Regularity for Hypergraphs

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May 10–24 2003

1 Introduction

The Regularity Method for hypergraphs is a newly emerging technique that grew out of the famous Regularity Lemma of Szemerédi for graphs [53]. The purpose of this Focused Research Group was to bring together the experts who developed the Regularity Method for hypergraphs with some other leading researchers in extremal hypergraph theory, so that all participants could learn the technical details of the new method, and so that new applications of the method to important extremal problems in hypergraphs could be found. The workshop was structured in a way that allowed a lot of informal discussion. Each session was led by a workshop participant, who usually spent some time giving something like a formal lecture on the session topic, but also strongly encouraged the other participants to contribute ideas, ask questions and make suggestions. No time limits were imposed on sessions and each typically lasted several hours.

This report outlines the topics of the discussion sessions, and closes with a short section highlighting the main accomplishments of the workshop. The topics are loosely arranged into three categories. In Section 2 the basics of the Regularity Method are described, Section 3 details some applications of the method, and in Section 4 other extremal hypergraph problems are discussed, in particular two Ramsey theoretic questions that were solved at BIRS by the participants working as a group. Each subsection heading notes the name of the participant who led the session on that topic. Finally in Section 5 the specific results and accomplishments of the workshop are noted.

2 The Regularity Method

2.1 The regularity lemma for graphs and hypergraphs

2.1.1 The Regularity Lemma for graphs (V. Rödl)

Because it was the inspiration for the new regularity method for hypergraphs, part of a session was devoted to the regularity lemma for graphs. While proving his famous Density Theorem [52], E. Szemerédi invented an auxiliary lemma which later proved to be a powerful tool in extremal graph theory. This lemma and its improved version named the Regularity Lemma [53], assert that an arbitrary large graph can be approximated by "random-like" graphs.

More precisely, let $G = (V, E)$ be a graph and A, $B \subset V$ be a pair of disjoint sets of vertices of G. Denote by $E(A, B)$ the set of all edges of G between A and B, i.e., $E(A, B) = \big\{\{a, b\} \in$

E: $a \in A$, $b \in B\}$, and let $e(A,B) = |E(A,B)|$. The *density* of the pair (A,B) is defined by $d(A,B) = e(A,B)/|A||B|$. The pair (A,B) is called *ε-regular* if for any $A' \subset A$, $B' \subset B$ with $|A'| \geq \varepsilon|A|$, $|B'| \geq \varepsilon|B|$, we have $|d(A,B) - d(A',B')| < \varepsilon$.

Theorem 1 (Szemerédi's Regularity Lemma [53]) *For every $\varepsilon > 0$ and $l > 0$, there exist integers L and n_0 such that any graph $G = (V,E)$ with $n \geq n_0$ vertices admits a partition $V = V_1 \cup \cdots \cup V_t$, where $|V_1| \leq |V_2| \leq \cdots \leq |V_t| \leq |V_1| + 1$, $l \leq t \leq L$, and all but at most $\varepsilon\binom{t}{2}$ pairs (V_i, V_j) are ε-regular.*

The lemma below and its generalizations are crucial for most applications of the Regularity Lemma. We refer to the combined use of the Regularity Lemma followed by the Counting Lemma as *The Regularity Method.*

Lemma 2 (Counting lemma) *For any ν, $d > 0$, there exists $\varepsilon > 0$ such that the following holds. Let F be a graph with vertex set $\{w_1, \dots, w_k\}$ and let $G = (V,E)$ be a graph and $V_1, \dots, V_k$ be disjoint subsets of V, all of size n. If for every edge $\{w_i, w_j\} \in E(F)$ the pair (V_i, V_j) is ε-regular with density d, then there are $(1 \pm \nu)d^{|E(F)|}n^k$ copies of F in G with w_i mapped onto a vertex of V_i for all $1 \leq i \leq k$.*

To understand this lemma it is best to look at the case in which F is the complete graph K_k. It is easy to see that if G' is a k-partite graph, where each partite set has size n, in which edges between partite sets are generated with probability d, then G' contains $(1 + o(1))d^{\binom{k}{2}}n^k$ copies of F. Lemma 2 says that if G is such that all pairs (V_i, V_j) are ε-regular with density d, then the number of copies of F in G is about the same as the number of copies in G'. In other words, regularity of G guarantees "random-like" behaviour in this sense.

2.1.2 The regularity lemma for hypergraphs (V. Rödl and J. Skokan)

Recall that in Szemerédi's Regularity Lemma the main structure which undergoes regularization is the edge set of a graph, and the auxiliary structure is a partition of the vertex set. Briefly, the 2-tuples (edges) are regularized with respect to the 1-tuples (vertices).

Unlike the graph case, there are several natural ways to define "regularity" for k-uniform hypergraphs. For example,

- for $k \geq 3$, if we naturally extend the concept of a regular pair and just regularize the k-tuples versus 1-tuples, as e.g. in [11, 23, 44], we obtain a **weak $\boldsymbol{\delta}$-regularity**. Then, one can easily prove the Regularity Lemma, but the natural analogue to the counting lemma fails to be true.
- A more refined approach is to consider an auxiliary partition of the l-tuples for each $l < k$ (concept of **$\boldsymbol{(\delta, 1)}$-regularity**). This was done in [8, 21]. However, there was no attempt to prove a companion counting statement and it is an open question whether it is even possible.

A breakthrough came when Frankl and Rödl [22] modified hypergraph $(\delta, 1)$-regularity for $k = 3$ and developed a concept of **$\boldsymbol{(\delta, r)}$-regularity**. They succeeded in proving both the regularity lemma and the counting lemma for the case $F = K_4^{(3)}$, where $K_4^{(3)}$ is the complete 3-uniform hypergraph on 4 vertices. The general 3-uniform hypergraph counting lemma corresponding to Lemma 2 for graphs was proved later by Nagle and Ródl [42].

Subsequently, the regularity lemma from [22] was extended by Rödl and Skokan [48] to k-uniform hypergraphs for arbitrary $k \geq 3$.

This workshop session discussed in detail all three above concepts of regularity, and the reasons why the rather complicated concept of (δ, r)-regularity is needed (including counterexamples to the counting statement for the weak δ-regularity). The speakers also described the statement of the hypergraph regularity lemma.

2.2 The counting lemma (J. Skokan)

As noted above, many applications of Szemerédi's Regularity Lemma for graphs are based on Lemma 2. To generalize this result for k-uniform hypergraphs, we consider the following random environment:

i) a vertex partition $\mathcal{H}^{(1)} = V_1 \cup \ldots \cup V_{k+1}$, $|V_1| = \ldots = |V_{k+1}| = n$,

ii) a random $(k+1)$-partite graph $\mathcal{H}^{(2)}$, the edges of which are generated with probability d_2,

iii) a random $(k+1)$-partite 3-uniform hypergraph $\mathcal{H}^{(3)}$, whose edges are chosen from triangles of $\mathcal{H}^{(2)}$ independently with probability d_3, and

iv) for $i = 4, \ldots, k$, a random $(k+1)$-partite i-uniform hypergraph $\mathcal{H}^{(i)}$, whose edges are chosen from copies of $K_i^{(i-1)}$ in $\mathcal{H}^{(i-1)}$ independently with probability d_i.

It is easy to show that under the above assumptions, the number of copies of $K_{k+1}^{(k)}$ in $\mathcal{H}^{(k)}$ is

$$(1 + o(1)) \prod_{j=2}^{k} d_j^{\binom{k+1}{j}} n^{k+1}, \tag{1}$$

where $o(1) \to o$ as $n \to \infty$. Any counting lemma should show that (1) is also true in the setup produced by the corresponding regularity lemma for k-uniform hypergraphs.

Guided by the hypergraph regularity lemma of Frankl and Rödl [22], Nagle and Rödl [41] proved a corresponding counting lemma for 3-uniform hypergraphs

Lemma 3 (Counting lemma for 3-uniform hypergraphs [41]) *Let $s \geq 3$ be an integer. For every $\mu > 0$ and $d_3 \in (0, 1]$ there exists $\delta > 0$ such that for every $d_2 \in (0, 1]$ there exist $\varepsilon > 0$ and integers r and m_0 such that the following assertion holds.*

If $\mathcal{G}$ is an s-partite graph with partition $V = \bigcup_{i=1}^{s} V_i$, where $\left|V_i\right| = m > m_0$ for $1 \leq i \leq s$, and $\mathcal{H}$ is an s-partite 3-uniform hypergraph with the same partition such that

(1) $\mathcal{G}$ is (ε, d_2)-regular, and

(2) $\mathcal{G}$ underlies $\mathcal{H}$, and $\mathcal{H}$ is (δ, d_3, r)-regular with respect to $\mathcal{G}$,

then $\mathcal{H}$ contains $(1 \pm \mu) d_2^{\binom{s}{2}} d_3^{\binom{s}{3}} m^s$ copies of $K_s^{(3)}$.

There is hope that the concept of (δ, r)-regularity will allow one to prove a generalization of Lemma 2 and Lemma 3, at least for $F = K_{k+1}^{(k)}$. This is stated as Conjecture 4 below.

Conjecture 4 (see Conjecture 1.16, p. 6 in [49]) *For any $\nu > 0$ and any $k \in \mathbb{N}$, the following is true: $\forall d_k\ \exists \delta_k\ \forall d_{k-1}\ \exists \delta_{k-1}\ \ldots\ \forall d_2\ \exists \delta_2\ \exists r \in \mathbb{N}$ such that if $\mathcal{H}^{(k)}$ is a k-uniform hypergraph and $\{\mathcal{G}^{(l)}\}_{l=1}^{k}$ is a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular $(k+1, k)$-complex, where $\boldsymbol{d} = (d_2, \ldots, d_k)$ and $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_k)$, with $\mathcal{G}^{(k)} = \mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{G}^{(k-1)})$ and $\mathcal{G}^{(1)} = W_1 \cup \cdots \cup W_{k+1}$, where $|W_i| = n$ for all i, then $\mathcal{H}^{(k)}$ contains at least*

$$(1 - \nu) \prod_{l=2}^{k} d_l^{\binom{k+1}{l}} \times n^{k+1}$$

copies of $K_{k+1}^{(k)}$.

For $k = 4$, Conjecture 4 was proved by Rödl and Skokan (see [49]). The techniques of [42] seem to be extendible to the counting of arbitrary 4-uniform hypergraphs F. The presentation in this session provided a detailed proof of Lemma 3 (focusing on $s = 4$) and outlined the major differences between this proof and the proof of Conjecture 4 for $k = 4$.

2.3 Alternative proof of the counting lemma (Y. Peng)

The proof of the counting lemma (Lemma 3) is rather technical, mostly due to the fact that the 'quasi-random' hypergraph arising after applying the regularity lemma of Frankl-Rödl is sparse and consequently is difficult to handle. Recently, Kohayakawa, Rödl and Skokan [36] found a simpler proof of the counting lemma in the easier dense case. Their result applies to k-uniform hypergraphs for arbitrary k.

Lemma 5 *[36] Let $s \geq 3$ be an integer. For every $\mu > 0$ and every $d \in (0, 1]$, there exist $\delta_0 > 0$ and $m_0 > 0$ such that the following holds. If*

(1) $\mathcal{G}$ is a complete s-partite graph with partition $V = \bigcup_{i=1}^{s} V_i$, where $|V_i| = m \geq m_0$ for $1 \leq i \leq s$, and

(2) $\mathcal{H}$ is an s-partite 3-uniform hypergraph with the same partition $V = \bigcup_{i=1}^{s} V_i$ and $\mathcal{H}$ is $(\delta, d, 1)$-regular with respect to $\mathcal{G}$, where $\delta \leq \delta_0$,

then $\mathcal{H}$ contains $(1 \pm \mu)d^{\binom{s}{3}} m^s$ copies of $K_s^{(3)}$.

The subject of this session was the paper "Counting small cliques in 3-uniform hypergraphs" by Peng, Rödl and Skokan [43], where for $k = 3$, the harder, sparse case is reduced to the dense case. In particular, it is shown that a 'dense substructure' randomly chosen from the 'sparse δ-regular structure' is δ-regular as well. This makes it possible to count the number of cliques (and other subhypergraphs) using the Kohayakawa-Rödl-Skokan result and provides an alternative proof of the counting lemma in the sparse case. Since the counting lemma in the dense case applies to k-uniform hypergraphs for arbitrary k, there is a possibility that the approach of this paper can be adapted to the general case as well.

2.4 Characterizing Hypergraph Quasi-randomness (B. Nagle)

An important development regarding Szemerédi's Lemma showed the equivalence between the property of ε-regularity of a bipartite graph G and an easily verifiable property concerning the neighborhoods of its vertices [1] (cf. [15]). This characterization of ε-regularity led to an algorithmic version of Szemerédi's Lemma [1].

Similar problems were also considered for hypergraphs. In [10], [27] and [36], various descriptions of quasi-randomness of k-uniform hypergraphs were given. These notions of hypergraph quasi-randomness coincided with a special case of the quasi-randomness provided by the Frankl-Rödl Regularity Lemma.

The hypergraph regularity of [22] renders quasi-random "blocks of hyperedges" (i.e. (δ, r)-regular triads) which are very sparse. This situation leads to technical difficulties in its application. Moreover, as was shown in [13], some easily verifiable conditions analogous to those considered in [10] and [36] fail to be true in the setting of [22]. However, in [13] and [12], some necessary and sufficient conditions for the Frankl-Rödl notion of hypergraph quasi-randomness were established. These conditions enabled the design in [12] of an algorithmic version of a hypergraph regularity lemma in [22].

To understand the above notions it is best to look at the graph analogues. In what follows, we consider a fixed bipartite graph Γ with bipartition $X \cup Y$. For fixed positive constants α and ε, we assume $d(X, Y) \sim_{\varepsilon} \alpha$, where by $a \sim_{\gamma} b$, we mean $(1 + \gamma)^{-1} \leq a/b \leq 1 + \gamma$. We denote by $\deg_{\Gamma}(x)$ the number of vertices that are neighbors of x in the graph Γ, and by $\deg_{\Gamma}(x_1, x_2)$ the number of vertices that are neighbors of both x_1 and x_2 in Γ.

The property of ε-regularity of Γ is a "global" property in the sense that it asserts a fact about every pair of reasonably large subsets of its vertex classes X and Y. An important development regarding Szemerédi's Lemma showed the equivalence between this global regularity property of Γ and a fairly simple "local" property concerning the neighborhoods of the vertices in X. Given positive reals α, ε and ε', consider the following two properties:

$\mathbf{G_1} = \mathbf{G_1}(\varepsilon)$ $\quad\Gamma$ is ε-regular with density $d(X,Y) \sim_\varepsilon \alpha$.

$\mathbf{G_2} = \mathbf{G_2}(\varepsilon')$ $\quad(i)$ $\deg_\Gamma(x) \sim_{\varepsilon'} \alpha|Y|$ for all but $\varepsilon'|X|$ vertices $x \in X$,

(ii) $\deg_\Gamma(x_1, x_2) \sim_{\varepsilon'} \alpha^2|Y|$ for all but $\varepsilon'|X|^2$ pairs $x_1, x_2 \in X$.

It was shown in [1] (cf. [15]) that properties $\mathbf{G_1}$ and $\mathbf{G_2}$ are equivalent in the following sense.

Theorem 6 (Alon, Duke, Lefmann, Rödl, Yuster, [1]) *For any $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that*

$$\mathbf{G_1}(\varepsilon') \Rightarrow \mathbf{G_2}(\varepsilon).$$

Similarly, for any $\varepsilon' > 0$, there exists $\varepsilon > 0$ such that

$$\mathbf{G_2}(\varepsilon) \Rightarrow \mathbf{G_1}(\varepsilon').$$

The equivalence of Properties $\mathbf{G_1}$ and $\mathbf{G_2}$ tells us that the notion of ε-regularity is equivalent to a condition concerning uniformity of degrees and codegrees. Since degrees and codegrees concern only vertices and pairs of vertices, and not large subsets as in the definition of ε-regularity, Property $\mathbf{G_2}$ is a "local" criterion for the regularity of graphs.

As mentioned earlier, the equivalence of Properties $\mathbf{G_1}$ and $\mathbf{G_2}$ played the crucial role in the algorithmic version of Szemerédi's Regularity Lemma in [1].

Theorem 7 (Constructive Regularity Lemma, [1]) *For every $\varepsilon > 0$ and every positive integer k, there exists an integer $Q = Q(\varepsilon, k)$ such that every graph G with $n > Q$ vertices admits an ε-regular partition into $t+1$ classes for some $k < t < Q$ and such a partition can be found in $O(M(n))$ sequential time, where $M(n)$ denotes the time needed for the multiplication of two $(0,1)$ matrices of size n.*

In the hypergraph setting, there is also a natural candidate for the "local" property $\mathbf{H_2}$ that should correspond to the global property $\mathbf{H_1}$ of regularity in the Frankl-Rödl sense. However, as shown in Dementieva, Haxell, Nagle and Rödl [12], unfortunately these two properties are not fully equivalent. Nevertheless, there is an equivalence in a special case, namely when the regularity parameter $r = 1$. This result was discussed in detail during the session.

Despite the inequivalence of properties $\mathbf{H_1}$ and $\mathbf{H_2}$, in [12], an algorithmic version of a special case of Frankl and Rödl's Hypergraph Regularity Lemma was obtained, namely, when $r = 1$.

Theorem 8 ([12]) *Let δ and γ with $0 < \gamma \leq 2\delta^4$, integers t_0 and ℓ_0 and function $\varepsilon(\ell) > 0$ be given. Let T_0, L_0, and N_0 be those constants guaranteed by the Frankl-Rödl Regularity Lemma. There exists a constant k so that, given any 3-uniform hypergraph $\mathcal{H} \subseteq [N]^3$, $N \geq N_0$, one may in time $O(N^k)$ find a $(\delta, 1)$-regular, $(\ell, t, \gamma, \varepsilon(\ell))$-partition of $\mathcal{H}$, for some t and ℓ satisfying $t_0 \leq t \leq T_0$ and $\ell_0 \leq \ell \leq L_0$.*

This restricted version of the lemma is still sufficient for some applications, for example for Theorem 17 described below. Therefore, for such applications, corresponding efficient algorithms exist.

The following problem attempts to make the Frankl-Rödl regularity lemma fully algorithmic.

Problem 9 *Let δ and γ with $0 < \gamma \leq 2\delta^4$, integers t_0 and ℓ_0 and functions $\varepsilon(\ell) > 0$, and $r(t, \ell)$ (integer valued) be given. Let T_0, L_0, and N_0 be those constants guaranteed by the Frankl-Rödl regularity lemma. Does there exist a constant k so that, given any 3-uniform hypergraph $\mathcal{H} \subseteq [N]^3$, $N \geq N_0$, one may in time $O(N^k)$ find a $(\delta, r(t, \ell))$-regular, $(\ell, t, \gamma, \varepsilon(\ell))$-partition of $\mathcal{H}$, for some t and ℓ satisfying $t_0 \leq t \leq T_0$ and $\ell_0 \leq \ell \leq L_0$?*

The combination of an algorithmic version of the Hypergraph Regularity Lemma and the Counting Lemma would be very helpful in solving many constructive hypergraph problems.

The current algorithmic version of the Hypergraph Regularity Lemma seen in Theorem 8 delivers only the special case $r = 1$. As one sees in the hypothesis of the Counting Lemma, however, one requires $r > 1$ to apply counting. As such, there is not a direct link between the current Theorem 8 and the Counting Lemma. A positive solution to Problem 9 would allow one to combine an algorithmic version of the Hypergraph Regularity Lemma with the Counting Lemma.

Very recently, exciting but partial progress on Problem 9 was obtained. It was shown by Dementieva, Haxell, Nagle and Rödl that one can indeed combine the current algorithmic version of the Hypergraph Regularity Lemma seen in Theorem 8 with a form of the Counting Lemma, despite the fact that Theorem 8 only delivers $r = 1$. It is hoped that further constructive applications may now ensue.

3 Applications of the Regularity Method

3.1 Thresholds for Ramsey properties of hypergraphs (M. Schacht)

We denote by $\mathbb{G}^{(k)}(n,p)$ the binomial random k-uniform hypergraph with n vertices and edges occurring independently with probability $p = p(n)$. It is well known that for many interesting properties $\mathcal{P}$ of hypergraphs there exists a critical function $\widetilde{p} = \widetilde{p}(n)$ around which the behaviour of $\mathbb{G}^{(k)}(n,p)$ suddenly changes with respect to $\mathcal{P}$. More precisely, we say $\widetilde{p} = \widetilde{p}(n)$ is a *threshold for the property* $\mathcal{P}$ if $\mathbb{G}^{(k)}(n,p)$ *asymptotically almost surely* (with probability tending to 1 as $n \to \infty$) satisfies $\mathcal{P}$ if $p \gg \widetilde{p}$ (i.e., $p(n)/\widetilde{p}(n) \to \infty$ as $n \to \infty$), while $\mathbb{G}^{(k)}(n,p)$ asymptotically almost surely fails to satisfy $\mathcal{P}$ for $p \ll \widetilde{p}$.

For two k-uniform hypergraphs $\mathcal{G}$ and $\mathcal{H}$ we use the arrow notation $\mathcal{G} \to (\mathcal{H})^e_r$ to abbreviate the following Ramsey type statement: *For every r-colouring of the edges of $\mathcal{G}$ there exists a monochromatic copy of $\mathcal{H}$.* Obviously, the property $\mathcal{G} \to (\mathcal{H})^e_r$ is increasing and hence as a consequence of [5] it has a threshold. The study of the threshold function for the Ramsey property $\mathbb{G}^{(2)}(n,p) \to (\mathcal{H})^e_r$ for fixed graphs $\mathcal{H}$ and fixed integers r was initiated by Łuczak, Ruciński, and Voigt in [38]. In [45, 46], Rödl and Ruciński solved the problem completely for graphs ($k = 2$). They proved the following for graphs $\mathcal{H}$ different than forests. For a fixed k-uniform hypergraph $\mathcal{H}$ with at least one edge we define the k-density $m_k(\mathcal{H})$ as follows

$$m_k(\mathcal{H}) = \max\{d_k(\mathcal{H}')\colon\ \mathcal{H}' \subseteq \mathcal{H} \text{ and } e_{\mathcal{H}'} \geq 1\}\,,$$

where $d_k(\mathcal{H}')$ is defined as

$$d_k(\mathcal{H}') = \begin{cases} \frac{e_{\mathcal{H}'}-1}{v_{\mathcal{H}'}-k} & \text{if} \quad e_{\mathcal{H}'} > 1 \\ \frac{1}{k-1} & \text{if} \quad e_{\mathcal{H}'} = 1 \end{cases}$$

and $e_{\mathcal{H}}$ ($v_{\mathcal{H}}$) denotes the number of edges (vertices) of $\mathcal{H}$.

Theorem 10 (Rödl–Ruciński) *For all graphs $\mathcal{H}$ with at least one cycle and for all integers $r \geq 2$, there are constants $0 < c < C$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\mathbb{G}^{(2)}(n,p) \to (\mathcal{H})^e_r\right) = \begin{cases} 0 & \text{if} \quad p \leq cn^{-1/m_2(\mathcal{H})} \\ 1 & \text{if} \quad p \geq Cn^{-1/m_2(\mathcal{H})} \end{cases}$$

The proof of Theorem 10 utilises the Szemerédi Regularity Lemma [53], despite the fact that the result deals with sparse random graphs.

The general problem for k-uniform hypergraphs $k > 2$ is still wide open. It is conjectured that Theorem 10 extends naturally to "most" hypergraphs $\mathcal{H}$ with 2 replaced by k. (There might be some class of exceptional hypergraphs similar to forests in the graph case.)

Conjecture 11 *For "most" k-uniform hypergraphs $\mathcal{H}$ and for all integers $r \geq 2$, there are constants $0 < c < C$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\mathbb{G}^{(k)}(n,p) \to (\mathcal{H})^e_r\right) = \begin{cases} 0 & \text{if} \quad p \leq cn^{-1/m_k(\mathcal{H})} \\ 1 & \text{if} \quad p \geq Cn^{-1/m_k(\mathcal{H})} \end{cases}$$

In [47] Rödl and Ruciński verified Conjecture 11 for $r = 2$ and $\mathcal{H} = K_4^{(3)}$, the complete 3-uniform hypergraph on four vertices. The proof of that case involves the hypergraph regularity lemma of Frankl and Rödl [22] for 3-uniform hypergraphs and a corresponding counting lemma which estimates the number of copies of $K_4^{(3)}$ in a regular 4-partite 3-uniform hypergraph (see Section 2). We believe that the counting tools described in Sections 2.2 and 2.3 ([41] and [43]) combined with the techniques of [46] can be applied to establish Conjecture 11 for 3-uniform hypergraphs $\mathcal{H}$ different from $K_4^{(3)}$. Moreover, the recent progress in developing the regularity lemma for k-uniform hypergraphs and the accompanying counting lemmas (see [48, 49]), hopefully, shed light in the study of thresholds for other Ramsey properties of random k-uniform hypergraphs.

Rödl, Ruciński, and Schacht previously worked on that problem. As a first step they were able to verify Conjecture 11 for arbitrary $k \geq 2$ and $r \geq 2$ in the case when $\mathcal{H}$ is a k-partite k-uniform hypergraph.

Theorem 12 (Rödl–Ruciński–Schacht) *For all integers $k \geq 2$ and $r \geq 2$ and every k-uniform k-partite hypergraphs $\mathcal{H}$ with at least one edge, there exists a constant $C > 0$ such that for every $p = p(n) \geq Cn^{-1/m_k(\mathcal{H})}$*

$$\lim_{n\to\infty} \mathbb{P}\left(\mathbb{G}^{(k)}(n,p) \to (\mathcal{H})_r^e\right) = 1\,.$$

We discussed their approach in this session of the workshop. In particular, due to a discussion initiated by Łuczak, we were interested whether this approach could be used to extend Theorem 12 to the corresponding Turán problem.

The Turán problem is a density version of the Ramsey type question. Here we are interested in the minimum size of a colour class to ensure a monochromatic copy of $\mathcal{H}$. Turán [54] first solved that problem for complete graphs $\mathcal{H}$ in the deterministic setting. Erdős, Stone, and Simonovits [16, 18] then generalised Turán's Theorem to arbitrary graphs $\mathcal{H}$.

Theorem 13 (Erdős–Stone–Simonovits) *For every graph $\mathcal{H}$ and $\eta > 0$, there exists an $n_0 > 1$ such that if an n-vertex graph $\mathcal{F}$ with $n \geq n_0$ contains more than*

$$\left(1 - \frac{1}{(\chi(\mathcal{H}) - 1)} + \eta\right)\binom{n}{2}$$

edges, where $\chi(\mathcal{H})$ is the chromatic number of $\mathcal{H}$, then $\mathcal{F}$ contains at least one copy of $\mathcal{H}$.

Theorem 13 can be viewed as a result for subgraphs $\mathcal{F}$ of the random graphs $\mathbb{G}^{(2)}(n,p)$ with $p \equiv 1$. Hence, naturally the question arises for which $p = p(n)$ Theorem 13 remains asymptotically almost surely (a.a.s.) true for subgraphs $\mathcal{F}$ of $\mathbb{G}^{(2)}(n,p)$ with $\binom{n}{2}$ replaced by $p\binom{n}{2}$ (the expected number of edges in the random graph $\mathbb{G}^{(2)}(n,p)$). It was shown that there exists a threshold for the Turán property discussed above (see, e.g., [28, Chapter 8]), even though it is not a monotone property. If $p = p(n)$ is such that the expected number of copies of subgraphs $\mathcal{H}'$ of $\mathcal{H}$ in $\mathbb{G}^{(2)}(n,p)$ is much smaller than the expected number of edges of $\mathbb{G}^{(2)}(n,p)$, then it is not hard to show that $\mathbb{G}^{(2)}(n,p)$ a.a.s. fails to satisfy the Turán property for $\mathcal{H}$. Conjecture 14 below, first formulated by Kohayakawa, Łuczak, and Rödl in [32], demonstrates the belief that this is the only obstacle.

Conjecture 14 (Kohayakawa–Łuczak–Rödl) *For every graph $\mathcal{H}$ containing at least one edge and $\eta > 0$, there exists a constant $C > 0$ such that if $p = p(n) \geq Cn^{-1/m_2(\mathcal{H})}$, then $\mathbb{G}^{(2)}(n,p)$ a.a.s. satisfies the following Turán type property. If $\mathcal{F}$ is a subgraph of $\mathbb{G}^{(2)}(n,p)$ with more than*

$$\left(1 - \frac{1}{(\chi(\mathcal{H}) - 1)} + \eta\right)p\binom{n}{2}$$

edges, then $\mathcal{F}$ contains at least one copy of $\mathcal{H}$.

So far, there are a few results in support of Conjecture 14. Any result concerning the tree-universality of expanding graphs, or any simple application of Szemerédi's regularity lemma for sparse graphs, gives Conjecture 14 for $\mathcal{H}$ a forest. The cases in which $\mathcal{H} = K_3$ and $\mathcal{H} = C_4$ are essentially proved in Frankl and Rödl [20] and Füredi [24], respectively, in connection with problems concerning the existence of some graphs with certain extremal properties. The case for $\mathcal{H} = K_4$ was proved by Kohayakawa, Łuczak, and Rödl [32] and the case in which $\mathcal{H}$ is a general cycle was settled by Haxell, Kohayakawa, and Łuczak [25, 26] (see also Kohayakawa, Kreuter, and Steger [31]). Very recently Gerke et al. settled the case $\mathcal{H} = K_5$. In [35] and [51] some weaker versions are obtained for arbitrary l and $\mathcal{H} = K_l$.

Due to the fruitful discussion during the workshop, mentioned earlier, some further progress towards Conjecture 14 was achieved. Rödl, Ruciński, and Schacht may extend their proof of Theorem 12 to the corresponding Turán problem. This, e.g., would verify Conjecture 14 for arbitrary bipartite graphs $\mathcal{H}$.

3.2 Dirac's theorem for hypergraphs (A. Ruciński)

A substantial amount of research in graph theory continues to concentrate on the existence of hamiltonian cycles. The following classical theorem of Dirac from 1952 [19] is one of the best known results in graph theory.

Theorem 15 (Dirac) *Every graph with $n \geq 3$ vertices and minimum degree at least $n/2$ contains a Hamiltonian cycle. Moreover, there is an example showing that this is best possible.*

The study of hamiltonian cycles in hypergraphs was initiated by Bermond where, however, a different definition than the one considered here was introduced. Here, by a *Hamiltonian cycle* in a 3-uniform hypergraph with n vertices we mean a spanning subhypergraph with n edges that admits an ordering $v_1, \ldots, v_n$ of the vertices so that all n triples $\{v_i, v_{i+1}, v_{i+2}\}$ (indices modulo n) are edges of the subhypergraph. Katona and Kierstead [29] proved that for a Hamilton cycle to exist, it is sufficient that all pairs belong to at least $5n/6$ edges. They also suggested that the following conjecture might be true.

Conjecture 16 *Every 3-uniform hypergraph with $n \geq 4$ vertices in which every pair of vertices belongs to at least $n/2$ edges contains a Hamilton cycle.*

The support for this conjecture stems from a construction of an edge-maximal, 3-uniform hypergraph with each pair degree at least $\lfloor n/2 \rfloor - 1$, not containing a hamiltonian cycle (see [29, Theorem 3]).

Rödl, Ruciński and Szemerédi have proved an asymptotic version of this conjecture. We say that a 3-uniform hypergraph H is an (n, γ)-graph if H has n vertices and every pair of vertices belongs to at least $(1/2 + \gamma)n$ edges.

Theorem 17 *Let $\gamma > 0$. Then, for sufficiently large n, every (n, γ)-graph contains a hamiltonian cycle.*

The proof of this Theorem is based on the hypergraph regularity lemma and its accompanying counting lemma for $k = 3$ (see Section 2). This proof was the main topic of the session.

During the workshop the three participants continued to work on the stronger version, where γ is totally eliminated, and are now about to complete the proof of the following extension of Theorem 17. A 3-uniform hypergraph on n vertices with every pair belonging to at least $n/2$ edges will be called a *Dirac* 3-graph.

Theorem 18 *For sufficiently large n, every Dirac 3-graph contains a hamiltonian cycle.*

4 Extremal Hypergraph Problems

4.1 Stability and structure in Turán problems (D. Mubayi)

In this session, several extremal problems concerning Turán questions in hypergraphs were discussed.

4.1.1 Ramsey-Turán problems for hypergraphs.

For an l-graph $\mathcal{G}$, the Turán number $\text{ex}(n, \mathcal{G})$ is the maximum number of edges in an n-vertex l-graph $\mathcal{H}$ containing no copy of $\mathcal{G}$. The limit $\pi(\mathcal{G}) = \lim_{n\to\infty} \text{ex}\ (n, \mathcal{G})/\binom{n}{l}$ is known to exist. The Ramsey-Turán density $\rho(\mathcal{G})$ is defined similarly to $\pi(\mathcal{G})$ except that we restrict to only those $\mathcal{H}$ with independence number $o(n)$. This definition is motivated by the fact that the densest graphs without a fixed graph usually have large independence sets; so what happens if we do not allow large independent sets? A result of Erdős and Sós [17] states that $\pi(\mathcal{G}) = \rho(\mathcal{G})$ as long as $\mathcal{G}$ is in some sense locally dense. Therefore a natural (first) question is whether there exist $\mathcal{G}$ for which $\rho(\mathcal{G}) < \pi(\mathcal{G})$.

Another variant $\tilde{\rho}(\mathcal{G})$ proposed in [17] requires the stronger condition that every set of vertices of $\mathcal{H}$ of size at least εn $(0 < \varepsilon < 1)$ has density bounded below by some threshold (we omit the precise formulation). By definition, $\tilde{\rho}(\mathcal{G}) \le \rho(\mathcal{G}) \le \pi(\mathcal{G})$ for every $\mathcal{G}$. However, even $\tilde{\rho}(\mathcal{G}) < \pi(\mathcal{G})$ is not known for very many l-graphs $\mathcal{G}$ when $l > 2$.

Let $\alpha \in (0, 1), l \ge 2$ and let $\mathcal{H}_n$ be an l-graph on n vertices. $\mathcal{H}_n$ is (α, ξ)-uniform if every ξn vertices of $\mathcal{H}_n$ span $(\alpha \pm \xi)\binom{\xi n}{l}$ edges. A recent result of Mubayi and Rödl implies the following:

Theorem 19 (Mubayi-Rödl) *For all $\tilde{\delta}$, there exist δ, r, n_0 such that, if $n > n_0$ and $\mathcal{H}_n$ is (α, δ)-uniform, then all but $\tilde{\delta}\binom{n}{r}$ r-sets of vertices of $\mathcal{H}_n$ induce a subsystem that is $(\alpha, \tilde{\delta})$-uniform.*

Theorem 19 has important consequences for Ramsey-Turán problems for hypergraphs. In particular, it allows one to prove a phenomenon similar to supersaturation for Turán problems for hypergraphs. This could perhaps be the first step of a “Ramsey-Turán” analogue (to hypergraphs) of the celebrated Erdős-Stone-Simonovits theorem of extremal graph theory.

A slightly weaker version of Theorem 19 was proved independently by Alon, de la Vega, Kannan, and Karpinski [3]. Their motivation was to obtain an efficient sampling method for approximating r-dimensional Maximum Constrained Satisfaction Problems. Another application of Theorem 19 is

Theorem 20 *Let $\mathcal{F}$ be a fixed l-graph, and $c > 0$. Then there is an n_0 and r' such that: If $\mathcal{H}$ is an n vertex l-graph ($n > n_0$) such that the deletion of any cn^l edges of $\mathcal{H}$ leaves an l-graph that admits no homomorphism into $\mathcal{F}$, then there exists $\mathcal{H}' \subset \mathcal{H}$ on r' vertices, that also admits no homomorphism into $\mathcal{F}$.*

The special case of Theorem 20 when $\mathcal{F}$ is a complete graph was also recently proved by Alon and Shapira [2]. We hope that Theorem 19 applies more generally to show that global properties of an l-graph imply some local structure. The following problem was posed during the session.

Problem 21 *Find other applications of Theorem 19.*

4.1.2 The structure of extremal hypergraphs.

The Turán problem for hypergraphs is one of the oldest unsolved problems in combinatorics. In all known examples, there exists a 3-graph containing no copy of $\mathcal{F}$ with close to $\text{ex}(n, \mathcal{F})$ edges with a reasonable structure. Our goal is to make this statement precise.

Definition 22 *A directed hypergraph of rank three is a hypergraph whose edges consist of one element sets, ordered two element sets, and three element sets.*

Let $\mathcal{H}$ be a directed hypergraph of rank three with vertex set $\{v_1, \ldots, v_h\}$. Then $\mathcal{G}$ is a *recursive blow up* of $\mathcal{H}$ if the vertices of $\mathcal{G}$ can be partitioned into $V_1 \cup \ldots \cup V_h$ and

- if $x \in V_i, y \in V_j, z \in V_k$, then $xyz \in \mathcal{G}$ if and only if $v_i v_j v_k \in \mathcal{H}$
- if $x, y \in V_i, z \in V_j$, then $xyz \in \mathcal{G}$ if and only if $(v_i, v_j) \in \mathcal{G}$
- the construction giving directed edges and triples as above is repeated in V_i if and only if $v_i \in \mathcal{H}$.

As an example, the standard construction of a 3-graph with density 5/9 and no copy of $\mathcal{K}_4^{(3)}$ is a recursive blow up of $\mathcal{H} = \{(x, y), (y, z), (z, x), \{x, y, z\}\}$. This and all other known examples are motivation for the following

Conjecture 23 *Let $\mathcal{F}$ be fixed a 3-graph. Then there exists a directed hypergraph of rank three $\mathcal{H} = \mathcal{H}(\mathcal{F})$ and an n_0 such that for all $\varepsilon > 0$: there is a 3-graph $\mathcal{G}$ on $n > n_0$ vertices*

- *with density at least $\pi(\mathcal{F}) - \varepsilon$,*
- *containing no copy of $\mathcal{F}$, and*
- *$\mathcal{G}$ is a recursive blow up of $\mathcal{H}$.*

A weaker version of Conjecture 23, obtained by interchanging the quantifiers $\exists\mathcal{H}, \forall\varepsilon$, can be proved by the Hypergraph Regularity Lemma. This was observed during the workshop by Rödl, and Simonovits.

4.1.3 Cycles in hypergraphs.

The Turán problem for cycles in graphs is notoriously hard. It seems natural to ask the same question for hypergraphs, but we need a meaningful definition of cycle. There are several possibilities (see, e.g., Duke [14]), one of which is the following: A t-cycle $\mathcal{C}_t$ in a hypergraph is a sequence of t distinct edges $A_1, \ldots, A_t$, with $A_i \cap A_j \neq \emptyset$ if and only if $j = i + 1$ (modulo t). For r-graphs, it is now a natural question to ask for $\mathrm{ex}(n, \mathcal{C}_t)$.

This definition was also initially motivated by the following question of Erdős: Determine $f_r(n)$, the maximum size of a family of r-sets of an n element set such that whenever $A \cap B = C \cap D = \emptyset$, we have $A \cup B \neq C \cup D$. When $r = 2$, this is just $\mathrm{ex}(n, C_4)$, and probably inspired Erdős' question. For 3-graphs the forbidden configuration in Erdős' problem is a $\mathcal{C}_4$.

Theorem 24 (Mubayi-Verstraëte [39]) *Let $\mathcal{C}_t$ be an r-uniform t-cycle. Then*

$$\left\lfloor \frac{t-1}{2} \right\rfloor \binom{n-1}{r-1} \leq \; ex(n, \mathcal{C}_t) \leq 3 \left\lfloor \frac{t-1}{2} \right\rfloor \binom{n-1}{r-1}.$$

Conjecture 25 (Mubayi-Verstraëte) *Let $\mathcal{C}_t$ be an r-uniform t-cycle. Then, as $n \to \infty$, $ex(n, \mathcal{C}_t) = (1 + o(1)) \left\lfloor \frac{t-1}{2} \right\rfloor \binom{n-1}{r-1}$.*

4.2 Sharp Turán results for 3-uniform hypergraphs (M. Simonovits)

The aim of this session was to discuss two new results in extremal hypergraph theory. In general, the Turán problem in hypergraphs is very difficult, and so any particular instance that can be solved is of significant interest.

The following theorem proves a conjecture of Mubayi and Rödl. Here the 3-uniform hypergraph $F_{3,2}$ consists of the vertices $\{a, b, c, d, e\}$ and the triples abc, abd, abe and cde.

Theorem 26 (Füredi, Pikhurko, Simonovits) *Let $\mathcal{H}$ be a 3-uniform hypergraph with n vertices that does not contain a copy of $F_{3,2}$. Then $\mathcal{H}$ has at most $(4/9 + o(1))\binom{n}{3}$ triples.*

This theorem is best possible, because there exists a hypergraph with $(4/9)\binom{n}{3}$ triples that does not contain a copy of $F_{3,2}$. A notable feature of the proof is that it first establishes a stability result for the problem. In other words, it is first shown that any $F_{3,2}$-free hypergraph that has close to $(4/9)\binom{n}{3}$ triples must have a structure that is very close to the extremal example mentioned above.

A second significant fact about this problem is that the extremal hypergraph has two classes of vertices, canonically joined to each other by triples, but with a high asymmetry: one of the classes is twice as large as the other. Such asymmetry in non-degenerate hypergraph extremal configurations is very rare. Surprisingly, similar asymmetric configurations were found for the hypergraph Ramsey problems that were studied at the workshop (Section 4.3), in several different contexts. It would be very interesting to understand the reasons for asymmetry in problems where the original conditions are symmetric.

The second theorem discussed in this session was the analogous result for the Fano plane, proved by Füredi and Simonovits. In this case the extremal configuration has $(3/4)\binom{n}{3}$ triples.

4.3 Hypergraph Ramsey numbers for paths (T. Łuczak)

This session focused on Ramsey numbers for paths in graphs and hypergraphs. As defined in Section 3.1, the Ramsey number $R_k(H)$ of a graph H is defined to be the smallest integer m such that every colouring of the edges of the complete graph K_m with k colours contains a monochromatic copy of H, that is, a copy whose edges are all the same colour. For a 3-uniform hypergraph H the Ramsey number $R_k(H)$ is defined analogously, for colourings of the complete 3-uniform hypergraph $K_m^{(3)}$. It was proved by Bondy and Erdős [4] that $R_2(P_n) \leq 2n-1$ for the path P_n with n vertices, and they conjectured that $R_3(P_n) \leq 4n-3$. A few years ago Łuczak [37] proved an asymptotic version of this conjecture, using Szemerédi's regularity lemma for graphs [53]. The aim of this session was to investigate whether the new regularity method for hypergraphs could be used to exend the classical Bondy-Erdős result to the much more complex problem of finding $R_2(P_n^3)$ asymptotically, where P_n^3 is a hypergraph path with n vertices. There are two natural definitions for a 3-uniform hypergraph path with vertices $v_1, \ldots, v_n$: the *loose path* LP_n^3 with triples $v_1v_2v_3, v_3v_4v_5, v_5v_6v_7, \ldots, v_{n-2}v_{n-1}v_n$, and the *tight path* TP_n^3 with triples $v_1v_2v_3, v_2v_3v_4, v_3v_4v_5, \ldots, v_{n-2}v_{n-1}v_n$.

This proposal became the subject for a series of sessions involving all seven of the workshop participants who were present for the first week, who with a great deal of work and discussion succeeded in solving both of these problems. The value of $R_2(LP_n^3)$ is asymptotically $5n/4$, while $R(TP_n^3)$ is asymptotically $4n/3$. The arguments turn out to be quite different in the loose and tight cases, and require different approaches. The loose path argument is simpler and can be solved with a weaker form of hypergraph regularity, whereas the tight case requires the full force of the new method. Some of the intermediate results from the proof of Theorem 17 and insights from this work were key in the solution to the tight path problem. These two results will be joint papers and are currently being written up by J. Skokan and P. Haxell.

5 Conclusions

Here we highlight some specific accomplishments of the Focused Research Group, with references to the sections where they are noted in more detail.

- Two new results on hypergraph Ramsey numbers for paths (Section 4.3) were proved by the seven participants present during the first week of the workshop.
- The progress of Rödl, Ruciński, and Szemerédi on Theorem 18 (Section 3.2) partly took place during the workshop.
- The progress of Rödl, Ruciński, and Schacht on Conjecture 14 (Section 3.1) was prompted by suggestions of Łuczak at the workshop.
- progress of Luczak and Simonovits on their joint project on the structure of graphs of large minimum degree not containing subgraphs from a given family.
- A question posed by Łuczak at the workshop asks for a characterization for the notion of (γ, δ, r)-graph regularity, which is an essential concept in the definition of the "local" hypergraph property $\mathbf{H_2}$ (see Section 2.4). During the workshop, Nagle and Rödl found an efficiently verifiable characterization of (γ, δ, r)-regular graphs.
- The progress on Problem 9 of Dementieva, Haxell, Nagle and Rödl noted in Section 2.4 partly took place during the workshop.
- Some progress on Conjecture 4 was made (see Section 2.2), with contributions from several workshop participants. Proving this conjecture now seems quite feasible.

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