Women in Noncommutative Algebra and Representation Theory 4 (WINART4)

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This report summarizes the organization, presentation highlights, and scientific progress made at the fourth workshop for women in noncommutative algebra and representation theory, held at the Banff International Research Station in Banff, Canada. The workshop brought together 42 participants from 12 countries (Argentina, Australia, Brazil, Canada, Colombia, France, Germany, Hungary, Mexico, the Netherlands, the United Kingdom, and the United States) at various stages of their careers, all with strong ties to the research themes of the event. The program featured a series of 20-minute introductory talks by participants, as well as dedicated group time for collaborative research projects. Final presentations were delivered by each group at the end of the week, showcasing significant progress and laying the groundwork for continued collaboration. The WINART4 workshop was a vibrant and productive event, and we look forward to future opportunities to build on its success.

1 Overview of the Field

Noncommutative algebra and representation theory are vibrant and deeply interconnected areas of mathematics that provide foundational tools for understanding symmetry, structure, and transformations across both pure and applied disciplines. In noncommutative algebra, the focus is on algebraic systems where multiplication is not required to commutea generalization that encompasses matrix algebras, operator algebras, and many algebras arising in mathematical physics and geometry. Representation theory, in turn, studies how algebraic objects act on vector spaces, translating abstract structures into concrete linear algebraic data. Together, these fields offer a rich language for exploring the hidden algebraic structure of geometry, topology, combinatorics, and number theory.

The WINART4 workshop centered on a range of cutting-edge topics at the intersection of these fields, including cluster algebras and categories, gentle algebras, frieze patterns, superalgebras, quantum groups, and vertex operator algebras. Each of these themes exemplifies how algebraic and combinatorial ideas come together in modern representation theory. Cluster algebras, introduced in the early 2000s, have reshaped large parts of algebra and geometry through their recursive and combinatorial structure, with deep ties to Lie theory, Teichmller theory, and integrable systems. Gentle algebrasfinite-dimensional algebras with a tractable combinatorial structurehave emerged as key examples in the representation theory of surfaces and infinite-type algebras. Quantum groups, born from the theory of quantum integrable systems, provide a deformation of classical Lie algebras and have inspired powerful generalizations in topology and category theory. Finally, diagrammatic and combinatorial frameworks such as frieze patterns and planar algebras continue to reveal surprising new paths for understanding the structure of representations and their categorifications.

2 Recent Developments and Open Problems

Recent years have witnessed remarkable advances in noncommutative algebra and representation theory, often driven by new interactions among algebraic, combinatorial, and geometric approaches. In cluster theory, the development of cluster categories has provided deep insights into categorification and mutations, while applications of cluster structures to Markov-type Diophantine equations and representation theory of infinite-type quivers continue to evolve. New infinite-rank Grassmannian cluster categories and their completions have opened paths toward understanding wild representation types through topological and homological tools.

Parallel progress in the theory of gentle algebras has led to the classification of certain derived and graded structures, with important implications for the broader study of homological dimensions and silting theory. Quantum groups remain central to many areas of modern mathematics, and recent work has strengthened their connections to canonical bases, categorification, and cluster algebras, with applications extending into low-dimensional topology and mathematical physics.

Despite these developments, many compelling open problems remain. One such problem, addressed by several WINART4 participants, is the classical FrobeniusMarkov uniqueness conjecture: whether each largest entry in a Markov triple uniquely determines that triple. Reformulated through the lens of cluster mutations, this longstanding question now invites new approaches grounded in representation theory. Similarly, generalizations of ConwayCoxeter frieze patternssuch as infinite and superfriezesraise open questions about their classification, periodicity, and geometric interpretations via triangulated surfaces and annuli.

Other challenges include extending classification results for derived equivalence classes of gentle and skew-gentle algebras, understanding periodicity and growth in infinite-type cluster structures, and constructing explicit categorifications of quantum and combinatorial invariants. These questions form part of a larger program of unifying algebraic, topological, and combinatorial methodsa vision strongly reflected in the group projects and collaborative efforts of WINART4 participants.

3 Objectives

- To have accessible introductory lectures by world experts in the themes of the workshop.
- To have each participant engaged in a stimulating research project and/or be involved in an expansive research program in noncommutative algebra and/or representation theory.
- To have each participant provide or receive training toward this research activity (before and at the workshop) and to have made significant progress in such directions by the end of the workshop.
- To set up mechanisms so that the collaborative research groups formed before/at the workshop can continue research after the workshop, so that their findings will be published eventually.
- To provide networking opportunities and mentoring for its participants at and beyond the workshop.

4 Organization

In this section, we describe the organization of WINART4 in detail, as future organizers of workshops at BIRS or other venues may be interested in adapting this format.

In September 2023, the organizers submitted a proposal to BIRS for the fourth workshop in the Women in Noncommutative Algebra and Representation Theory (WINART) series. The format followed earlier WINART workshops and was modeled after similar events such as Women in Number Theory (WIN) and Women in Topology (WIT). Participants were divided into collaborative research groups, each led by two experts, with the goal of fostering sustained research engagement before, during, and after the workshop.

Soon after the proposal was accepted (in late 2023), the organizers confirmed participation from group leaders and created a webpage on the WINART network site:

`https://women-in-ncalg-repthy.org/`

This site included:

- a description of the workshop format and goals,
- information about each group leaders research interests and project description,
- funding and accommodation details, and
- an application form for prospective participants.

The application form, circulated via professional mailing lists and social media, requested: name, email, affiliation, position, year of Ph.D. (or expected), top three research group preferences, reasons for those preferences, and a summary of previous research experience. The application deadline was early July 2024, and we received an overwhelming number of applications from across the globe.

The process for selecting the non-leader participants was as follows. To ensure that we had a strong pool of graduate students and postdoctoral researchers, we pre-invited a select group of early-career mathematicians with a broad range of research interests before the application was launched. This step was necessary in part to meet institutional deadlines for grant and funding applications. As in previous WINART workshops, the public application was then circulated widely through academic listservs, professional societies, and social media. The resulting applicant pool was both large and highly qualified. Depending on the goals and funding needs of future WINART events, organizers may again find it helpful to combine a limited number of pre-invitations with an open call for applications.

By mid-September 2024, participants were notified of their group placements. This email included:

- names and websites of group members,
- a project description and suggested pre-reading,
- guidance for preliminary Zoom meetings, and
- information on lodging, meals, travel support, and child care (which is excellent at BIRS).

Due to continued interest in WINART, all applicants were invited to remain active in the broader WINART network via the website above. Several strong applicants not selected for WINART4 have already expressed interest in future workshops.

A few pre-workshop cancellations were addressed using a strong alternate list. One research group, unable to attend in person, gathered at the University of Minnesota and participated fully via Zoom joining discussions, research work, and final presentations remotely.

WINART4 officially began Sunday, March 23, 2025, with check-in and dinner at the Banff Centre, followed by an informal social gathering in BIRS Lounge. Each weekday began with breakfast at the Vistas Dining Room, and Monday morning included a brief welcome from the BIRS Station Manager.

The formal daily structure was as follows:

- **Morning:** Research time from 9:0011:30 AM (with coffee/tea at 10:00),
- **Afternoon:** Lunch from 11:301:00 PM; participant talks or research time from 1:005:30 PM (with a tea break at 3:00),
- **Evening:** Informal gatherings in the TCPL foyer or breakout rooms, and dinner from 5:307:30 PM.

Six participants gave 20-minute research talks from 1:002:00 PM on Monday, Tuesday, and Thursday: Melody Molander, Natasha Rozhkovskaya, J. Daisie Rock, Azzurra Ciliberti, Kayla Wright, and Elise Catania.

Wednesday afternoon was reserved for informal collaboration, hiking, and exploration of Banff. Throughout the week, research groups often continued working into the evenings.

Final research presentations were delivered in two sessions. On Thursday afternoon, Groups 13 (HarrisPatrias, BittmannYldrm, anakFedele) presented; on Friday morning, Groups 48 (CarboneJurisich, Colmenarejo-Tymoczko, GibneyMakarova, Gratzpenko, SerhiyenkoValdivieso-Daz) gave their summaries. Several groups included remote participants in their reports using Zoom.

Initially, the organizers considered rotating breakout rooms to ensure equal access to larger spaces. However, by request of the group leaders, rooms were fixed for the entire week so that boards and notes could remain in place. This decision helped maximize research continuity and comfort.

Throughout the workshop, informal discussions, mentoring, and spontaneous collaborations flourished across groups and career stages. Many participants remarked that WINART4 was one of the most intellectually productive and supportive environments of their careers. Several collaborative papers and follow-up meetings are already in progress as a result of the connections made during the workshop.

The complete schedule is included below.

WINART4 Schedule

Sunday, March 23

- 16:0017:30 Check-in (Professional Development Centre)
- 17:3019:30 Dinner (Vistas Dining Room)
- 20:0022:00 Informal gathering (BIRS Lounge)

Monday, March 24

- 07:0008:45 Breakfast
- 08:4509:00 Welcome by BIRS Staff
- 09:0011:30 Research Time
- 10:0010:30 Coffee Break
- 11:3013:00 Lunch
- 13:0013:30 Melody Molander: *Planar Algebras and their Corresponding Categories*
- 13:3014:00 Natasha Rozhkovskaya: *Action of Infinite-Dimensional Algebraic Structures on Symmetric Functions*
- 14:0017:30 Research Time
- 15:0015:30 Coffee Break
- 17:3019:30 Dinner
- 19:3021:00 Informal gathering (TCPL foyer)

Tuesday, March 25

- 07:0008:45 Breakfast
- 08:4511:30 Research Time
- 10:0010:30 Coffee Break
- 11:3013:00 Lunch
- 13:0013:30 J. Daisie Rock: *Semi-Discrete Cluster Categories*
- 13:3014:00 Azzurra Ciliberti: *Cluster Algebras of Type B and C*
- 14:0017:30 Research Time
- 15:0015:30 Coffee Break
- 17:3019:30 Dinner
- 19:3021:00 Informal gathering

Wednesday, March 26

- 07:0008:45 Breakfast
- 08:4511:30 Research Time
- 10:0010:30 Coffee Break
- 11:3013:00 Lunch
- 13:0017:30 Free Afternoon
- 17:3019:30 Dinner
- 19:3021:00 Informal gathering

Thursday, March 27

- 07:0008:45 Breakfast
- 08:4511:30 Research Time
- 10:0010:30 Coffee Break
- 11:3013:00 Lunch
- 13:0013:30 Kayla Wright: *SL_3 and SL_4 Webs in Grassmannian Cluster Algebras*
- 13:3014:00 Elise Catania: *A Toric Analogue for Greenes Rational Function of a Poset*
- 14:0015:00 Research Time
- 15:0015:30 Coffee Break
- 16:3016:50 Group 1 Report (Harris and Patrias)
- 16:5017:10 Group 2 Report (Bittmann and Yldrm)
- 17:1017:30 Group 3 Report (anak and Fedele)
- 17:3019:30 Dinner
- 19:3021:00 Informal gathering

Friday, March 28

- 07:0008:45 Breakfast
- 08:0009:30 Research Time
- 09:3009:45 Group 4 Report (Carbone and Jurisich)
- 09:4510:00 Group 5 Report (Colmenarejo and Tymoczko)
- 10:0010:30 Coffee Break
- 10:3010:50 Group 6 Report (Gibney and Makarova)
- 10:5011:10 Group 7 Report (Gratz and penko)
- 11:1011:30 Group 8 Report (Serhiyenko and Valdivieso-Daz)
- 11:3013:30 Lunch

As with prior WINART workshops, participants were free to adjust their group schedules to suit their own pace and focus. Many groups worked late into the evening or took advantage of the outdoor spaces and informal lounges to continue discussion.

Overall, the schedule provided a productive rhythm that balanced intensive research time with space for reflection, exchange of ideas, and mentoring. Several groups formed strong new collaborations, and many participants expressed that WINART4 was among the most productive and inspiring events of their academic careers.

5 Presentation Highlights

- **Monday, March 24, 13:00–13:30 — Melody Molander: Planar Algebras and their Corresponding Categories**
  Subfactor planar algebras first were constructed by Vaughan Jones as a diagrammatic axiomatization of the standard invariant of a subfactor. Planar algebras can be conveniently encoded by diagrams in the plane. These diagrams satisfy some skein relations and have an invariant called an index. The Kuperberg Program asks to find all diagrammatic presentations of subfactor planar algebras. This program has been completed for index less than 4. In this talk, I will introduce subfactor planar algebras and find presentations for subfactor planar algebras of index 4 associated with the affine A Dynkin diagram. Then I will show that categories arising from these planar algebra presentations are also describing categories of representations.

- **Monday, March 24, 13:30–14:00 — Natasha Rozhkovskaya: Action of infinite-dimensional algebraic structures on symmetric functions**
  $W_{1+\infty}$ is the central extension of the Lie algebra of differential operators on the circle. It has applications in integrable systems and two-dimensional quantum field theory. There is an interest in the action of this algebra on symmetric functions coming from the studies of such integrable systems as, for example, the KP hierarchy. Our goal is to describe the action on the basis of symmetric functions in terms of generating functions of multiplication operators.

- **Tuesday, March 25, 13:00–13:30 — J. Daisie Rock: Semi-discrete cluster categories**
  In 2006, BMRRT introduced us to the wonderful world of cluster categories. In 2015, Igusa and Todorov gifted us continuous cluster categories. Until now, these constructions have largely remained separate. Last year, Paquette-R-Yldrm introduced generalized thread quivers, which introduce non-discreteness to the notion of thread quivers, which come from Berg and van Roosmalen. In this ongoing work, we discuss how to smash the continuous and discrete worlds together and obtain something that sits inbetween. Ongoing joint work with Charles Paquette and Emine Yldrm.

- **Tuesday, March 25, 13:30–14:00 — Azzurra Ciliberti: Cluster algebras of type B and C: from Combinatorics to Representation Theory**
  We begin by recalling the combinatorial definitions by generators and relations of cluster algebras of type A, B and C with principal coefficients in a triangulation of a regular polygon. Then, we present a formula expressing cluster variables of type B and C in terms of cluster variables of type A. This formula allows us to provide the cluster expansion of cluster variables of type B and C in terms of perfect matchings of certain modified snake graphs. Finally, we associate a symmetric quiver Q with any cluster of these cluster algebras. In this framework, cluster variables of type B (resp. C) correspond to orthogonal (resp. symplectic) indecomposable representations of Q.

- **Thursday, March 27, 13:00–13:30 — Kayla Wright: $\mathbb{SL}_3$ and $\mathbb{SL}_4$ Webs in Grassmannian Cluster Algebras**
  The Grassmannian Gr(k,n) of k-planes in an n-dimensional space is a well-loved algebraic variety and seems to be the keeper of many fascinating combinatorial problems. One way Gr(k,n) can be endowed with a cluster algebra structure is through the combinatorics of plabic graphs. Though its cluster algebra structure is defined combinatorially, generators and bases for these algebras are not well understood for $k \geq 3$. We will explore how webs seem to be the missing piece of combinatorics, focusing on $k = 3$ and 4, specifically using the new machinery of Gaetz, Pechenick, Pfannerer, Striker, Swanson

hourglass webs. In particular, we will discuss web duality, as defined by Fraser, Lam and Le, and show how it can be used to understand Laurent expansions of cluster variables as generating functions of higher dimer covers. This will be based on joint work with two other WINART participants: Esther Banaian and Elise Catania, as well as Christian Gaetz, Miranda Moore, and Gregg Musiker.

- **Thursday, March 27, 13:30–14:00 — Elise Catania: A Toric Analogue for Greene's Rational Function of a Poset**
  Given a finite poset, Greene introduced a rational function obtained by summing certain rational functions over the linear extensions of the poset. This function has interesting interpretations, and for certain families of posets, it simplifies surprisingly. In particular, Greene evaluated this rational function for strongly planar posets in his work on the Murnaghan–Nakayama formula. In 2012, Develin, Macauley, and Reiner introduced toric posets, which combinatorially are equivalence classes of posets (or rather acyclic quivers) under the operation of flipping maximum elements into minimum elements and vice versa. In this work, we introduce a toric analogue of Greene's rational function for toric posets, and study its properties. In addition, we use toric posets to show that the Kleiss–Kuijf relations, which appear in scattering amplitudes, are equivalent to a specific instance of Greene's evaluation of his rational function for strongly planar posets. Also in this work, we give an algorithm for finding the set of toric total extensions of a toric poset.

6 Scientific Progress Made

6.1 Kostant's Partition Function and Multiplex Juggling

Group leaders: Pamela E. Harris and Rebecca Patrias.
Group members: Kimberly P. Hadaway, Kimberly J. Harry, Lucy Martinez, Miriam Norris.

Introduction: In this short report we give progress report on the project on Kostant's partition function and multiplex juggling sequences for the exceptional Lie algebras. We begin by giving the connection to representation theory of Lie algebras which motivates our problem of interest and follow this with some of our ongoing progress.

Representation Theory: The following problem arises in representation theory of complex semisimple Lie algebras: What is the multiplicity, denoted $m(\lambda, \mu)$, of the weight μ in the irreducible representation with dominant highest weight λ, which we denote by $L(\lambda)$? This problem dates back to Hermann Weyl, (Mathematische Zeitschrift, 1925 [3]), and it continues to attract the attention of present day mathematicians. The first approaches addressing this question stemmed from formulas such as the Weyl character formula. In 1948, Kostant developed his well-known formula for computing the multiplicity of a weight in an irreducible highest weight representation (Amer. J. Math., 1959 [2]). This formula consists of an alternating sum over the Weyl group and involves a partition function. The partition function is known as Kostant's partition function and it counts the number of ways a weight (vector) can be written as a nonnegative integer linear combination of the positive roots (a fixed set of vectors). Despite the availability of such a formula, using it for computational purposes can be quite daunting, due to the fact that the number of terms appearing in the alternating sum is factorial in the rank of the Lie algebra, and the value of the Kostant partition function involved is very often unknown. These complications and the computational complexity involved in such a formula have motivated Pamela E. Harris's research in this field and were the basis for the collaboration of this team, which began at BIRS in March 2025.

Connection to Multiplex Juggling Sequences: In the paper *Kostant's partition function and magic multiplex juggling sequences*, C. Benedetti, C. R. H. Hanusa, P. E. Harris, A. Morales, and A. Simpson [1] establish a combinatorial equivalence between Kostant's partition function and (magic) multiplex juggling sequences, providing a juggling framework to calculate Kostant's partition functions and a partition function framework to compute the number of juggling sequences. This equivalence yields applications to polytopes, posets, positroids, and weight multiplicities, thus opening numerous directions for our future research. In our work at BIRS, we began the work of extending the juggling framework of this paper to the exceptional Lie algebras. We have succeeded in doing this for the exceptional Lie algebra $\mathfrak{g}_2$ while at BIRS. We are in the writing stages and have some initial ideas that appear to be promising for us to give juggling analogs for all remaining exceptional Lie algebras.

Based on our initial investigation at BIRS, it appears that our juggling framework for the Kostant partition function for the exceptional Lie algebra $\mathfrak{g}_2$ provides evidence for a unifying juggling framework much more generally for all vector partition functions using parts which are positive roots of Lie types including the classical and the exceptional Lie algebras. The specialization of our juggling framework will not only recover known juggling frameworks for Kostant's partition function in the classical Lie types, but it will also provide the analogous result for the exceptional Lie types, which was our goal for this collaboration.

6.2 Grassmannians, cluster categories and completions

Group leaders: Sira Gratz and Špela Špenko.
Group members: Charley Cummings, Ellen Kirkman, Janina Letz, Daisie Rock.

Report: Constructing new triangulated categories from old is a notoriously challenging problem. In recent work, Neeman [7] defines the completion of a triangulated category $\mathcal{T}$ with respect to a metric on the category, which is again a triangulated category, with triangulated structure imposed by the original triangulated structure on $\mathcal{T}$. This process is inspired by topological completions of metric spaces: One works in the ind-completion of $\mathcal{T}$, and formally adds the colimits of compactly supported Cauchy-sequences. In this project, we calculate metric completions for categories exhibiting Grassmannian cluster combinatorics of infinite rank. Specifically, we are interested in the categories of maximal Cohen-Macaulay modules (MCM) over certain graded hypersurface singularities, as studied in a previous WINART project [4]. Specifically, we work over a field k, set $S = k[x, y]$, viewed as a graded ring with x in degree 1 and y in degree -1, and consider the hypersurfaces $S/(x^k)$ for $k \geq 2$—which includes the hypersurface singularity $S/(x^2)$ of type A_∞—and $S/(x^2y)$—the hypersurface singularity of type D_∞. Setting R to be any of these rings with the grading inherited from S, we set out to realise its category $\mathsf{MCM}_{\mathbb{Z}}(R)$ of graded MCMs as a completion of its subcategory of generically free MCMs. This is inspired by [5] which computes the completion of a class of cluster categories, which includes the $R = S/(x^2)$ case as a special example, via a classical topological completion.

We attack this problem from different perspectives:

1. We treat the type D_∞ hypersurface singularity separately, and explicitly compute homomorphism spaces in $\mathsf{MCM}_{\mathbb{Z}}(S/(x^2y))$, with the goal of finding a combinatorial model, and a framework in which to explicitly compute completions.

2. We consider two distinct "approximations" of $\mathcal{T} = \mathsf{MCM}_{\mathbb{Z}}(S/(x^k))$: Firstly, we expand the grading by considering S as a $(\mathbb{Z} \times \mathbb{Z})$-graded ring with x in degree $(1, 0)$ and y in degree $(0, -1)$. This allows us to view $\mathcal{T}$ as an orbit of the bounded derived category $\mathsf{D^b}(\mathsf{gr}(k[y]A_{n-1}))$, where y is in degree 1. Secondly, we forget part of the grading, by considering the alternative $\mathbb{Z}$-grading on S by putting x in degree 1 and y in degree 0. This allows us to obtain a functor from $\mathcal{T}$ to the bounded derived category $\mathsf{D^b}(\mathsf{mod}k[y]A_{n-1})$.

3. On the way, we study the dual construction of completions of a triangulated category within its pro-completion, via Cauchy-cosequences, and compare it to the metric completion, specifically in the case of self-dual triangulated categories.

6.3 Caldero-Chapoton Map for Gentle Algebras

Group leaders: Khrystyna Serhiyenko and Yadira Valdivieso.
Group members: Esther Banaian, Ilaria Di Dedda, Azzurra Ciliberti, Kayla Wright.

Report: Our WINART project explores the rich connections between representation theory of algebras and geometry/combinatorics of surfaces. More specifically, a celebrated result in this direction is that gentle Jacobian algebras provide a categorification of surface cluster algebras. The setup for this result is as follows: given a triangulation of a surface, one can associate a Jacobian algebra J such that the combinatorics of the arcs in the surface reflects the representation theory of J as well as the associated cluster algebra [9, 10]. One important aspect of this is that arcs γ on the surface correspond to J-modules M_γ. Our project focuses on a map called the Chaldero-Chapoton map, commonly called the CC map for short. The CC map of a J-module

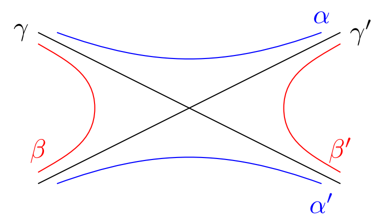


Figure 1: Resolution of a crossing.

M is defined by the following formula:

$$CC(M) = x^{\text{ind}(M)} \sum_{\mathbf{e}} \chi(Gr_{\mathbf{e}}(M)) x^{B\mathbf{e}},$$

where the sum is taken over all dimension vectors $\mathbf{e}$ of submodules of M. This CC map is important in cluster theory as it yields Laurent polynomials in x_i's which are elements of the cluster algebra [8, 14]. Moreover, the CC map satisfies the Ptolemy relations which define the associated surface cluster algebra. That is, given two arcs γ, γ' that cross, let $\alpha, \alpha', \beta, \beta'$ be the four arcs obtained by resolving this crossing. This means that α, α' and β, β' form opposite sides of the quadrilateral in the surface that contains γ, γ' as its diagonals, see Figure 6.3. Then the CC map of these arcs satisfies the following identity:

$$CC(M_\gamma)CC(M_{\gamma'}) = CC(M_\alpha)CC(M_{\alpha'}) + CC(M_\beta)CC(M_{\beta'}).$$

More recently, given any gentle algebra Λ one can associate a surface $\mathbf{S}(\mathbf{\Lambda})$ with some additional data such that it provides a geometric model for the bounded derived category $\mathcal{D}^b(\Lambda)$ of Λ [11, 12, 13]. In particular, an arc γ on the surface yields a complex of projective Λ-modules P_γ that come from the crossings of the arc with the laminations on $\mathbf{S}(\mathbf{\Lambda})$. Moreover, a crossing between two arcs γ, γ' corresponds to two morphisms $P_\gamma \to P_{\gamma'}$ and $P_{\gamma'} \to P_\gamma$ between the associated objects of $\mathcal{D}^b(\Lambda)$, while resolving a crossing in the two possible ways corresponds to computing the cones of these morphisms. Our goal is to study the analog of Ptolemy relations in this setting and identify quantities that satisfy these relations.

Our first main result shows that the Ptolemy relations hold on the level of the Grothendieck group. That is, given $P_\gamma \in \mathcal{D}^b(\Lambda)$, let $x(P_\gamma)$ denote the associated element in the Grothendieck group $K_0(\mathcal{D}^b(\Lambda))$, which corresponds to an alternating sum of x_i's for every indecomposable projective modules $P(i)$ appearing in the complex P_γ. One can think of $x(P_\gamma)$ as the generalization of the g-vector or the index ind(M) appearing in the CC map. With this notation we are able to show the following statement.

Theorem 1. *Let γ, γ' be two finite arcs in $\mathbf{S}(\mathbf{\Lambda})$ that cross in the interior. Let $\alpha, \alpha', \beta, \beta'$ be the arcs obtained by resolving this crossing, as in Figure 6.3. Then*

$$x(P_\gamma)x(P_{\gamma'}) = x(P_\alpha)x(P_{\alpha'}) + x(P_\beta)x(P_{\beta'}).$$

Next, we explore relations between homology of the complexes and Ptolemy relations. We conjecture that the dimension vectors of the homology, which we denote by $y(P_\gamma)$ also satisfy Ptolemy relations, so we obtain an analogous formula as in the theorem above. Unlike in the classical case of surface triangulations and cluster algebras, neither $x(P_\gamma)$ nor $y(P_\gamma)$ determine the complex P_γ. Therefore, our future goal is to associate a function to P_γ analogous to the CC map that would capture the complex completely and satisfy the Ptolemy relations.

6.4 Springer fibers and their generalizations, Hessenberg varieties

Group leaders: Laura Colmenarejo and Julianna Tymoczko
Group members: Elise Catania, Sheila Sundaram, Tamanna Chatterjee, Mitsuki Hanada.

Springer fibers and their generalizations, Hessenberg varieties, are at the nexus of combinatorics, geometry, linear algebra, and representation theory. They sit inside the flag variety, which can be viewed either as the collection of nested linear subspaces in a fixed n-dimensional complex vector space, or as the quotient of invertible matrices by upper-triangular matrices. It has long been known that the geometry of flag varieties is deeply connected with the combinatorics of the symmetric group. This field of research, Schubert calculus, has led to enormously powerful interactions between representation theory, commutative algebra, algebraic geometry, and combinatorics.

The Springer fiber of a square matrix X consists of the "eigenflags" of that matrix, namely the flags for which X restricts to an endomorphism on each of the nested subspaces. Its geometry is subtle and complicated, interweaving the combinatorics of partitions (inherited from the Jordan decomposition of X) with the combinatorics of permutations (inherited from the flag variety). Adding even more complexity, Hessenberg varieties loosen the condition under which X acts on each subspace using another combinatorial constraint (essentially a Dyck path).

A *web* is a directed planar graph with boundary. In our project we use webs and other combinatorial tools to analyze representation theoretic and geometric questions about Springer fibers and Hessenberg varieties.

Webs for $\mathfrak{sl}_2$ are noncrossing matchings, which biject with $2 \times n$ standard Young tableaux as well as many other Catalan objects. If we model 2-row Springer fibers with noncrossing matchings, we gain information about cell closures and gluings.

For the $\mathfrak{sl}_3$ case, much less is known. In particular, modeling the top-dimensional cells of 3-row Springer fibers using $\mathfrak{sl}_3$ webs is more complicated.

Before meeting in person, we looked at the following topics and readings:

- Springer fibers. Our primary resource was "The geometry and combinatorics of Springer fibers" by Tymoczko (available at: `https://arxiv.org/pdf/1606.02760`).

- Webs, which represent quantum invariant vectors. These are explicitly constructed in "Web bases for $sl(3)$ are not dual canonical" by KhovanovKuperberg (available at: `https://msp.org/pjm/1999/188-1/pjm-v188-n1-p07-s.pdf`).

- The basis for the web space is indexed by rectangular standard Young tableaux, and the relationships between these two combinatorial constructions are interesting. Two references we used to explore this relationship were "Promotion and cyclic sieving via webs" by PetersonPylyavskyyRhoades (available at: `https://arxiv.org/pdf/0804.3375`), and "A simple bijection between standard $3 \times n$ tableaux and irreducible webs for sl_3" by Tymoczko (available at: `https://www.emis.de/journals/JACO/Volume35_4/9441547648513778.fulltext.pdf`).

These meetings were very helpful as they provided us with an outline of ideas to discuss at the workshop.

During the workshop week at BIRS, we investigated the different combinatorial models, including webs, strandings, tableaux, and the Bruhat order. We also compiled data both by hand and using Sage. Our goal was to look at the following questions:

- Given a web coming from a standard Young tableau of shape $3 \times n$, which standard Young tableaux of the same shape *strand it*?

- Give combinatorial ways to compute web depth in terms of the strandings.

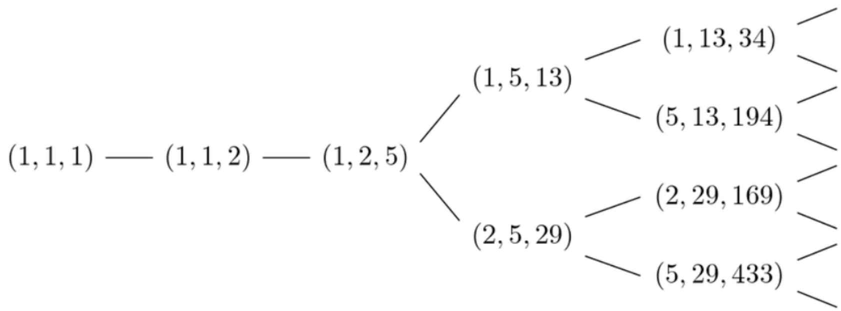
6.5 Mirror deformation of Markov numbers

Group leaders: La Bittmann and Emine Yldrm
Group members: Perrine Jouteur, Melody Molander, Ezgi Kantarc Ouz.

Report: The Markov numbers are the positive integer solutions of the following famous Diophantine Equation

$$x^2 + y^2 + z^2 = 3xyz. \qquad (ME_0)$$

This equation is called *Markov equation* defined by Markov in the late 19th century [20]. The solution triples consisting of positive integers are called *Markov triples*. One solution triple we can easily get is $(1, 1, 1)$. It is very well known that all the other solution triples can be obtain by a *Vieta jump* which is an operation of replacing an entry x on a triple with $x' = (y^2 + z^2)/x$. Thus, we obtain a new Markov triple



tree placeholder.png

Figure 2: First levels of Markov Tree $\mathbb{T}_0$

(x', y, z). Note that the Vieta jump can be applied to any coordinate as the equation is symmetric on x, y and z. Visualising the operations as edges on a graph, we get the *Markov Tree* $\mathbb{T}_0$.

Each triple has a maximum and they grow as we move further from the solution $(1, 1, 1)$. These maximums are important to understand the Lagrange spectrums. Frobenius conjectured in 1913 that each of these maximums is unique; more precisely, they do not occur as a maximum in another Markov triple. This conjecture is still open after a century (see Aigner's book [15]).

The Markov Equation can be translated into the language of cluster algebras and the Vieta jumps correspond to cluster mutations. Consider the quiver below in Figure 3

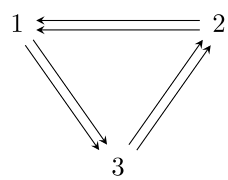


Figure 3: Markov quiver

We label each vertex of the quiver with a variable; say we call the initial cluster (x, y, z). If we mutate at any of the vertices, we then get a new cluster $(x, y, \frac{x^2+y^2}{z})$, corresponding exactly to a Vieta jumps in the Markov Equation. Markov triples correspond to the clusters for the cluster algebra coming from the quiver in Figure 3. Setting the initial cluster variables to 1 gives rise to all Markov numbers. This connection made it possible to expand the study of Markov numbers and led to other generalized versions of the Markov equation (see for instance [18, 19, 16]).

Our work is inspired by a question from Frdric Chapoton. We look at a certain q-deformation that exhibits algebraically and combinatorially interesting properties, while carrying a strong connection to the original Markov Equation. We introduced the *Squared Deformed Markov Equation* as follows.

$$X^2 + Y^2 + Z^2 + (q + q^{-1})(XY + YZ + XZ) = 3(1 + q + q^{-1})XYZ. \qquad (ME_{q+q^{-1}})$$

A solution to $ME_{q+q^{-1}}$ is given by a triple of Laurent polynomials $(X(q), Y(q), Z(q))$ with positive coefficients. As a normalization condition, we require our polynomials to be *degree-symmetric*: if the maximum degree of q occurring with a non-zero coefficient is q^t, then the minimum degree is q^{-t}. First of all, we show that all solutions is obtained from the initial solution $(1, 1, 1)$ via mutations which results in a tree structure $\mathbb{T}_{q+q^{-1}}$.

We note that this deformation is distinct from the deformations previously explored in [23]. The word *mirror* refers to an interesting property of the solutions. We have

$$(X(q), Y(q), Z(q)) = (x(q)x(q^{-1}), y(q)y(q^{-1}), z(q)z(q^{-1}))$$

where $x(q)$, $y(q)$ and $z(q)$ are polynomials in q and $(x(1), y(1), z(1))$ is a Markov triple. The product

$X(q) = x(q)x(q^-1)$ can be rewritten as $q^{-\deg(x)}x(q)\tilde{x}(q)$ where $\tilde{x}$ is the *mirror image* of x, i.e., x with its coefficients reversed.

We define a mutation operation directly on the polynomial triples $(x(q), y(q), z(q))$. This leads us a new q-deformation for Markov numbers that we call the *mirror deformation*. This new discovery leads us to prove interesting combinatorial properties.

Finally, we would like to mention that the equation $ME_{q+q^{-1}}$ can be modified to get other generalizations of the Markov equation.

(i) Setting $(q + q^{-1})^2 = 0$ in the mirror Markov equation, then one gets super Markov numbers [21];

(ii) Setting q to integers, one obtains Gyoda-Matsushita [19] Markov equations;

(iii) Setting q to complex numbers, we discover a connection to complex Markov numbers in the sense of [17].

(iv) Setting $q + q^{-1} = \lambda_p = 2\cos\pi/p$, we get the generalized Markov equation coming from cluster algebras of orbifold surfaces.

6.6 Periodic Infinite Super-Friezes

Group leaders: İlke Çanakçı and Francesca Fedele
Group members: Amanda Burcroff, Monica Garcia, Viktória Klász.

Integral friezes were first introduced by Coxeter in [26] as arrays of integers of the form

$$\begin{array}{cccccccccccccccccccc}
\dots & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \dots & \\
& \dots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \dots \\
\dots & a_1 & & a_2 & & a_3 & & a_4 & & a_5 & & a_6 & & a_7 & & a_8 & & \dots & \\
& \dots & b_1 & & b_2 & & b_3 & & b_4 & & b_5 & & b_6 & & b_7 & & b_8 & & \dots \\
\dots & & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \dots &
\end{array}$$

that satisfy a local rule: for any four neighbours a, b, c, d forming a *diamond* $\begin{smallmatrix} & b & \\ a & & d \\ & c & \end{smallmatrix}$, we have that $ad - bc = 1$.

Integral friezes that end with a row of 1s followed by a row of 0s after n many non-trivial rows are called *finite integral friezes of width* n and Conway and Coxeter proved in [25] that they are in bijection with triangulations of convex $(n+3)$-gons.

Building on the above, friezes have been studied and generalised in various directions. One particularly interesting generalisation arises from cluster algebras of type A and their connection to decorated Teichmller theory. In this setting, the entries of the frieze are not integers but the generators of the algebranamely, the *cluster variables*.

Choosing an initial triangulation of a convex $(n+3)$-gon with boundary segments set to 1, in the corresponding decorated Teichmüller space, the lambda lengths of arcs between the polygon vertices can be expressed in terms of the initial ones and they satisfy the so-called *Ptolemy relations*. A special case of such a relation is illustrated in Figure 4, where a, b, c, d are lambda lengths and the two arcs denoted by 1 are boundary.

All cluster variables can be expressed as Laurent polynomials of the ones corresponding to the arcs of the initial triangulation and the integral friezes can be recovered by specialising to 1 the cluster variables corresponding to the initial triangulation.

The notion of integral and algebraic friezes has also been expanded to cover infinite cases, that is friezes of infinite width. In [24], the authors prove that *periodic infinite friezes* come from triangulations of annuli. As before, such triangulations give both friezes whose entries are cluster variables in the corresponding cluster algebra and integral friezes (when the cluster variables corresponding to the arcs in the initial triangulation are set to 1).

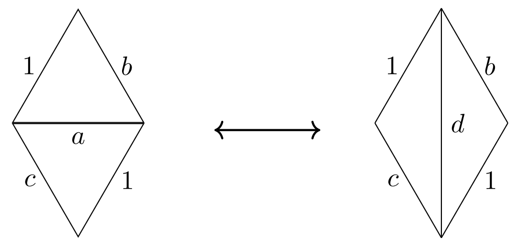


Figure 4: Ptolemy relation: $ad = bc + 1$.

As a further generalisation of friezes, [27] introduced the notion of (finite) *superfriezes*. Roughly speaking, a *superalgebra* is a $\mathbb{Z}_2$-graded algebra, whose even variables commute with everything and odd variables anticommute with each other. Similarly to the way we associate a cluster algebra to a surface S, with generators corresponding to lambda lengths in the decorated Teichmüller space of S, in [29] Penner and Zeitlin associate a superalgebra to S with even generators corresponding to super lambda lengths in the decorated super Teichmüller space of S and odd generators corresponding to triangles. These generators obey a generalisation of the Ptolemy relations: the so-called *super Ptolemy relations*. Similarly to the classic case, in [28] the authors use them to build a corresponding superfrieze for type $\mathbb{A}$: a diamond and the corresponding variables are illustrated in Figure 5,

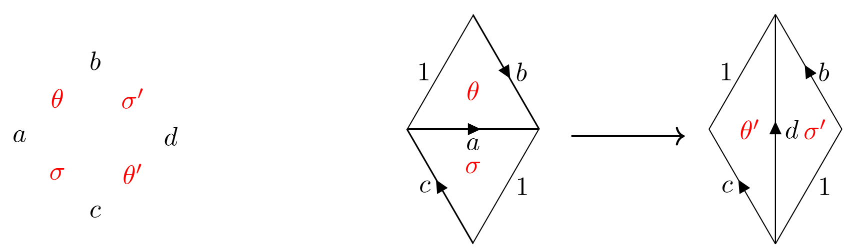


Figure 5: A diamond in the superfrieze and the variables on the surface.

where $ad - bc = 1 + \theta\theta'$, $a\theta' - c\theta = \sigma$ and $b\theta' - d\theta = \sigma'$.

The above led us to a natural question: do periodic infinite super-friezes exist? If so, do they come from triangulations of annuli? Our project aims to answer these questions.

6.7 Defining Hecke-like Operators on Vertex Operator Algebras with modular traces

Group leaders: Lisa Carbone and Elizabeth Jurisich
Group members: Maryam Khaqan and Natasha Rozhkovshkaya.

Hecke operators serve as fundamental tools in the intersection of modular forms, representation theory, and vertex operator algebras. In the case of monstrous moonshine Hecke operators are applied to the relevant modular functions (which are hauptmoduls) to obtain recursions vital to the computational part of the proof of the monstrous moonshine conjectures [30], [32] or [36] for an overview. Motivated by the moonshine case, and the importance of modularity in vertex algebra theory, our group worked on creating Hecke-like operators on vertex algebras and other related objects.

Starting with the foundational example of the Moonshine $\mathbb{M}$-module $V^\natural$ [31], where $\mathrm{Aut}(V^\natural) = \mathbb{M}$ is the Monster simple group with McKay-Thompson series T_g for $g \in \mathbb{M}$, we define generalized Hecke-like operators on $V^\natural$ considered as a graded $\mathbb{M}$-module. This same definition can be extended extended to other suitable pairs of vertex operator algebras V and finite groups G, where V is a graded G-module with modular graded traces. Additionally, we explored certain non-vertex algebra cases.

We define a new operator for "suitable" pairs (V,G) with $V \in R(G)[q]$ (for example the moonshine module $V^\natural$ and $\mathbb{M}$). The suitability condition on the module we assume here is modularity of the graded dimension (or trace), of weight k. We define

Definition: The ith Hecke-Adams operator on V is given by

$$\mathcal{T}(i)\left[\sum_{j\in\mathbb{Z}} V_j q^j\right] = \sum_{j\geq -1} \gamma_i(V_j)q^j \tag{1}$$

$$\gamma_i(V_j) = \sum_{d|(i,j)} d^{k-1}\Psi^d(V_{ij/d^2}).$$

Here $\Psi^d : R(G) \to R(G)$ is the dth Adams operation on the λ-ring $R(G)$ [34]. This definition, when the trace of g acting on the formula is taken, aligns with the generalized Hecke operator definition in [35]. We have naturally extended their definition to an operator on the ring $R(G)[[q]]$ as summarized in [33].

At Banff the group worked on identifying cases of "suitable" vertex algebra and group pairs, and also potential non-VOA settings. The fundamental properties of VOAs that ensure the modularity of the associated graded traces [37] are crucial for understanding the context in which Hecke operators may be defined.

We studied three types of examples where our operators can or are conjectured to be defined:

1. Moonshine Examples: G a finite group, V a graded G-module that is a Vertex Algebra or superalgebra with $Aut(V) = G$.

2. Examples with G a finite group, V a graded G-module with no explicit or known VOA structure. McKay-Thompson series are assumed modular and may be completely replicable. We consider examples such as those G. Mason's introduced in [38].

3. Examples of VOA's V with modular or weakly modular characters, and some choice of group,including the trivial group and the full automorphism group. These examples will include C2-cofinite cases and those vertex algebras satisfying the conditions Zhu's theorem [37].

Examples of VOAs in the above categories include the Moonshine VOA $V^\natural$, the Leech Lattice VOA V_Λ, the Niemeier Lattice VOAs, the Baby Monster VOA. The non-VOA examples excepting the ones suggested by [38] are conjectural.

6.8 Associative Algebras for VOSAs

Group leaders: Angela Gibney and Svetlana Makarova
Group members: Darlayne Addabbo, Lilit Martirosyan, and Ava Mock

Introduction: Vertex operator algebras (VOAs) and their supersymmetric generalizations provide a mathematical framework for studying two-dimensional conformal field theories and related structures. A well-known construction, dating back to the early work of Tsuchiya–Kanie [59], Tsuchiya–Ueno–Yamada [61], and Beilinson–Feigin–Mazur [60], takes as input a classical vertex operator algebra (VOA), together with n modules over it, together with an algebraic curve with n marked points, and outputs a vector space of coinvariants (and its dual vector space of conformal blocks). For affine VOAs at positive integer levels, and discrete series Virasoro VOAs, studied by [59, 61, 60], vector spaces spaces of conformal blocks are famously isomorphic to spaces of (generalized) theta functions (see eg. [62]), and they were shown to fit together to form vector bundles over the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable n-pointed curves of genus g.

The definitions for such sheaves of coinvariants and conformal blocks were extended, for rational and C_2-cofinite VOAs of CFT-type, on moduli of smooth curves (see e.g. [63]), and on moduli spaces of n-pointed curves with singularities by Nagatomo and Tsuchiya in [64]. Extensions to pointed stable curves higher genus and more general VOAs have recently been found by Damiolini–Gibney–Tarasca and Damiolini–Gibney–Krashen [65, 66, 68, 67]. Much is understood about these vector bundles [69, 70, 71, 72, 73].

Current and future work: Our long-term goal is to use theory developed in [67] in order to further extend sheaves of coinvariants and conformal blocks to vertex operator super algebras (VOSAs). VOSAs are ubiquitous, including recent realizations in terms of homology of numerous moduli spaces.

Key tools from [67] include the so-called mode transition algebra $\mathfrak{A}$, and d-th mode transition sub-algebras $\mathfrak{A}_d$, whose construction relies crucially on an associative algebra introduced by Zhu [74]. Now known as Zhu's algebra, $\mathrm{A}(V)$ was defined by Zhu [74] as a quotient of V by a specific subspace, where he established a one-to-one correspondence between simple admissible V-modules and irreducible $\mathrm{A}(V)$-modules. Analgous "higher level" Zhu algebras $\mathrm{A}_d(V)$ were defined in [75], where they showed that V is rational if and only if all the $\mathrm{A}_d(V)$ are finite and semi-simple.

Extending these ideas to the supersymmetric setting, Kac and Wang [76] defined an associative algebra structure on an analogous quotient $\mathrm{A}(V)$ for VOSAs and established a bijective correspondence between simple admissible V-modules and the irreducible representations of $\mathrm{A}(V)$. They further described fusion rules and discussed rationality for several examples, including VOSAs associated to representations of affine Kac-Moody superalgebras, Neveu-Schwarz algebras, and those generated by free fermionic fields.

In both the classical and super cases, $\mathrm{A}(V)$ and its higher level generalizations (and twists) have been realized through different constructions. Notably, He [77], following work of [78] in the twisted super setting, proved that in the classical case the $\mathrm{A}_d(V)$ are isomorphic to a certain quotient of the degree zero part of the universal enveloping algebra for the Lie algebra associated to V. These quotient realizations of the Zhu algebra for a classical VOA are used in [67] to define the mode transition algebra $\mathfrak{A}$ and d-th mode transition subalgebras $\mathfrak{A}_d$.

With this in mind, the first step of our project, carried out at the 2025 WINART workshop at Banff, was to obtain such a quotient construction in the supersymmetric setting, showing that for a VOSA V, the Zhu algebra $A^S(V) = A_0^S(V)$ and its higher level super Zhu algebra $A_d^S(V)$ for each d, is isomorphic to a particular sub-quotient of the degree zero part of the universal enveloping algebra associated to V. For our applications, the universal enveloping algebra we work with is different than what is used in [78]. Given this and our interest in characteristic free applications, we found it useful to work everything out explicitly and with full details.

Our current work involves applying these quotient constructions for the Zhu algebra to the development of the theory of the mode transition algebra and d-th mode transition sub-algebras in the supersymmetric context.

To illustrate, as was shown for instance in [67], there are adjoint functors between the categories of $A_d(V)$-modules and certain V-modules which we have extended to the super setting. The mode transition algebra is defined via the $d = 0$ functor, and the d-th mode transition algebras are related to the higher level Zhu algebras. The d-th mode transition algebras act on the degree d parts W_d of admissible V-modules W. In the classical setting, for VOAs, there are a number of applications especially in cases where the d-th mode transition algebras admit units that act as identity elements on W_d for all d and W (these are strong units).

A medium term aim is to show in the super setting that, like for classical VOAs, the adjoint $d = 0$ functors define equivalent categories if and only if the $\mathfrak{A}_d$ have strong units for all d. The long plan is to apply these in the development of the theory of sheaves of coinvariants and conformal blocks for representations of VOSAs.

7 Outcome of the Meeting

As described above, very significant scientific progress was made before and during the WINART workshop. Moreover, a number of common themes emerged across groups (e.g. involving certain homological, Lie-theoretic, diagrammatic techniques), which will be explored between members of different groups after the workshop. In any case, every group set concrete plans to continue research activities, and all look forward to staying in touch with their group members and other participants in the future.

Plans are already underway for WINART5 and for the continuation of collaborations begun at BIRS. Follow-up activities may include special sessions at AMS or CMS meetings, and collaborative visits through research-in-teams programs.

References

[1] C. Benedetti Velásquez, C. R. H. Hanusa, P. E. Harris, A. Morales, and A. Simpson. *Kostant's partition function and magic multiplex juggling sequences*, Ann. Comb. **24** (2020), no. 3, 439–473; MR4151625

[2] B. Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math., USA **81** (1959), 973–1032.

[3] H. Weyl, *Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Trans- formationen, I, II, III,* Nachtrag, Mathematische Zeitschrift 23 (1925), 271–309, 24 (1926), 328–376, 377–395, 789–791.

[4] J. August, M.-W. Cheung, E. Faber, S. Gratz, and S. Schroll, “Categories for Grassmannian Cluster Algebras of Infinite Rank,” *International Mathematics Research Notices*, vol. 2024, no. 2, pp. 1166–1210, 2023. DOI: 10.1093/imrn/rnad004.

[5] C. Cummings and S. Gratz, “Metric completions of discrete cluster categories,”
preprint, arXiv:2407.17369 [math.RT], 2024.

[6] F. W. Lawvere, “Metric spaces, generalized logic, and closed categories,” *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 43, pp. 135–166, 1974.

[7] A. Neeman, “Metrics on triangulated categories,” preprint, arXiv:1901.01453v3 [math.RT], 2021.

[8] Philippe Caldero and Frederic Chapoton, Cluster algebras as Hall algebras of quiver representations, *Comment. Math. Helv.* **81** (2006), no. 3, 595–616.

[9] Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky, Quivers with potentials and their representations. I. Mutations, *Selecta Math. (N.S.)* **14** (2008), no. 1, 59–119.

[10] Sergey Fomin, Michael Shapiro, and Dylan Thurston, Cluster algebras and triangulated surfaces. I. Cluster complexes, *Acta Math.* **201** (2008), no. 1, 83–146.

[11] Sebastian Opper, Pierre–Guy Plamondon, and Sibylle Schroll, A geometric model for the derived category of gentle algebras, arXiv:1801.09659, 2018.

[12] Fabian Haiden, Ludmil Katzarkov, and Maxim Kontsevich, Flat surfaces and stability structures. *Publ. Math. IHES* **126** (2017), 247318.

[13] Yankı Lekili, and Alexander Polishchuk, Derived equivalences of gentle algebras via Fukaya categories. *Math. Ann*. **376** (2020), 187225.

[14] Yann Palu, Cluster characters for 2-Calabi-Yau triangulated categories, *Ann. Inst. Fourier (Grenoble)* **58** (2008), no. 6, 2221–2248.

[15] M. Aigner, *Markov's Theorem and 100 Years of the Uniqueness Conjecture: A Mathematical Journey from Irrational Numbers to Perfect Matchings*, 2013. DOI: 10.1007/978-3-319-00888-2.

[16] E. Banaian and A. Sen, “A Generalization of Markov Numbers,” preprint, arXiv:2210.07366 [math.CO], 2022. URL: https://arxiv.org/pdf/2210.07366.

[17] Z. Greenberg, D. Kaufman, and A. Wienhard, “SL_2-like properties of matrices over noncommutative rings and generalizations of Markov numbers,” preprint, arXiv:2402.19300 [math.CO], 2024. URL: https://arxiv.org/pdf/2402.19300.

[18] Y. Gyoda, “Positive integer solutions to $(x+y)^2+(y+z)^2+(z+x)^2 = 12xyz$,” preprint, arXiv:2109.09639v4 [math.CO], 2022. URL: https://arxiv.org/pdf/2109.09639.

[19] Y. Gyoda and K. Matsushita, “Generalization of Markov Diophantine equation via generalized cluster algebra,” *Electronic Journal of Combinatorics*, vol. 30, no. 4, Paper No. 4.10, 20 pages, 2023. DOI: 10.37236/11420.

[20] A. Markoff, “Sur les formes quadratiques binaires indfinies,” *Mathematische Annalen*, vol. 15, no. 34, pp. 381–406, 1879.

[21] G. Musiker, “Super Markov Numbers and Signed Double Dimer Covers,” preprint, arXiv:2503.21872 [math.CO], 2025. URL: https://www.arxiv.org/pdf/2503.21872.

[22] S. Morier-Genoud and V. Ovsienko, "q-deformed rationals and q-continued fractions," *Forum of Mathematics. Sigma*, vol. 8, Paper No. e13, 55 pages, 2020. DOI: 10.1017/fms.2020.9.

[23] L. Leclere and S. Morier-Genoud, "q-Deformations in the modular group and of the real quadratic irrational numbers," *Advances in Applied Mathematics*, vol. 130, 2021. DOI: 10.1016/j.aam.2021.102223.

[24] K. Baur, M. J. Parsons, and M. Tschabold. *Infinite friezes*. European J. Combin., 54:220237, 2016.

[25] J. H. Conway and H. S. M. Coxeter. *Triangulated polygons and frieze patterns*. Math. Gaz., 57(400):8794, 175183, 1973.

[26] H. S. M. Coxeter. *Frieze patterns*. Acta Arith., 18:297310, 1971.

[27] S. Morier-Genoud, V. Ovsienko and S. Tabachnikov. *Introducing supersymmetric frieze patterns and linear difference operators*. Math. Z., 281:1061-1087, 2015.

[28] G. Musiker, N. Ovenhouse and S. W. Zhang, An expansion formula for decorated super-teichmller spaces, SIGMA 17:0-80, 2021.

[29] R. C. Penner and A. M. Zeitlin. *Decorated super-teichmüller space*. Journal of Differential Geometry, 111(3):527566, 2019.

[30] R. E. Borcherds, "Monstrous moonshine and monstrous Lie superalgebras," *Inventiones Mathematicae*, vol. 109, no. 1, pp. 405–444, 1988.

[31] I. Frenkel, J. Lepowsky, and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Applied Mathematics, vol. 134, Academic Press, Boston, 1988.

[32] J. H. Conway and S. P. Norton, "Monstrous moonshine," *Bulletin of the London Mathematical Society*, vol. 11, no. 3, pp. 308–339, 1979.

[33] E. Jurisich, J. Lepowsky, and R. L. Wilson, "Realizations of the Monster Lie algebra," *Selecta Mathematica*, vol. 1, no. 2, pp. 339–395, 1995.

[34] D. Knutson, *Lambda-Rings and the Representation Theory of the Symmetric Group*, Lecture Notes in Mathematics, vol. 308, Springer, Berlin, 1973.

[35] D. Alexander, C. Cummins, J. McKay, and C. Simons, "Completely replicable functions," in *Groups, Combinatorics and Geometry*, pp. 87–98, Cambridge University Press, 1992. Edited by M. W. Liebeck and J. Saxl.

[36] T. Gannon, *Moonshine Beyond the Monster: The Bridge Connecting Algebra, Modular Forms and Physics*, Cambridge University Press, Cambridge, 2007.

[37] Y. Zhu, "Modular invariance of characters of vertex operator algebras," *Journal of the American Mathematical Society*, vol. 9, no. 1, pp. 237–302, 1996.

[38] G. Mason, "Finite groups and Hecke operators," *Mathematische Annalen*, vol. 283, no. 3, pp. 381–409, 1989.

[39] A. Beauville, "Conformal blocks, fusion rules and the Verlinde formula," in *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry*, Israel Math. Conf. Proc., vol. 9, pp. 75–96, Bar-Ilan Univ., 1996.

[40] A. Beilinson, B. Feigin, and B. Mazur, "Introduction to algebraic field theory on curves," Unpublished manuscript, 1991.

[41] C. Damiolini and A. Gibney, "On global generation of vector bundles on the moduli space of curves from representations of vertex operator algebras," *Algebraic Geometry*, vol. 10, no. 3, pp. 298–326, 2023.

[42] C. Damiolini, A. Gibney, and D. Krashen, "Factorization presentations," preprint, 2022. URL: https://arxiv.org/pdf/2207.05110.pdf.

[43] C. Damiolini, A. Gibney, and D. Krashen, “Conformal blocks on smoothings via mode transition algebras,” preprint, arXiv:2307.03767, 2023.

[44] C. Damiolini, A. Gibney, and N. Tarasca, “Conformal blocks from vertex algebras and their connections on $\mathcal{M}_{g,n}$,” *Geometry & Topology*, vol. 25, no. 5, pp. 2235–2286, 2021.

[45] C. Damiolini, A. Gibney, and N. Tarasca, “Vertex algebras of CohFT-type,” in *Facets of Algebraic Geometry. Vol. I*, London Math. Soc. Lecture Note Ser., vol. 472, pp. 164–189, Cambridge Univ. Press, Cambridge, 2022.

[46] C. Damiolini, A. Gibney, and N. Tarasca, “On factorization and vector bundles of conformal blocks from vertex algebras,” *Annales Scientifiques de l’cole Normale Suprieure (4)*, vol. 57, no. 1, pp. 241–294, 2024.

[47] C. Dong, H. Li, and G. Mason, “Vertex operator algebras and associative algebras,” *Journal of Algebra*, vol. 206, no. 1, pp. 67–96, 1998.

[48] N. Fakhruddin, “Chern classes of conformal blocks,” in *Compact Moduli Spaces and Vector Bundles*, Contemp. Math., vol. 564, pp. 145–176, American Mathematical Society, Providence, RI, 2012.

[49] E. Frenkel and D. Ben-Zvi, *Vertex Algebras and Algebraic Curves*, Mathematical Surveys and Monographs, vol. 88, American Mathematical Society, Providence, RI, 2nd edition, 2004.

[50] X. He, “Higher level Zhu algebras are subquotients of universal enveloping algebras,” *Journal of Algebra*, vol. 491, pp. 265–279, 2017.

[51] V. Kac and W. Wang, “Vertex operator superalgebras and their representations,” in *Mathematical Aspects of Conformal and Topological Field Theories and Quantum Groups (South Hadley, MA, 1992)*, Contemp. Math., vol. 175, pp. 161–191, American Mathematical Society, Providence, RI, 1994.

[52] A. Marian, D. Oprea, and R. Pandharipande, “The first Chern class of the Verlinde bundles,” in *String-Math 2012*, Proc. Sympos. Pure Math., vol. 90, pp. 87–111, American Mathematical Society, Providence, RI, 2015.

[53] A. Marian, D. Oprea, R. Pandharipande, A. Pixton, and D. Zvonkine, “The Chern character of the Verlinde bundle over $\mathcal{M}_{g,n}$,” *Journal fr die reine und angewandte Mathematik (Crelles Journal)*, vol. 732, 2017.

[54] K. Nagatomo and A. Tsuchiya, “Conformal field theories associated to regular chiral vertex operator algebras. I. Theories over the projective line,” *Duke Mathematical Journal*, vol. 128, no. 3, pp. 393–471, 2005.

[55] A. Tsuchiya and Y. Kanie, “Vertex operators in the conformal field theory on $\mathbb{P}^1$ and monodromy representations of the braid group,” *Letters in Mathematical Physics*, vol. 13, no. 4, pp. 303–312, 1987.

[56] A. Tsuchiya, K. Ueno, and Y. Yamada, “Conformal field theory on universal family of stable curves with gauge symmetries,” in *Integrable Systems in Quantum Field Theory and Statistical Mechanics*, Adv. Stud. Pure Math., vol. 19, pp. 459–566, Academic Press, Boston, MA, 1989.

[57] J. Van Ekeren, “Modular invariance for twisted modules over a vertex operator superalgebra,” *Communications in Mathematical Physics*, vol. 322, no. 2, pp. 333–371, 2013.

[58] Y. Zhu, “Modular invariance of characters of vertex operator algebras,” *Journal of the American Mathematical Society*, vol. 9, no. 1, pp. 237–302, 1996.

[59] A. Tsuchiya and Y. Kanie, “Vertex operators in the conformal field theory on $\mathbb{P}^1$ and monodromy representations of the braid group,” *Letters in Mathematical Physics*, vol. 13, no. 4, pp. 303–312, 1987. DOI: 10.1007/BF00401159.

[60] A. Beilinson, B. Feigin, and B. Mazur, "Introduction to Algebraic Field Theory on Curves," Unpublished manuscript, 1991.

[61] A. Tsuchiya, K. Ueno, and Y. Yamada, "Conformal field theory on universal family of stable curves with gauge symmetries," in *Integrable Systems in Quantum Field Theory and Statistical Mechanics*, Adv. Stud. Pure Math., vol. 19, pp. 459–566, Academic Press, Boston, MA, 1989.

[62] A. Beauville, "Conformal blocks, fusion rules and the Verlinde formula," in *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry*, Israel Math. Conf. Proc., vol. 9, pp. 75–96, Bar-Ilan Univ., 1996.

[63] E. Frenkel and D. Ben-Zvi, *Vertex Algebras and Algebraic Curves*, Mathematical Surveys and Monographs, vol. 88, American Mathematical Society, Providence, RI, 2nd edition, 2004. DOI: 10.1090/surv/088.

[64] K. Nagatomo and A. Tsuchiya, "Conformal field theories associated to regular chiral vertex operator algebras. I. Theories over the projective line," *Duke Mathematical Journal*, vol. 128, no. 3, pp. 393–471, 2005. DOI: 10.1215/S0012-7094-04-12831-3.

[65] C. Damiolini, A. Gibney, and N. Tarasca, "Conformal blocks from vertex algebras and their connections on $\overline{\mathcal{M}}_{g,n}$," *Geometry & Topology*, vol. 25, no. 5, pp. 2235–2286, 2021. DOI: 10.2140/gt.2021.25.2235.

[66] C. Damiolini, A. Gibney, and N. Tarasca, "On factorization and vector bundles of conformal blocks from vertex algebras," *Annales Scientifiques de l'cole Normale Suprieure (4)*, vol. 57, no. 1, pp. 241–294, 2024.

[67] C. Damiolini, A. Gibney, and D. Krashen, "Conformal blocks on smoothings via mode transition algebras," preprint, arXiv:2307.03767, 2023.

[68] C. Damiolini, A. Gibney, and D. Krashen, "Factorization Presentations," preprint, 2022. URL: https://arxiv.org/pdf/2207.05110.pdf.

[69] N. Fakhruddin, "Chern classes of conformal blocks," in *Compact Moduli Spaces and Vector Bundles*, Contemp. Math., vol. 564, pp. 145–176, American Mathematical Society, Providence, RI, 2012. DOI: 10.1090/conm/564/11148.

[70] A. Marian, D. Oprea, and R. Pandharipande, "The first Chern class of the Verlinde bundles," in *String-Math 2012*, Proc. Sympos. Pure Math., vol. 90, pp. 87–111, American Mathematical Society, Providence, RI, 2015. DOI: 10.1090/pspum/090/01521.

[71] A. Marian, D. Oprea, R. Pandharipande, A. Pixton, and D. Zvonkine, "The Chern character of the Verlinde bundle over $\overline{\mathcal{M}}_{g,n}$," *Journal fr die Reine und Angewandte Mathematik*, vol. 732, pp. 1–43, 2017. DOI: 10.1515/crelle-2015-0003.

[72] C. Damiolini, A. Gibney, and N. Tarasca, "Vertex algebras of CohFT-type," in *Facets of Algebraic Geometry. Vol. I*, London Math. Soc. Lecture Note Ser., vol. 472, pp. 164–189, Cambridge University Press, Cambridge, 2022.

[73] C. Damiolini and A. Gibney, "On global generation of vector bundles on the moduli space of curves from representations of vertex operator algebras," *Algebraic Geometry*, vol. 10, no. 3, pp. 298–326, 2023. DOI: 10.14231/ag-2023-010.

[74] Y. Zhu, "Modular invariance of characters of vertex operator algebras," *Journal of the American Mathematical Society*, vol. 9, no. 1, pp. 237–302, 1996. DOI: 10.1090/S0894-0347-96-00182-8.

[75] C. Dong, H. Li, and G. Mason, "Vertex operator algebras and associative algebras," *Journal of Algebra*, vol. 206, no. 1, pp. 67–96, 1998. DOI: 10.1006/jabr.1998.7425.

[76] V. Kac and W. Wang, “Vertex operator superalgebras and their representations,” in *Mathematical Aspects of Conformal and Topological Field Theories and Quantum Groups (South Hadley, MA, 1992)*, Contemp. Math., vol. 175, pp. 161–191, American Mathematical Society, Providence, RI, 1994. DOI: 10.1090/conm/175/01843.

[77] X. He, “Higher level Zhu algebras are subquotients of universal enveloping algebras,” *Journal of Algebra*, vol. 491, pp. 265–279, 2017. DOI: 10.1016/j.jalgebra.2017.08.014.

[78] J. Van Ekeren, “Modular invariance for twisted modules over a vertex operator superalgebra,” *Communications in Mathematical Physics*, vol. 322, no. 2, pp. 333–371, 2013. DOI: 10.1007/s00220-013-1758-2.