Convex Geometric Analysis

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The main goal of the workshop was to bring researchers from different fields of Convex Geometric Analysis to exchange new ideas, to inform on new results and to consider new directions essential for further developments and applications. This goal was achieved and overachieved. We brought together senior experts and we ensured the participation of a significant number of young researchers – in fact, more than a half of all talks were given by people from this latter group. The subject was treated in a very broad sense, and some leading people from related fields (such as Classical Convexity and Asymptotic Combinatorics) were invited and contributed to the success of the meeting. By the request of some participants we also had an informal seminar lecture (see below) which attracted many participants and continued much more than an hour.

Below we collect together the abstracts of the talks, organized in the thematic groups, corresponding to the schedule of the workshop.

Franck Barthe: Orlicz Hypercontractive semigroups

This is a joint work with Patrick Cattiaux and Cyril Roberto.

The usual Ornstein Uhlenbeck semigroup is known to be hypercontractive (it is a contraction from L_2 into a smaller L_p space, and p increases with time). We study the analogue question for Heat semigroups of measures between exponential and Gaussian. An analogue of Gross theorem is presented, relating hypercontractivity in Orlicz spaces to general F-Sobolev inequalities. These Sobolev inequalities are analysed in connection with previous Sobolev type inequalities for these measures, as the ones of Latala and Oleszkiewicz. Applications to concentration and isoperimetric inequalities will be discussed too.

Bo'az Klartag: Approaches to the slicing problem

We will discuss some recent partial progress regarding the slicing problem. The slicing problem asks whether any n-dimensional convex body of volume one, has at least one hyperplane section, whose n-1 dimensional volume is larger than some positive, universal constant, independent of the dimension. This question is known to be equivalent to the question of the universal boundedness of the isotropic constant of centrally symmetric convex bodies. A few directions and possible tools to handle this problem will be described. We will focus on two main issues. The first is the use of geometric symmetrization techniques, and the second is the proof that any centrally symmetric n-dimensional convex body K, has a perturbation T such that the isotropic constant of T is bounded, and the Banach-Mazur distance between K and T is smaller than $c \log n$, where $c > 0$ is a numerical constant.

Vitali Milman: Explicit versus random

In this talk connections of the Asymptotic theory with Complexity Theory were exploited. The new notion of *Simplicity* was introduced which describes exactly the reverse direction to the standard notion of *Complexity*. Let a family of "simple" procedures be described and a family of "simple" objects are specified. Starting with some (supposedly complicated) object we would like to estimate the minimal number N of *simple* steps (i.e. steps from the family of simple procedures) that may be applied to our object in order to bring it to some other object that has been defined as a simple one. Then we say that N is a simplicity of our object. (Note that it is exactly the opposite direction of transformations which are used in defining the Complexity). So, in the process of constructing an algorithm which estimates Complexity we are starting with a simple object (system) and recover the original structure; but in estimating Simplicity, we "destroy" all specific information of our system to come to a simple one with very little specific information. A lot of recent results of the Asymptotic Theory of Convexity are directed to this goal: how quickly we may destroy all specification of a given (arbitrary, and *a priori* very complicated) object (normed space, or a convex body) and to derive some, say, isomorphic copy of an euclidean space (or an ellipsoid). There are really a number of breackthrough in this direction. For example, it is proved by Klartag that just $5n$ Minkowski symmetrizations are sufficient to bring any convex body in $\mathbb{R}^n$ to a body very close to an euclidean ball. Or, only $3n$ Steiner symmetrizations are enough to bring an arbitrary body to a neighbourhood of a euclidean ball (Klartag, Milman), and many others.

After describing this scheme and a number of examples, we moved to another complexity related subject. Standardly, we are describing some very interesting features of spaces and bodies (say, euclidean subspaces of very large dimensions, or euclidean quotients of subspaces of proportional dimension) through random selection of corresponding subspaces in a specified euclidean structure. To estimate the complexity of such features it would be right to demonstrate an explicit construction which leads to these properties. However, such explicit constructions are unknown. Then, we suggest to estimate a complexity through finding a small number of random steps which should be complemented by a number of explicit simple and short constructions. Then this number of (remaining) random steps will tell us what is the remaining complexity ("randomized complexity") of the feature we are studing. We consider a very famous example of the space ℓ_1^n: It is known that this space contains isomorphic copies of euclidean subspaces of any dimension proportional to n (with the isomorphic constant depending only on this proportion); this is so-called Kashin decomposition. We analys different ways for such an euclidean embedding, the problem which very recently attracted significant attention, including important talks on this conference, and we demonstrate some ways of reducing known "randomized complexity" by considering partially explicit steps using Walsh matrix (joint work with Artstein and Friedland).

Assaf Naor: Vertex expansion, edge expansion and the observable diameter.

Joint work with Yuval Rabani and Alistair Sinclair.

In this talk we will show that the edges of any n-point vertex expander can be replaced by new edges so that the resulting graph is an edge expander, and such that any two vertices that are joined by a new edge are at distance $O(\sqrt{\log n})$ in the original graph. This result is optimal, and is shown to have various geometric consequences. In particular, it is used to give a nearly optimal lower bound on the ratio between the observable diameter and the diameter of doubling metric measure spaces which are quasisymmetrically embeddable in Hilbert space.

Alexander Litvak: Behaviour of the smallest singular value of a random matrix and applications to geometry.

This is a report on the joint work with Alain Pajor, Mark Rudelson and Nicole Tomczak-Jaegermann.

We study behaviour of the smallest singular value of a rectangular random matrix, i.e., matrix whose entries are independent random variables satisfying some additional conditions. We prove a deviation inequality and show that such a matrix is a "good" isomorphism on its image. Then we show applications to geometry of random polytopes and to the problem of finding Euclidean subspaces of convex bodies.

Staszek Szarek: Saturation constructions in normed spaces

This is a joint work with Nicole Tomczak-Jaegermann.

Questions on how to detect possible regularities in the structure of a finite-dimensional normed space or improve and simplify this structure, by passing to its subspaces or quotients or, conversely, to what degree the structure of the entire space can be recovered from the knowledge of its subspaces or quotients, have constituted over the years one of the driving directions in the asymptotic theory of normed spaces. Many background results, starting with the fundamental Dvoretzky's theorem (especially in the form proved by Milman), through the Quotient of a Subspace theorem of Milman and its byproducts and relatives, show that one can achieve very considerable regularity for global invariants of a space by passing to a quotient or a subspace.

In this talk we present several results which clarify this circle of ideas. Here is a sample result: *Given finite dimensional normed space V there exists another space X with $\log \dim X = O(\log \dim V)$ and such that every subspace (or every quotient) of X, whose dimension is not "too small," contains a further subspace isometric to V. Moreover, some geometric properties of the space V can be "lifted" to X.* This sheds new light on the structure of such large subspaces or quotients (or, equivalently, of large sections or projections of convex bodies) and allows to solve several problems stated in the 1980s by V. Milman.

Semyon Alesker: The multiplicative structure on valuations.

We describe a canonical multiplicative structure on (a dense subspace of) continuous valuations. Then we discuss its properties and results from convexity and integral geometry staying behind these properties. If the time permits, we will discuss some applications.

Dario Cordero-Erausquin: On the convergence of Information in the Central Limit Theorem.

This is a joint work with Keith Ball.

The goal of the present work is to give uniform bounds for the converge in the Central Limit Theorem of quantities from information theory such as (Shannon) entropy and (Fisher) information. Only partial results were known, for instance under spectral gap assumptions. We show that under moment conditions (moment of order $2+\varepsilon$ is enough)the information converges polynomially (with uniform rate). The proof uses the variational formulation for the information of the sum of two random variables discovered by Ball, Barthe and Naor. The different regimes that appear in the proof bring new light on the behavior of the information along the Central Limitprocess.

Roman Vershynin: Gromovs isoperimetry of waists and its use in asymptotic convex geometry

The isoperimetry of waists on the sphere is a recent result of Gromov. We will describe a simple way to use it in asymptotic convex geometry. For example, it implies the following "local versus global result. If two bodies K and L have nicely bounded sections, then the intersection of random rotations of K and L is nicely bounded. For $L =$ subspace, this yields a new "deterministic versus random phenomenon: if K has one nicely bounded section, then most sections of K are nicely bounded. The latter phenomenon was also independently discovered by Giannopoulos, Milman and Tsolomitis.

Shiri Artstein: On convexified packing and entropy duality

This is a joint work with Vitali Milman, Staszek Szarek and Nicole Tomczak-Jaegermann.

The notion of convexified packing (and convex separation) of convex sets will be introduced, and shown to satisfy a duality theorem: the convexified packings of a body by another body, and of its polars 9taken in the opposite direction) are comparable. A number of instances will be mentioned in which a relationship between convex separation and the usual separation can be established by geometric considerations. This will lead to a generalization of the recent duality result (by the first three authors [AMS]) from the case when one body is an ellipsoid, to the setting of two arbitrary bodies under only mild geometric assumptions about one of the underlying norms.

Robert McCann: A convex action principle giving steepest descent into a nonconvex landscape

This is a joint work with Nassif Ghoussoub.

Physical dynamics interpolate naturally between the dissipative and conservative extremes, in which friction either dominates or can be neglected. *Gradient flows* and *Hamiltonian systems* represent the archetypal examples of these two extremes. The orbits of a Hamiltonian system correspond to the critical paths of an *action* functional, but variational characterizations for the trajectories of a gradient flow are less familiar. For steepest descent into a *convex valley* such characterizations were formulated by Brezis-Ekeland and Auchmuty. Here we refine their approach, taking advantage of the Bolza self-duality introduced by Ghoussoub-Tzou, to formulate a *convex* variational principle for steepest descent into a valley which is merely semi-convex.

Van Vu: Random polytopes: High moments and beyond

Let K be a convex body with volume one in $\mathbb{R}^d$. Consider a selection of n random points in K (chosen with repsect to the unfirom distribution). The convex hull K_n of these points is a random polytope.

Basic parameters (such as the volume, number of vertices, number of facets etc) of random polytopes have been studied for many years. There is a huge amount of strong results about the expectation of these parameters. On the other hand, not too much has been obtained about the distribution. For instance, determining the higher moments is a major problem.

In this talk, we introduce a new method, which allows us to obtain useful information about the distribution. Using this, we can, among others, derive fairly accurate bounds for the high moments and prove limit theorems.

Mark Rudelson: Random processes via the combinatorial dimension

This is a report on a joint work with Roman Vershynin.

Let F be a class of real valued functions defined on a probability space X. For a given $t > 0$ we introduce a dimension $v(F, t)$ measuring the complexity of F in terms of the existence of specific patterns in F. More precisely, the combinatorial dimension $v(F, t)$ is the largest dimension of a structure in F, which is similar to a discrete cube of size t. This characteristic plays a crucial role in determining whether the class F satisfies the uniform Law of Large Numbers and the uniform Central Limit Theorem.

Combinatorial dimension provides a sharp estimate of the metric entropy of the function class. This allows to prove two basic combinatorial conjectures on random processes.

1. A class of functions satisfies the uniform Central Limit Theorem if the square root of its combinatorial dimension is integrable.

2. The uniform entropy is equivalent to the combinatorial dimension under minimal regularity.

Michael Krivelevich: Models of Random Graphs

In this survey talk I will discuss several, old and new, models of random graphs. The main aim of the talk is to familiarize the audience with the variety of models of random graphs, while stressing their differences and similarities and emphasizing common research approaches and methodology. Between the models I plan (or rather hope) to discuss are:

- binomial random graphs G(n,p) and the Erdos-Renyi model G(n,m);
- graph processes and hitting times;
- random regular graphs;
- network reliability model;
- adding random edges (smoothed analysis);
- random lifts;
- preferential attachment models.

Although no previous research experience with random graphs will be assumed, a genuine interest in the subject would be appreciated.

Gideon Schechtman: An observation regarding the dependence on ε in Dvoretzky's theorem

Recall that Dvoretzky's theorem says that there is a function $c(\varepsilon) > 0$ such that for all $n \geq 1$ and all $\varepsilon > 0$, every n-dimensional normed space contains a subspace $(1+\varepsilon)$-isomorphic to ℓ_2^k, for all $k < c(\varepsilon)\log n$. It was well-known that one may take $c(\varepsilon) \geq c\,\varepsilon^2$. In this talk it is shown that this estimate can be improved to

$$c(\varepsilon) > c\,\frac{\varepsilon}{(\log(1/\varepsilon))^2},$$

where $c > 0$ is an absolute constant.

Deane Yang: Moment-entropy Inequalities

This is a joint work with Erwin Lutwak and Gaoyong Zhang.

We establish a new link between the dual L^p Brunn-Minkowski theory and probability theory by using p-th moments to associate a star body to each $\mathbb{R}^n$-valued random variable and defining the dual mixed volume of a random variable with a star body. Using this, the authors generalized the fundamental dual Minkowski inequality for star bodies to an inequality of dual mixed volumes of star bodies and random variables. This in turn gives a fundamental inequality between the Renyi entropy of a random variable and its associated star body. Combining this with the L_p affine isoperimetric inequality of centroid bodies establishes a moment-entropy inequality of random variables that implies the well-known Blaschke-Santaló inequality of convex bodies.

Daniel Hug: On the L_p Minkowski problem

This is a joint work with Erwin Lutwak, Deane Yang, and Gaoyong Zhang.

A classical result of the Brunn-Minkowski theory is Minkowski's existence theorem. For a given centred and non-degenerate Borel measure μ on the unit sphere $\mathbb{S}^{n-1}$ it yields the existence of a unique (up to translation) convex body $K \subset \mathbb{R}^n$ such that the top order surface area measure of K equals μ. In the middle of the last century, Firey extended the Minkowski combination of convex bodies and thus laid the foundations of the Brunn-Minkowski-Firey (or L_p) theory. Subsequently, various elements of the classical theory such as Minkowski's inequality have been established in the L_p setting.

In this talk, we consider an L_p extension of Minkowski's existence theorem. A first proof making use of the machinery of PDE's is due to Chou and Wang. We describe two different elementary approaches to an L_p version of Minkowski's existence theorem. For this we first study polytopal solutions to the discrete-data L_p Minkowski problem.

Rolf Schneider: Size and limit shape of some random polytopes

This is a joint work with Daniel Hug. Familiar ways of generating a random polytope are either taking the convex hull of random points or the intersection of random halfspaces. In the first case, finitely many independent uniform points in a given convex body are an often studied setup. In the second case, which we consider here, an equally natural approach consists in taking a homogeneous Poisson hyperplane process and intersecting the halfspaces which are bounded by the hyperplanes of the process and contain a fixed point, say 0. Here, geometry comes in via the direction distribution of the hyperplane process. For the random polytope thus obtained, we study the existence of weak limits of the conditional shape distribution, given that the 'size' of the polytope is large. We show how the answer depends on the direction distribution and on the way how the 'size' is measured.

Olivier Guédon: Concentration of Mass on the Schatten Classes.

This is a report on a recent work with Grigoris Paouris.

Let $1 \leq p \leq \infty$ and $\widetilde{B(S_p^n)}$ be the unit ball of the Schatten trace class of matrices on $\mathbb{C}^n$ or on $\mathbb{R}^n$, normalized to have Lebesgue measure equal to one. We prove that

$$\lambda\left(\left\{T \in \widetilde{B(S_p^n)} : \frac{\|T\|_{HS}}{n} \geq c_1\,t\right\}\right) \leq \exp\left(-c_2\,t\,n^{k_p}\right)$$

for every $t \geq 1$, where $k_p = \min\{2, 1 + p/2\}$, $c_1, c_2 > 1$ are universal constants and λ is the Lebesgue measure. This concentration of mass inside a ball of radius proportional to n follows from an almost constant behaviour of the L_q norms (with respect to the Lebesgue measure on $\widetilde{B(S_p^n)}$) of the Hilbert-Schmidt operator norm of T.

Carsten Schuett: Approximation of Convex Bodies by Polytopes

In this joint work with Monika Ludwig and Elisabeth Werner, we study the approximation of convex bodies by polytopes. There are extensive investigations of this problem with the additional assumption that the polytope is either contained in the convex body or contains the convex body. Here we study the approximation without this additional assumption.

Monika Ludwig: A characterization of the intersection body transform

The intersection body transform associates with each convex body its intersection body. It is well know that it intertwines with the general linear group and that it is a valuation with respect to radial addition. We describe a classification of radial valuations on convex bodies that commute with the general linear group and obtain a characterization of the intersection body transform.

Hermann Koenig: Geometric inequalities for a class of exponential measures

This is a joint work with Nicole Tomczak-Jaegermann.

In this talk we consider versions of some geometric inequalities (of an isomorphic-type) for a natural class of exponential log-concave measures, replacing the usual volume in $\mathbb{R}^n$. These are versions of Milman's inverse Brunn-Minkowski inequality and Bourgain–Milman's inverse Santaló inequality, which have played an important role in the convex geometric analysis and the asymptotic theory of normed spaces during the last fifteen years. Both these inequalities can be viewed as a consequence of the existence, for any symmetric convex body in $\mathbb{R}^n$, of a special ellipsoid, called nowadays an M-ellipsoid, which, in a sense, reflects volumetric properties of the body. We show that the same ellipsoid also reflects, in an analogous way, properties of the body with respect to a large class of exponential (log-concave) measures on $\mathbb{R}^n$. This class contains in particular the Gaussian measure on $\mathbb{R}^n$.

An informal seminar was given by **Shiri Artstein** on her joint work with B. Klartag and V. Milman on the geometry of log-concave measures.