25w5405: Operator Systems and their Applications

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1 Overview of the Field

Operator systems were first introduced by Arveson in the seminal paper [Arv69], providing the groundwork for a plethora of subsequent developments in non-commutative analysis. In the past decade we have witnessed a surge of interest in these mathematical objects, much of which was inspired by applications to *mathematical physics*, *zero-error quantum information theory* and *non-local games*. The aim of the workshop has been to bring together researchers with diverse background and make progress in research questions that require a combination of skills and viewpoints.

I. Non-commutative analysis and mathematical physics. Originally defined as subspaces of bounded operators on a Hilbert space containing the identity and closed under taking adjoints, operator systems were characterised abstractly in 1976 in terms of families of matricial cones via the, by now classical, Choi-Effros Theorem. Moving away from the underlying Hilbert spaces allowed for fruitful categorical constructions, such as tensor products [KPTT11], quotients [KPTT13] and inductive limits [MT18]. While, within operator algebras, both the norm closed (*C*-algebras*) and weak* closed (*von Neumann algebras*) sub-branches of the theory are well-developed, the understanding of weak*-closed operator systems has received relatively little attention when compared to norm closed ones. In particular, there has been virtually no development of the weak* tensor theory, important for operator system approximation properties such as *local reflexivity*.

Motivated by the Gelfand-Naimark Theorem, Arveson's program [Arv69] relied on the success of interpreting operator systems as non-commutative function systems. An important breakthrough thereof was obtained by Davidson and Kennedy in [DK15], where the existence of non-commutative *Choquet boundaries* was established in full generality, settling a long-standing open problem of Arveson. The description of the Choquet boundary and its closure, i.e., the non-commutative *Shilov boundary*, for specific operator systems has been a driving force in the field with interesting connections with theoretical physics. For example the interplay of the Shilov boundary with tensor categories relates to the study of quantum correlation boxes [FKPT14], and it has been used to provide equivalent reformulations of Kirchberg's Conjecture and thus of Tsirelson's Problem [Kav11].

A first step towards the study of *groupoid operator systems* was made by Connes and van Suijlekom [CS22] by associating operator systems to *tolerence relations* that lack transitivity, a principle going back to Poincaré, according to which physical measurements A, B and C may satisfy the relations $A = B$ and $B = C$ but, due to inherent measurement errors, fail to satisfy the relation $A = C$. The idea was first exploited in [CS21], where operator systems were used to model phenomena in the absence of complete physical data. Initiating an in-depth study of groupoid operator systems and identifying their C*-envelopes is expected to reveal connections with extension problems for partially defined positive definite functions and operators, and to draw the first steps towards a classification theory for general operator systems.

The importance of *stable equivalence* of operator systems (that is, isomorphism after tensoring with the compact operators) was physically justified in [CS21, CS22]. Following a question posed in [CS21], stable equivalence was mathematically formalised through the development of a full *Morita theory* in the operator system category by Eleftherakis, Kakariadis and Todorov [EKT21], which complements its C*-algebra counterpart, pioneered by Mackey and Rieffel, and important in C*-algebraic K-theory and C*-algebra classification. Following [CS21], Farenick studied *Toeplitz operator systems* [Far21], which appeared in [CS21] as a physical model of truncation of classical Fourier series.

II. Quantum stochastic processes. In the cornerstone work [FNW92], states on a quantum spin chain or, in mathematical terms, on a uniformly hyperfinite (UHF) C*-algebra, were proposed as a fruitful model for quantum stochastic processes. The *finitely correlated states* [FNW92] correspond to stochastic processes whose future behaviour can be perfectly predicted from the knowledge of finitely many degrees of freedom in the past. It was demonstrated in [FNW92] that the properties of such a stochastic process are mathematically encoded in a finite dimensional operator system, a quotient [KPTT13] of the dual of the underlying UHF algebra. Despite some subsequent advances in this area (see [MW16]), a number of questions remain open; in particular, no mathematical framework of measuring the degree to which a state fails to be finitely correlated, has been proposed. *Completely compact maps* from operator space theory, and *nuclear operator systems* [KPTT13] offer a promising route forward, an approach that combines approximation properties in operator systems [KPTT13].

III. Non-commutative graph theory. The *confusability graph* of a classical information channel was introduced by Shannon in the 1950's; it encodes the zero-error transmission properties of the channel and allows to translate problems about the channel's zero-error capacities into problems in asymptotic combinatorics. The concept was quantised in [DSW13], where the authors proposed the thesis that operator subsystems of M_n be considered as (non-commutative) confusability graphs of *quantum channels* (that is, trace preserving completely positive maps) with domain M_n.

Non-commutative combinatorics has since been shaped as a branch of operator system theory (see e.g. [BTW21]); much in the spirit of non-commutative geometry, questions about classical graphs can in many cases be translated into questions about the corresponding *graph operator systems* [DSW13] which, in their own turn, are a type of groupoid operator systems. Many classical graph parameters, such as the *Lovász number*, admit natural quantisations [BTW21, DSW13], and have physical interpretations in terms of properties of associated quantum channels.

Following their introduction [DSW13], non-commutative graphs were generalised and the wider notion of a *quantum graph* was studied in close connection to quantum groups; operator theoretically, these are operator systems that admit bimodule actions over finite dimensional C*-algebras [BCEHPSW20].

IV. Non-local games. *Non-local games* are cooperative two-player games, of direct relevance to quantum information theory as a model for witnessing finite dimensional entanglement. It was noticed in the work [LMPRSSTW20] that their perfect strategies, corresponding to different quantum mechanical models, can be described via states on different tensor products [KPTT11] of two encoding operator systems, canonically associated to each of the players. A similar development was achieved for *quantum non-local games* (see [BHTT23]); in these paradigms, the recently resolved in the negative [JNVWY20] *Connes Embedding Problem (CEP)* is substantiated by the inequality of the commuting and the minimal tensor products [KPTT11] of the encoding operator systems.

CEP was settled in [JNVWY20] by exhibiting a non-local game, whose *quantum commuting value* is strictly larger than its *quantum value*. Non-commutative graphs (see Subsection III) give rise to quantum homomorphism games [BHTT23], offering a continuum of tentative game value tests for CEP as, in contrast to classical graphs, non-commutative ones form a continuous manifold.

2 Recent Developments and Open Problems

The objective of the workshop has been to target target four entities, descibred below, by bringing together participants with expertise in the areas of *operator algebras (OA)*, *mathematical physics (MP)* and *quantum information theory (QIT)*. The participants formed research groups, working during the workshop towards these objectives. Here we describe those four key areas, and specific progress per group is described in the following sections.

Entity 1. Approximation properties and applications: OA + MP. (a) Development of weak* tensor product theory and applications to local reflexivity; (b) Definition of approximately finitely correlated states (AFCS) and their characterisation in terms of operator systems possessing finiteness properties; (c) Fine calibration of AFCS's via operator mapping ideals.

Entity 2. Non-commutative analysis for physical applications: OA + MP. (a) Identification of C*-envelopes of groupoid operator systems; (b) Positive extension problems for functions via groupoid operator systems; (c) Examination of the operator systems of tolerence relations from the viewpoint of Morita theory and tensor theory.

Entity 3. Quantum graphs: OA + MP + QIT. (a) Definition of quantum tolerance relations; their relevance to quantum physics; (b) Characterisation of the zero-error entanglement assisted capacity of an operator system in the commuting operator model; (c) First steps in quantum graph limit theory; connections to quantum channels with memory.

Entity 4. Non-local games: OA + QIT. (a) Intrinsic identification of the quantum commuting and the quantum value of quantum graph homomorphism/isomorphism games; (b) Development of a quantum NPA hierarchy; (c) Clarification of the meaning of Morita equivalent non-commutative graphs for the corresponding quantum graph isomorphism games.

3 Presentation Highlights

With participants being from a variety of backgrounds (in particular, operator algebras, quantum information theory, and mathematical physics), as well as different career stages (early career researchers and well-established experts on the topic), the workshop program started with three one-hour plenary lectures providing overviews of the field and the major open problems and areas of interest. These lectures helped ensure all participants had a good understanding of the distinct aspects of operator systems and their applications, and ensured common scientific ground to allow for fruitful discussions between participants who are members of different communities.

Later in the day, there were talks given by early career researchers, providing them a chance to expose their work to the larger group. These talks included dissemination of results generalizing the Duquet-Le Merdy's theorem characterizing the absolutely dilatable Schur multipliers over a measure space to the non-commutative setting; a discussion on how certain correlations, called "quansal", emerge in the asymptotic limit of computational nonsignalling strategies for compiled nonlocal games; a characterization of equivariant Fock covariant injective representations for product systems; and a definition and discussion surrounding the property of graph regularity for quantum graphs.

The morning of the second day of the workshop started with presentations of research questions and the formation of research breakout groups, with the aim of initiating collaborative research during the workshop, to be continued during the months following the workshop. The progress made by the groups is discussed in Section 4. The morning and lunch break gave participants time to enrol in a group, discuss the open problems or general lines of inquiry and potential methodology for solving the problems or pushing the topic further, and switch groups if desired.

The afternoon of the second day was filled with talks ranging from generalizing the framework of operator systems to general conic systems, and an investigation into game strategy transport and the existence of strategies for quantum games using an operator system approach.

Prior to dinner, there was a panel discussion led by three early-career researchers (Lara Ismert, Travis Russell, and Camila Sehnem) on the topic of Establishing a Career in Research. This panel discussion was complemented by another panel discussion on Thursday on the same topic but led by more senior academics

(Doug Farenick, Mahya Ghandehari, and Adam Skalski). The questions were similar for both panel groups, and included questions surrounding the job market, grant writing, dealing with career setbacks, work/life balance, the transition from student/postdoc to an independent researcher, and more.

Wednesday marked the halfway point of the workshop, and was primarily devoted to research time for the research breakout groups in the morning, and team-building hiking and exploration of Banff during the free afternoon.

The fourth day of the workshop included a mix of talks from early career researchers and more established researchers, as well as more time for research in groups. Talks included the development of multipartite nonlocality via extending two-player nonlocal games to multiplayer settings defined on graph vertices, a discussion of some results related to the question of whether spectral truncations converge as more spectral data is taken into account, and quantum graphs and their associated quantum Cuntz–Krieger algebras. The second panel discussion mentioned above was held just prior to dinner.

There were several talks on the final day of the workshop as well as time for research groups to wrap up and make future plans.

4 Outcomes of the Meeting

Much work was done through the six research groups: Hyperrigid operator systems (led by Raphael Clouatre and Ken Davidson), Noncommutative geometry (led by Malte Leimbach and Walter van Suijlekom), C*-envelopes and subquotient systems (led by Samuel Harris and Vern Paulsen), Groupoid Fourier algebras (led by Mahya Ghandehari and Camila Sehnem), Quantum graphs and index (led by Adam Skalski and Mateusz Wasilewski), and Decision problems for nonlocal games (led by Connor Paddock and William Slofstra).

We created a dedicated webpage as a hub for the research groups from the workshop: `https://sites.google.com/view/operator-systems-applications/ncg?authuser=0`. We plan to have a virtual meetup/reunion with all participants six months following the workshop. We provide descriptions of the progress of each group below.

4.1 Hyperrigid operator systems

Summary. In 2011, Arveson [Arv11] introduced a notion of hyperridigity for operator systems, having to do with a certain asymptotic rigidity property for $*$-representations. He conjectured that an operator system is hyperrigid precisely when all irreducible $*$-representations of the ambient C*-algebra belong to the noncommutative Choquet boundary.

This conjecture has generated a large amount of research activity in the last 15 years. The last 2 years witnessed especially rapid development: a counterexample was produced by Bilich and Dor-On [BD24], and subsequently an amended version of the conjecture was established by Clouâtre and Thompson [CT24]. However there remains the question of whether there is a clean and natural set of intrinsic conditions equivalent to Arveson's original notion of hyperrigidity. Also the commutative case remains open. Accordingly, the main goals of this project are:

- to gain a deeper understanding of the Bilich–Dor-On counterexample, and to identify a new obstruction to hyperrigidity;
- to make progress towards the resolution of the conjecture in the commutative case, based on recent work of Davidson and Kennedy.

Scientific progress. Raphaël Clouâtre and Ken Davidson gave a 10-minute overview of Arveson's hyperrigidity conjecture and some particularly relevant recent partial results. During the week, several discussions were held with other participants. Most of our efforts were concentrated on the Bilich–Dor-On construction. We now have a more conceptual understanding of what truly goes wrong there, and have identified a general condition capturing the "pathology" of the counterexample.

Some time was also spent discussing the commutative situation. For instance, the topological regularity and universal measurability of the boundary projection of Clouâtre and Saikia [CS23] was investigated. Moreover, the newly developed tools from noncommutative convex analysis of Davidson and Kennedy were examined [DK19],[DK21]. These discussions are still ongoing between Clouâtre, Davidson and Kennedy.

4.2 Noncommutative Geometry

Summary. This research group focused on operator systems in groupoid C*-algebras. A key example was given in terms of a positivity domain Ω (or a bond) in a groupoid G. The elements supported on Ω naturally gave rise to an operator system $E(\Omega)$ in the groupoid C*-algebra $\mathrm{C}^*(G)$. Key examples are given by Fourier truncations of topological groups G If Ω is a generating set for G it is not too difficult to realize that $\mathrm{C}^*(G)$ is a C*-extension of $E(\Omega)$. This naturally yields the following question:

Question. Under what conditions on Ω and G is $C^*(G)$ the C^*-envelope of $E(\Omega)$.

If this were the case, we expect to express a new invariant —more precisely, the propagation number— of $E(\Omega)$ in terms of groupoid theoretical data. Indeed, it is expected that prop$(E(\Omega))$ is encoded by combinatorial data of the Cayley graph $\mathrm{Gr}(G, \Omega)$ for the subset Ω in the groupoid G. Here the vertices of $\mathrm{Gr}(G, \Omega)$ are the elements of G and a tuple $(\gamma, \eta) \in \bigcup_{x \in G^{(0)}} G_x \times G_x$ is an edge of $\mathrm{Gr}(G, \Omega)$ if and only if $\gamma\eta^{-1} \in \Omega$.

Scientific progress. During the research group sessions at BIRS there were very fruitful discussions, with quite a large group (7) of actively participating mathematicians present at the workshop. It started with the observation that a good approach to answering the main question was first to simplify the situation and consider first discrete groups, then action groupoids (for a discrete group), and subsequently consider étale groupoids. The first two steps indicated a path to approach the case of an étale groupoid, which was worked out in subsequent communication, after the workshop. Moreover, it appears a connection can be made to the notion of hyperrigidity. For the propagation number it remains to further work out the details for the Cayley graph argument.

4.3 Subquotient systems and C*-envelopes

Summary. In the theory of operator systems the quotients are taken not just by subspaces, but by kernels of ucp maps. Their significance was identified by Kavruk, Paulsen, Todorov and Tomforde [KPTT13] in relation to finite approximation properties such as nuclearity, exactness, the operator system local lifting property (OSLLP), the weak expectation property (WEP), and the double commutant expectation property (DCEP). The research group focused in quotients of the form $\mathcal{S}/\mathcal{K}$ for $\mathcal{S} \subseteq M_n$ an operator system and $\mathcal{K}$ a kernel, and their C*-envelopes. The intuition is that these C*-algebras are not just finitely generated but they also have some “finite determinancy”.

There are some well-known C*-algebras that arise in this way such as the full group C*-algebra of the free group $\mathrm{C}*(\mathbb{F}_n)$ [FP12] and the Cuntz C*-algebra $\mathcal{O}_n$ [Zhe14]. One of the questions considered is whether the graph C*-algebras fit in this context. That is, given a graph $\mathcal{G}$ find $\mathcal{S} \subseteq M_n$ and a kernel $\mathcal{K}$ of $\mathcal{S}$ so that the C*-envelope of $\mathcal{S}/\mathcal{K}$ is the graph C*-algebra $\mathcal{O}_{\mathcal{G}}$. A further question is to find properties on the pair $(\mathcal{S}, \mathcal{K})$ such that the C*-envelope of their quotient is nuclear, exact or has the WEP.

Aubrun–Davidson–Hermes–Muller–Paulsen–Rahaman [ADHMPR24] show that a finite dimensional operator system $\mathcal{S}$ has the OSLLP if and only if

$$\lim_{k\to\infty} \sup\{\|\phi\|_{cb} \,|\, \phi : \mathcal{S} \to \mathcal{T}, UKP\} = 1,$$

where the supremum is over all operator systems $\mathcal{T}$ and unital k-positive maps from $\mathcal{S}$ to $\mathcal{T}$. Taking motivation of the currently known examples the research group considered the characterisation of the subquotient operator systems that have the OSLLP.

Scientific progress. During the workshop the group met several times and worked on the research questions. They further initiated a forum to discuss ideas and ways forward. There are initial results on low dimensions as well as studying the structure of quotients of operator systems endowed with the maximal or the minimal positive cone structure. Furthermore it has been verified that quotients of M_n have the OSLLP.

In a further direction, the group looked at the dual of subquotients. A duality has been established between subsystems of a finite dimensional C*-algebra $\mathcal{A}$ and quotients of $\mathcal{A}$ by kernels inside the kernel of a fixed trace. With this at hand it can be shown that the dual of any WEP detector $\mathcal{S}$ that is a quotient of M_n is a nuclearity detector for C*-algebras. There is work in progress and the group communicates on a regular basis.

4.4 Groupoid Fourier Algebras

Summary. The Fourier and Fourier-Stieltjes algebras over locally compact groupoids have been defined in a way that parallels their construction for groups. In the case of locally compact groups, these Banach algebras exhibit desirable functorial properties, which allow us to investigate their important Banach algebraic features. Recently, similar functorial properties for the Fourier algebra of locally compact étale groupoids were observed. As a result, certain hereditary properties of the Fourier algebras of groupoids could be deduced from relevant features of their isotropy subgroups. For instance, if an étale groupoid G admits a non-amenable isotropy subgroup, then $A(G)$ does not contain a bounded approximate identity. This result can be viewed as a partial analog to Leptin's theorem for the groupoid setting.

One goal of this project was to explore various Banach algebraic properties of the Fourier and Fourier-Stieltjes algebras of groupoids, akin to the results available in the group setting. Characterizing étale groupoids whose Fourier algebras admit bounded approximate identities is an example of such efforts. Another goal was to investigate relations between the Fourier (and Fourier-Stieltjes) algebra and the C*-algebra of (certain classes of) groupoids. Given that our definition of the Fourier and Fourier-Stieljes algebras are based on continuous representations of groupoids, the relation between these algebras and the groupoid C*-algebras is not a priori clear.

Scientific progress. A small group (Ghandehari, Sehnem, and Wiersma) worked on this project during our time at BIRS. Wiersma and Ghandehari have previously worked on the Fourier algebras of locally compact groups. Sehnem has experience with C*-algebras of groupoids. We started our discussions with how the Haar system of a groupoid relates to the Haar system of its subgroupoids. Understanding this relation allows us to relate elements of the Fourier algebra of a groupoid and those of its subgroupoids. In particular, we looked at the case of transformation groupoids; the functorial properties of the Fourier algebra associated with this class of groupoids are not studied in the earlier work in this field. We have made partial progress on lifting Banach algebraic features of the Fourier algebra of a subgroupoid to the groupoid itself, in this particular case. We plan to continue our discussions in May 2025, when the team members attend the Canadian Operator Algebras Symposium (COSY) in Waterloo.

4.5 Quantum graphs and index

Summary. Suppose that $A \subset B$ is a (say unital) inclusion of C^*-algebras. We can then ask about the 'size of B over A', formally called the *index of the inclusion*. This notion appeared very naturally in the von Neumann algebra theory in the study of subfactors. Fix (for the moment) a conditional expectation $E : B \to A$ and consider the two following numbers:

$$\text{Ind}(E) := \inf\{C > 0 : CE - \text{id}_B \text{ is positive}\},$$

$$\text{Ind}(E) := \inf\{C > 0 : CE - \text{id}_B \geq 0 \text{ is completely positive}\}$$

(the actual notion of the index involves also minimising over the choice of a conditional expectation). A key theorem of Frank and Kirchberg (see [FK98]) says that for (A, B, E) as above we have

$$\text{Ind}(E) < \infty \Leftrightarrow \text{Ind}_{CP}(E) < \infty.$$

The proof of the above result passes through biduals, and involves the notion of basis, crucially studied in the von Neumann algebraic context.

Recently a notion of (both positive and completely positive) index was introduced for operator systems by Araiza, Griffin and Sinclair [AGS24]. This raises immediately the following question: suppose that $X \subset Y$ is a (unital) inclusion of operator systems, and $\phi : Y \to X$ is a ucp map. We can then naturally define as above the notions of Ind(ϕ) and Ind$_{CP}(\phi)$. The problem we would like to understand is the following (we just focus on the non-trivial implication). Is it true that

$$\text{Ind}(\phi) < \infty \Rightarrow \text{Ind}_{CP}(\phi) < \infty?$$

There is an analogous extension of the index for bimodules, which is related to the study of infinite quantum graphs, for which we also do not know whether finiteness of the complete version of the index is equivalent to finiteness of the usual index (see [Was24, Proposition 3.27, Remark 3.28]).

Scientific progress. We spent two sessions working on the task at hand, mostly discussing the subtleties of the problem and identifying further related questions. We have in particular focused on the quantum graph problem mentioned above, which can be reformulated as follows: suppose that we have two Hilbert bimodules H_1, H_2, over the direct product algebra $\prod_{i\in I} M_{n_i}$ of matrix algebras. Suppose that H_1 and H_2 are 'Banach space equivalent'; must they also be 'operator space equivalent'? If we could find a positive answer to the problem listed above, also here the answer would be yes. We have shown that the answer is yes if the size of matrices is uniformly bounded; beyond that we have essentially reduced the question to the following explicit matrix inequality problem.

Question. Suppose that we have K matrices $V_1, \ldots, V_K \in M_N$ which form an orthonormal set with respect to the normalised trace on M_N, and for any scalars $c_1, \ldots, c_K \in \mathbb{C}$ we have

$$\| \sum_{i,j=1}^{K} c_i \bar{c}_j V_i V_j^* \| \leq M \sum_{i=1}^{n} |c_i|^2.$$

What is then the maximal value of the norm $\| \sum_{i=1}^{K} V_i V_i^* \|$? In particular, is it uniformly bounded?

The problem appears elusive; we have checked various concrete orthonormal bases, and the bound appears to be 'uniform' – but the general situation remains unclear.

4.6 Decision problems for non-local games

Summary. Given a nonlocal game $\mathcal{G}$, the Navescues-Pironio-Acin (NPA) hierarchy gives a monotone (decreasing) sequence of upper bounds $\{val(\mathrm{NPA}(\mathcal{G}, \mathrm{k}))\}_{\mathrm{k}\in\mathbb{N}}$ on the commuting operator value $\omega_{qc}(\mathcal{G})$ of the game [NPA08]. Fixing a constant $\theta \in (0, 1]$, the problem of deciding if $\omega_{qc}(\mathcal{G}) \geq \theta$ is in coRE since running the NPA hierarchy gives a computer program which halts if $val(\mathrm{NPA}(\mathcal{G}, \mathrm{k})) < \theta$ for some $k = 1, 2, 3, \ldots$. In particular, a result of Slofstra [Sol20] shows that this problem is *undecidable* for a $\mathbb{Z}_2$-linear constraint system (LCS) nonlocal game when $\theta = 1$.

On the other hand, we know very little about the complexity of deciding if $\omega_{qc}(\mathcal{G}) > \theta$ for a chosen $\theta \in [0, 1)$. In particular, if this problem is undecidable (and in RE), then there is a Turing machine M which on input $\mathcal{G}$ halts if $\omega_{qc}(\mathcal{G}) > \theta$. A common misconception is that the NPA hierarchy *always* achieves the value $\omega_{qc}(\mathcal{G})$ at some finite level $N \in \mathbb{N}$. However, if this were true, then it could solve the problem "$\omega_{qc}(\mathcal{G}) > \theta$". Therefore, if this problem is undecidable, it would follow that this commonly held belief is false.

Question. Is it true that for any nonlocal game $\mathcal{G}$ there is a $N \in \mathbb{N}$ such that $val(\mathrm{NPA}(\mathcal{G}, \mathrm{N})) = \omega_{\mathrm{qc}}(\mathcal{G})$?

Scientific progress. One approach to this problem is to consider synchronous nonlocal games. A result of Mehta, Slofstra, and Zhao shows that there is a computable mapping from Turing machines to elements of the scenario algebra $M \mapsto \alpha_m \in \mathbb{C}\mathbb{Z}_{n_A}^{*m_A}$ such that deciding if α_m is trace positive is coRE-hard [MSZ23]. Recall that if $\Phi_{\mathcal{G}}^s \in \mathbb{C}\mathbb{Z}_{n_A}^{*m_A}$ is a synchronous game polynomial, then the synchronous commuting operator value is achieved by a tracial state $\tau(\Phi_{\mathcal{G}}^s) = \omega_{qc}^s(\mathcal{G})$. We remark that in the non-perfect case, it is possible that $\omega_{qc}^s(\mathcal{G}) < \omega_{qc}(\mathcal{G})$ and $\omega_{qc}(\mathcal{G})$ is not necessarily achieved by a tracial state on $\Phi_{\mathcal{G}}^s$. (The perfect case is one where the strategy wins with probability one.) Therefore, this approach concerns the problem of determining whether $\omega_{qc}^s(\mathcal{G}) > \theta$ is RE-hard given a synchronous game $\mathcal{G}$ and $\theta \in [0, 1)$.

Letting $\theta = 1/2$, it follows that $\omega_{qc}^s(\mathcal{G}) > \frac{1}{2}$ if there exists a tracial state $\tau(\frac{1}{2} - \Phi_{\mathcal{G}}^s) < 0$, and by the [MSZ23] result deciding if $(\frac{1}{2} - \Phi_{\mathcal{G}}^s)$ is *not* trace positive is in RE. Therefore, to establish the existence of a nonlocal game $\mathcal{G}$ for which the NPA hierarchy never reaches the synchronous quantum commuting value, it would suffice to alter the embedding of [MSZ23, Equation 5.1] to obtain $M \mapsto \alpha(m) = \frac{1}{2} - \Phi_{\mathcal{G}}^s(m)$ for some synchronous game $\mathcal{G}$. Notably, it appears as though the relations on the algebra $\mathcal{A}_{\mathcal{S}}(m)$ could all be expressed through relations of a Boolean constraint system (BCS) nonlocal game. Such games are related intimately with synchronous nonlocal games, see for instance [PS23]. This direction is currently being pursued by a collaboration involving the individuals listed above.

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