaking of extremal functions.** Due to the work of Aubin(1976), Talenti(1976), Lieb(1983), and Chou-Chu(1993), for $a \geq 0, a \leq b < a+1$, all extremal functions of the inequalities are radially symmetric. Some recent work have partially clarified the symmetry property of extremal functions for the remaining parameter region.

Theorem *(Catrina-Wang, 2001) There is a function $h(a)$ defined for $a \leq 0$, satisfying $h(0) = 0$, $a < h(a) < a+1$ for $a < 0$, and $a+1-h(a) \to 0$ as $-a \to \infty$, such that for (a,b) satisfying $a < 0$ and $a < b < h(a)$, the extremal functions for $S(a,b)$ are non-radial.*

The curve $h(a)$ is sharpened by Felli-Schneider(2003) to $h(a) = 1+a-\frac{N}{2}(1-\frac{N-2-2a}{\sqrt{(N-2-2a)^2+4(N-1)}})$.

Theorem *(Lin-Wang, 2004) For (a,b) satisfying $a < 0$ and $a < b < h(a)$, any extremal function u to $S(a,b)$ is axially symmetric about a line through the origin. Moreover, up to a rotation, $u(x)$ only depends on the radius r and the angle θ_N between the x_N-axis and $\vec{ox}$, and on each sphere $\{x \in \mathbf{R}^N \mid |x| = r\}$, u is strictly decreasing as the angle θ_N increases.*

• **Sharp versions of the improved Hardy inequalities.** When restricted to bounded domains, the right hand side of (**??**) can add additional terms, i.e., Hardy-Sobolev inequalities with remainder terms. The following is the improved weighted Hardy inequality which gives the sharp version of the improved Hardy inequality due to Brezis-Vazguez(1997) and Vazguez-Zuazua(2000), as well as generalizes theirs to the weighted versions. These inequalities are useful tools for elliptic and parabolic equations having singular potentials.

Theorem *(Wang-Willem, 2003) Let $N \geq 1$, $a < \frac{N-2}{2}$, and $\Omega \subset\subset B_R(0)$ for some $R > 0$. Then there exists $C = C(a,\Omega) > 0$ such that for all $u \in C_0^\infty(\Omega)$*

$$\int_\Omega |x|^{-2a}|\nabla u|^2 dx - \left(\frac{N-2-2a}{2}\right)^2 \int_\Omega |x|^{-2(a+1)}u^2 dx \geq C \int_\Omega \left(\ln \frac{R}{|x|}\right)^{-2} |x|^{-2a}|\nabla u|^2 dx.$$

When $0 \in \Omega$ the inequality is sharp in the sense that $\left(\ln \frac{R}{|x|}\right)^{-2}$ can not be replaced by $g(x) \ln \left(\frac{R}{|x|}\right)^{-2}$ with g satisfying $|g(x)| \to \infty$ as $|x| \to 0$.

• **Further questions.** i.) The symmetry of extremal functions for parameters $a \leq 0, h(a) \leq b < a+1$. ii.) Related issues for the L^p versions of the weighted Hardy-Sobolev inequalities.

Title: Multiple Bubbles For Nonlinear Elliptic Equations with Critical Nonlinearity

Speaker: Juncheng Wei Department of Mathematics Chinese University of Hong Kong

Abstract:

We consider the following nonlinear elliptic equation

$$(1) \qquad \Delta u - \mu u + u^q = 0 \text{ in } \Omega, \quad u > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded and smooth domain in R^N, $\mu > 0$ and $q = \frac{N+2}{N-2}$.

Problem (**??**) has been studied by many authors in recent years . We mention the following results of Gui-Wei: Let $q < \frac{N+2}{N-2}$. Given arbitrary two positive integers K, l, there exists a $\mu_{k,l}$ such that for $\mu > \mu_{k,l}$, there exists a solution to (**??**) with $k-$interior spikes and $l-$boundary spikes. In this talk, we show similar phenomena for the critical exponent case.

Our first result concerns the case of μ large and $N \geq 7$. (Joint work with C.-S. Lin.) We show that at a positive nondegenerate local minimum point Q_0 of the mean curvature, (we may assume that $Q_0 = 0$), for any fixed integer $K \geq 2$, there exists a $\mu_K > 0$ such that for $\mu > \mu_K$, the above problem has $K - bubble$ solution u_μ concentrating at the same point Q_0. More precisely, we show that u_μ has K local maximum points $Q_1^\mu, ..., Q_K^\mu \in \partial\Omega$ with the property that $u_\mu(Q_j^\mu) \sim \mu^{\frac{2}{N-2}}, Q_j^\mu \to Q_0, j = 1, ..., K$, and $\mu^{\frac{3-N}{N}}(Q_1^\mu, ..., Q_K^\mu)$ approach an optimal configuration of the following functional

$(*)$ *Find out the optimal configuration that minimizes the following functional:*

$$R[Q_1, ..., Q_K] = c_1 \sum_{i=1}^{K} \varphi(Q_j) + c_2 \sum_{i \neq j} \frac{1}{|Q_i - Q_j|^{N-2}}.$$

where $c_1, c_2 > 0$ are two generic constants and $\varphi(Q) = Q^T \mathbf{G} Q$ with $\mathbf{G} = (\nabla_{ij} H(Q_0))$.

This result shows that the bubbling accumulations phenomenon can occur for $N \geq 7$. (We remark that when $N = 3$, it was proved by Y.Y. Li that no bubbling accumulations should occur.)

Our second result concerns μ and lower dimension case $N = 4, 5, 6$. (Joint work with O.Rey.) We show that for $N = 4, 5, 6$ and any positive integer K such that $K \neq 2$, there exists $\mu_K > 0$ such that for $0 < \mu < \mu_K$, the above problem has a nontrivial solution which blows up at some K interior points in Ω, as $\mu \to 0$. The locations of the blowing

up points are related to the domain geometry. No assumption on the symmetry or the geometry or the topology of the domain is needed.

On some 1-d forward-backward parabolic equations

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We take a quasiconvexity approach to some one-dimensional first order forward-backward parabolic equations in the form

$$u_t = \sigma(u_x)_x$$

including the Perona-Malik equation in image processing. We rephrase the problem as one related to the quasiconvex hull of a graph in the space of real 2×2 matrices so that the problem can be converted to one on systems of first order inhomogeneous partial differential inclusions. We construct approximate solutions by using simple laminates and the approximate solutions can then be viewed as solutions of a perturbation problem by $W^{-1,\infty}$ functions. The sequences of the approximate solutions generates Young measure-valued solutions whose centers of mass are not solutions. We also discuss the weak exact solutions of the above equations.