WORKSHOP ON MONGE-AMPÈRE TYPE EQUATIONS AND APPLICATIONS, AUGUST 2-7, 2003

August 3, Sunday

9:00-10:00am, **Joel Spruck** (Baltimore), *Alexandrov type inequalities for Cartan-Hadamard manifolds.*

In his study of elliptic equations and geometry, the great
Russian geometer A.D. Alexandrov introduced the simple but important idea of the lower (upper) contact set and the associated normal mapping (generalized gradient map). For a function $u \in C^0(\Omega)$ (Ω a bounded domain in R^n), the lower contact set

$$\Gamma^- = \{y \in \Omega | u(x) \geq u(y) + p \cdot (x - y) \text{ ,for all } x \in \Omega \text{ ,for some } p = p(y) \in R^n\} .$$

Note that u is convex if and only if $\Gamma^- = \Omega$ and if $u \in C^1(\Omega)$ $, p = Du(y)$ in the above definition. Further if $u \in C^2(\Omega)$, the hessian $D^2u \geq 0$ on Γ^-. Associated to $y \in \Omega$ is $\chi(y)$, the set of slopes of lower supporting planes to the graph of u at y (so $\chi(y) = \emptyset$ if $y \notin \Gamma^-$). It is not difficult to see that for $u \in C^2(\Omega \cap C^0(\overline{\Omega})$,

$$|\chi(\Omega)| = |\chi(\Gamma^-| \leq \int_{\Gamma^-} \det D^2u(y)\ dy$$

The extension of these ideas to Riemannian manifolds is surprisingly difficult. In this lecture we will discuss this extension for a large class of Cartan-Hadamard manifolds where we replace the linear functions in the definition of the lower contact set by Busemann functions associated to a given point at infinity. The proof uses in an essential way the ergodic theory of the geodesic flow to compute the Jacobian for the generalized gradient map. In a similar way, we derive for a compact hypersurface inequalities for a generalized Gauss map on the outer contact set. Our inequalities are sharp for radial functions and geodesic balls in H^n.

10:00-10:30am **Coffee Break**

10:30-11:30am, **Guofang Wang** (Leipzig), *Metrics of k-positive on locally conformally flat manifolds.*

In this talk, we discuss metrics with a so-called k-positivity condition. We first give more such metrics by using a fully nonlinear equation. Then we discuss a cohomological vanishing theorem for manifolds with a metric of k-positive.

11:30am-2:00pm **Lunch Break**

2:00-3:00pm, **Matt Gursky** (Southbend), *Some connections between fully nonlinear and higher order equations in geometry.*

We will describe some higher order equations which arise in conformal geometry, and explain their connection to some second order, fully nonlinear equations. We will give examples where the fully nonlinear theory can be used to address existence questions for certain higher order equations, and other examples where higher order equations are used to regularize fully nonlinear equations.

3:00-4:00pm, **Xinan Ma** (Shanghai), *Convex solutions of elliptic PDE in classical differential geometry.*

In this talk, we first review some works in the literature on the convexity of solutions of quasilinear ellpitic equation in R^n, especially the deformation technique of Caffarelli-Friedman and Korevaar-Lewis via the strong minimun principle. Then we discuss generalizations to a large general fully nonlinear elliptic equations. Some applications will be given to problems in classical differential geometry and convex bodies, for example Christoffel-Minkowski problem, prescribed Weingarten curvature and prescribed curvature measure in convex body.

4:00-4:30pm **Coffee Break**

4:30-5:30pm, **Yves Martinez-Maure** (Paris), *The Minkowski Problem for hedgehogs (geometrical differences of convex bodies)*

In differential geometry, the Minkowski problem is that of the existence, uniqueness and regularity of closed convex hypersurfaces of $\mathbf{R}^{n+1}$ whose Gauss curvature is prescribed as a function of the unit normal vector. More generally, the Minkowski problem concerns the existence and uniqueness of convex bodies of $\mathbf{R}^{n+1}$ whose area measure of order n is prescribed on the unit sphere $\mathbf{S}^n$. This classical Minkowski problem is equivalent to the question of solutions of a Monge-Ampère PDE of elliptic type on $\mathbf{S}^n$.

Hedgehogs of $\mathbf{R}^{n+1}$ are the geometrical realizations of formal differences of convex bodies of $\mathbf{R}^{n+1}$. In the case of differences of convex bodies of class C^2_+, these geometrical realizations are (possibly singular and self-intersecting) envelopes parametrized by their Gauss map. They constitute a vector space $\mathcal{H}^{n+1}$ in which one can study a given convex body by splitting it judiciously (i.e. according to the problem under consideration) into a sum of hedgehogs.

The Minkowski problem has a natural extension to hedgehogs. For non-convex hedgehogs, this extended Minkowski problem is equivalent to the question of solutions of a Monge-Ampère PDE of mixed type on $\mathbf{S}^n$.

This talk will expound the present state of our knowledge on the subject and will consider certain particular cases. In particular, we shall illustrate the interest of hedgehogs by showing how to construct counter-examples to an old conjectured characterization of the 2−sphere related to this extension of Minkowski's problem.

August 4, Monday

9:00-10:00am, **Xujia Wang** (Canberra), *Regularity of Monge-Ampere equations and applications in affine geometry*

We will briefly review some recent applications of interior regularity of Monge-Ampere equations in affine geometry. We then discuss a recent result by Trudinger and myself on boundary regularity of Monge-Ampere equations and its application.

10:00-10:30am **Coffee Break**

10:30-11:30am, **John Urbas** (Canberra), *Hessian equations on compact Riemannian manifolds.*

We prove the existence of smooth solutions of a class of Hessian equations on a compact Riemannian manifold without imposing any curvature restrictions on the manifold. The results generalize previous work of Delanoë (1981) on Monge-Ampère equations on arbitrary compact Riemannian manifolds, and of Li (1990) and Delanoë (2002) concerning Hessian equations on manifolds of nonnegative sectional.

11:30am-11:45am **Group Photo.**

11:45am-2:00pm **Lunch Break**

2:00-3:00pm, **Alice Chang** (Princeton), *Q-curvature on conformal compact Einstein manifold.*

We will first survey some works of R. Graham-Zworski and C. Fefferman-R.Graham relating the Q-curvature of Branson in conformal geometry to the behavior of scattering matrix on conformal compact Einstein manifold. We then will discuss some recent, preliminary results of Chang-Qing-Yang on the subject.

3:00-4:00pm, **Yu Yuan** (Seattle), *Counterexamples for Local Isometric Embedding.*

We construct smooth metrics on 2-manifold with nonpositive Gauss curvature which cannot be C^3 locally isometrically embedded in R^3. Moreover, the Gauss curvature of the metric can be made negative except for one point. This is joint work with Nikolai Nadirashvili.

4:00-4:30pm **Coffee Break**

4:30-5:30pm, **Eric Sawyer** (Hamilton), *A higher dimensional partial Legendre transform and regularity of degenerate Monge-Ampere equations*

August 5, Tuesday

9:00-10:00am, **Qing Han** (Southbend), *On the local isometric embedding of surfaces in* R^3*.*

In this talk, we shall discuss the local isometric embedding of surfaces in R^3 if Gauss curvature changes its sign. We shall prove that the 2-dim metric g admits a local isometric embedding in R^3 if its Gauss curvature changes its sign across a curve at a fixed order. This generalizes a previous result by C.-S. Lin when such an order is 1. In order to prove the result, we shall change the Darboux equation into a first order differential system. Our assumption implies that the linearized system is symmetric semi-positive.

10:00-10:30am **Coffee Break**

10:30-11:30pm, **Jeff Viaclovsky** (Cambridge), Nonlinear PDE in conformal geometry.

I will discuss various conformal deformation problems involving symmetric functions of the eigenvalues of the Ricci tensor.

11:30am-2:00pm **Lunch Break**

Free Afternoon.

August 6, Wednesday

9:00-10:00, **Mohammad Ghomi** (Atlanta), *Prescribing Curvature via the h-principle*

Using methods of the h-principle of Gromov, specifically the holonomic approximation theorems, we prove that any compact hypersurface with boundary immersed in Euclidean space is regularly homotopic to a hypersurface whose principal directions have a prescribed topological type, and whose principal curvatures are prescribed to within an arbitrary small error. Further we describe how to construct regular homotopies which control the principal curvatures and directions of hypersurfaces. These results have been obtained in recent joint work with Marek Kossowski, and generalize theorems of Gluck and Pan on positively curved surfaces in 3-space, which had been proved by explicit constructions. Also they are somewhat reminiscent of the classical continuity method used to obtain some of the recent results on locally convex hypersurfaces with boundary. A brief survey of these results will be presented as well.

10:00-10:30am **Coffee Break**

10:30-11:30am, **Bo Guan** (Knoxville), *Locally convex hypersurfaces of constant curvature with boundary.*

11:30am-2:00pm **Lunch Break**

2:00-3:00pm, **Guozhen Lu** (Detroit), *Convex functions in the subelliptic setting and applicaions.*

3:00-4:00pm, **Maria Gonzalez** (Princeton), *Singular sets of a class of fully non-linear equations in conformal geometry.*

I am interested in the singularities for the k-th symmetric function of the eigenvalues of the Schouten tensor in conformal geometry. I will cover two theorems that arise from the 'almost' divergence structure of σ_k.

4:00-4:30pm **Coffee Break**

4:30-5:30pm, 10:30-11:30am, **Song-Ying Li** (Irvine), *Regularity of complex Monge-Ampere equations and applications.*

August 7, Thursday

9:00-10:00am, **Wilfrid Gangbo** (Atlanta), *Geometric properties of the set of probability densities of prescribed second moments.*

The set of probability densities can be endowed with the so-called Wasserstein metric. The geodesic between two probability densities can be explicitly written using the convex function that appear in the Monge-Ampere equation involving these densities. Motivated by applications in kinetic theory, we analyze the induced geometry of the set of densities satisfying the constraint on the variance and means, and we determine all of the geodesics on it. It turns out, for example, that the entropy is uniformly strictly convex on the constrained manifold, though not uniformly convex without the constraint.

10:00-10:30am **Coffee Break**

10:30-11:30am **Robert McCann** (Toronto), *Semigeostrophic fluid flow in an elliptical ocean basin.*

This talk describes joint work with Adam Oberman concerning exact solutions to the semi-geostrophic model for atmospheric and oceanic flows. This rotating fluid model involves an active scalar transport problem, much like the vorticity formulation of Euler's equation, but with Monge-Ampere instead of Poisson's equation relating stream function to advected scalar. Affine invariance of the determinant provides exact solutions in which both pressure and stream function remain quadratic but evolve nonlinearly. The finite dimensional dynamics are expressed canonically in Hamiltonian form.

11:30am-2:00pm. **Lunch Break**