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NC Function Theory: The Non-commutative Frontier of Analysis and Algebra

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1 Overview of the Field

From an analytic viewpoint, Non-commutative (NC) Function Theory originated in the works of J.L. Taylor in the 70's as part of his research in multivariate spectral theory and functional calculus for tuples of linear operators [43–45]. NC function theory can be viewed as an extension of classical complex analysis to several NC variables [1, 26]. On the other hand, the theory of non-commutative rational functions and their linear representations or 'realizations', was developed earlier in Non-commutative Algebra and in Systems and Control Theory. Namely, S.A. Amitsur proved that the set of all NC rational functions is the *universal skew field of fractions of the free algebra* [2]. This is a universal localization of the free algebra, the non-commutative ring of all polynomials in several NC variables. The general theory of universal localizations of non-commutative rings was developed in detail by P.M. Cohn, who introduced various classes of non-commutative rings which admit universal localizations, including *semi-free ideal rings* or *semifirs* [6]. In particular, the rings of all free formal power series (FPS), rational free FPS and algebraic free FPS are semifirs and hence have universal skew fields of fractions. Analytic NC function theory was developed further by Kaliuzhnyi-Verbovetskyi and Vinnikov, who showed, in particular, that the elements of the *free skew field* of NC rational functions can be interpreted as holomorphic and analytic NC functions in the sense of Taylor [24–26].

Function theory and classical operator theory have been deeply intertwined since the middle of the previous century. The theory of Hardy spaces of analytic functions in the complex unit disk, for example, has played a seminal role in the development of modern Operator Theory. Classical algebraic ideas, such as extensions and modules, have, of course, often appeared in the study of operator algebras. Difficulties arising from topological considerations have, however, often limited the applicability of advanced algebraic tools and constructions in operator algebra theory and other branches of analysis. The pioneering work of J.L. Taylor stands out as a breakthrough example of the power that can be harnessed by combining Functional Analysis and Algebra. In the early 1970's, following his success in constructing joint spectra and holomorphic functional calculus for several commuting operators, Taylor attempted to tackle the problem of joint spectra for several non-commuting operators by applying sophisticated algebraic techniques and topological vector spaces to produce a categorical version of the spectrum and potential functional calculi. The free algebra of complex polynomials in several NC variables arises naturally as a central object in the study of functional calculus of several non-commuting linear operators on a Hilbert space, via evaluation at tuples of operators. Free polynomials can thus be viewed as functions on the $d-$dimensional *NC universe*, the disjoint union of $d-$tuples of square $n \times n$ matrices of all sizes, n, over a fixed field. Taylor observed that evaluations of free polynomials in several NC variables on tuples of matrices of a given size possess three basic properties: they respect the grading (matrix size), joint similarities, and direct sums. A *free non-commutative function* is then any function on an *NC set*, *i.e.* a subset of the NC universe which is closed under direct sums, which obeys these three key axioms. Perhaps, due to their complexity, Taylor's papers did not have an immediate impact. However, this work of Taylor has become extremely influential in the last decade, with several groups of researchers advancing what is now known as NC Function Theory, including, most notably, Kaliuzhnyĭ-Verbovetskiĭ–Vinnikov and Agler–McCarthy. The works of Muhly-Solel and of Popescu, which interpret the full Fock space as a *free Hardy space* of NC functions in a certain open unit row-ball of the NC universe, also fit nicely into the framework outlined by Taylor. Moreover, in the setting of Free Probability, an important branch of Operator Algebra Theory, D.V. Voiculescu rediscovered some of Taylor's NC Function Theory results in the early 21^{st} century. As first shown by Kaliuzhnyĭ-Verbovetskiĭ and Vinnikov, locally bounded NC functions on level-wise open NC sets are automatically holomorphic in the sense that they are Gâteaux and Fréchet differentiable at any point, and they are analytic in the sense that they have certain Taylor-type power series expansions (so-called *Taylor–Taylor series*) about any point in their NC domains with non-zero radii of convergence. NC Function Theory can therefore be viewed as a non-commutative extension of classical analytic function theory, and many classical complex analysis and several complex variable results extend naturally to this NC setting including: the Maximum Modulus Principle, Schwarz Lemma, Liouville's Theorem, Löwner's Theorem, the Monodromy Theorem, the Oka–Weil Theorem, the concept of a complex analytic manifold and much more. In the last decade, the application of algebraic and analytic tools in NC Function Theory has opened up a whole new frontier in non-commutative mathematics, providing a bridge between Free Algebra and Free Analysis. For example, J.E. Pascoe has proven a non-commutative Inverse Function Theorem and established an NC version of the Jacobian conjecture on invertibility of polynomial mappings. M.L. Augat has extended Pascoe's free Jacobian Theorem to obtain a Free Ax–Grothendieck Theorem: The compositional inverse of any injective free polynomial mapping is a free polynomial mapping. Moreover, while Pascoe's methods are largely analytic, Augat's proof is almost entirely algebraic in nature, applying results on free skew fields and rings of generic matrices.

An NC function, $\mathfrak{r}$, is rational if and only if it has a finite–dimensional "linear representation" or *realization*, (A, b, c). Namely, $A = (A_1, \cdots, A_d)$ is a $d-$tuple of square, finite matrices, $A_j \in \mathbb{C}^{n \times n}$, $b, c \in \mathbb{C}^n$ and (A, b, c) is said to be a realization of $\mathfrak{r}$ if $\mathfrak{r}(X_1, \cdots, X_d) = I_m \otimes b^* \left(I_m \otimes I_n - \sum X_j \otimes A_j\right)^{-1} I_m \otimes c$. Realizations originated in the work of Schützenberger in automata theory, and were further developed by Cohn and Reutenauer in the context of NC Algebra [7, 8, 39]. The technique was rediscovered independently by Fliess in Systems and Control Theory as well as by Haagerup and Thorbjørnsen in the setting of Free Probability Theory [13, 14, 16, 17]. Recently, NC rational function theory and its applications to NC Algebraic Geometry, Operator Convex Analysis, and Free Probability Theory have flourished under the application of a blend of algebraic and analytic techniques, as is explicit in the recent research successes of Helton–Klep–Volčič, Kaliuzhnyĭ-Verbovetskiĭ–Vinnikov and their collaborators [18, 20, 21, 29]. The expression, $L_A(Z) := I - \sum Z_j \otimes A_j$, appearing in the realization of an NC rational function is called a *linear pencil*. Helton, Klep, and Volčič employed determinantal varieties of linear pencils and free polynomials (elements of the free algebra) to study factorization into irreducible factors, and Klep–Pascoe–Volčič have established a positive solution to the NC analogue of Hilbert's 17th problem for the free skew field of NC rational functions, extending Helton's corresponding result for the free algebra [19, 27, 47].

Considering pencils with self-adjoint coefficients or, alternatively, allowing for adjoints leads one to the realm of Matrix Convex Analysis and NC Real Algebraic Geometry [23, 31]. The positivity domain of a linear pencil, *i.e.* the set of all $d-$tuples of matrices obeying the *linear matrix inequality* (LMI), $L_A(Z) \geq 0$, is a matrix–convex set, a central object in matrix and operator convexity theory. It was shown by Helton and McCullough that a *free semi-algebraic set* is matrix–convex if and only if it is the positivity domain of a linear pencil [22]. LMI domains are, as seen in the work of Davidson, Dor-On, Shalit, and Solel, duals of matrix ranges of tuples of operators [10]. Helton, Klep, McCullough, and Schweighofer have demonstrated that non-commutative convexity can be applied to solve relaxations of classical problems such as the problem of containment of LMI domains of Ben-Tal and Nemirovski [18].

In the purely algebraic setting, Bell and Smertnig [4] have recently resolved a long-standing conjecture of Reutenauer on NC rational Pólya series. Namely, they have proved that *NC rational Pólya series*, *i.e.* power series expansions of NC rational functions with non-zero coefficients contained in a finitely-generated subgroup of the multiplicative group of the field, are precisely the unambiguous rational series introduced by Reutenauer [5]. The work of Reichstein and Vonessen [35] on polynomial identity rings and non-commutative algebraic geometry has been a significant influence in the operator–algebraic investigations of Griesenauer, Muhly, and Solel and has led, in particular, to the study of C^*-envelopes of operator algebras arising from matrix concomitants [15, 36, 37]. Concomitants originate in (matrix) invariant theory, as developed by Artin, Procesi, and Le Bruyn [33]. Invariant theory is important in NC Function Theory due to the natural similarity action of invertible matrices on the NC universe. Derksen, Klep, Makam, and Volčič have, for example, harnessed invariant theory to resolve a conjecture on the similarity of tuples of matrices [12].

With firm foundations built, the time is ripe to exploit the links forged between NC Function Theory, Free Analysis, Free Algebra, and Operator Algebra Theory. Many algebraic problems, including an NC rational extension of the Grothendieck–Ax Theorem, the characterization of the commutant or centralizer of an NC rational function, and a version of Artin's approximation theorem for the free skew field may be amenable to both Free Algebra and Free Analysis techniques. New insights into existing NC algebraic results, such as Schofield's Theorem on relations for NC rational functions and the centralizer theorems of Bergman and Cohn, may benefit from a Free Analysis perspective. Development of 'free' analogues of several complex variables results such as the Cousin problems and their solution via Cartan's theorems may be more receptive to Free Analysis.

2 Presentation Highlights

Our morning plenary talks on related topics in NC Algebra and Free Analysis were quite successful in laying the foundations for a common discourse between algebraists and analysts. Some presentation highlights include:

- **Harm Derksen: Invariant theory of quiver representations and non-commutative rank.**

  This provided an interesting connection between invariant theory, quiver representations and computation of the inner or non-commutative rank of matrices over the free skew field of all NC rational functions. In particular, the talk presented how invariant theory gives rise to bounds for identity testing in NC function theory, and how quiver representations relate to the realization theory of NC rational functions.

- **Meric Augat: Operator realizations of non-commutative analytic functions.**

  An NC rational function is rational if and only if it has a finite–dimensional realization. By considering more general classes of infinite–dimensional realizations, the speaker and his collaborators characterize NC entire and meromorphic functions in terms of their realizations. This talk was based on the recent paper [3], and outlined a promising future direction for investigating classes of analytic NC functions through the operator theory of the coefficients in their realizations (e.g., when are the coefficients compact, or quasi-nilpotent, and so on).

- **Orr Shalit: Spectral radii for matrices over operator spaces and applications.**

  The study of algebras of bounded NC functions on subvarieties of NC unit balls led Salomon, Shalit and Shamovich to associate a spectral radius function ρ_E with every finite-dimensional operator space E. Concretely, if A is a tuple of matrices, then $\rho_E(A)$ is defined via a certain tensor power limit formula, which reduces to Gelfands spectral radius formula when E is one-dimensional. When E is the row operator space, ρ_E coincides with the joint spectral radius studied by Bunce, Popescu, and others.

  In a recent preprint, Shalit and Shamovich introduced a notion of spectral radius, ρ_E, for an arbitrary finite–dimensional operator space, E, and proved that $\rho_E(A) < 1$ if and only if A is jointly similar to a tuple that lies in the NC unit ball corresponding to E [40]. For example, when E is the row operator space, this means that A is jointly similar to a strict row contraction.

- **Be'eri Greenfeld: When is an almost-solution, almost a solution?**

  Given a tuple of matrices, $A_1, \cdots, A_n$ which "almost" satisfy a polynomial equation $p(X_1, \cdots, X_n) = 0$, in the sense that $p(A_1, ..., A_n)$ has small rank this talk investigated when one can perturb $A_1, ..., A_n$ by small-rank matrices to obtain a genuine solution? Stability of solutions was explored, along with related problems, using representation-theoretic techniques and tools from asymptotic metric algebra. Connections to amenability and soficity of associative and Lie algebras were also discussed.

- **Victor Vinnikov: A walk through the free skew field.**

  This plenary talk gave a nice introduction to the free skew field of NC rational functions from the NC function theory perspective, through their evaluations on matrices of arbitrary dimensions. The difference–differential calculus, Taylor–Taylor series expansions, and realization theory were discussed.

- **Jason Bell: Noncommutative Factorization.**

  This provided an introduction and overview of some of the recent developments in factorization theory in noncommutative domains. A particular emphasis was given to noncommutative unique factorization domains, such as formal power series.

- **Daniel Smertnig: Noncommutative rational series.**

  Prof. Smertnig gave an interesting introduction to the semi-free ideal ring of all NC rational series from the point of view of cellular automata and NC algebra. Characterizations of interesting subclasses by their growth and by arithmetic properties were provided.

- **Mike Jury: Non-commutative optimal polynomial approximants.**

  This talk gave a constructive and more algebraic proof of the fact that an NC rational function is cyclic for the left free shifts on the full Fock space if and only if it is non-singular in the NC unit row-ball.

- **Eric Evert: Inclusion constants for matrix convex sets relevant to quantum incompatibility.**

  Matrix convex sets are dimension free generalizations of classical convex sets which extend classical convex sets to include tuples of self-adjoint $n \times n$ matrices of all sizes n. As it turns out, given a classical convex set C, the extension of C to a matrix convex set is not unique. In fact there are typically infinitely many matrix convex sets which agree with C when restricted to their first level. Of particular note are the minimal and maximal matrix convex sets, $\mathcal{W}^{\min}(C)$ and $\mathcal{W}^{\max}(C)$, generated by C. Given a convex set C, a major direction of research in matrix convexity is to determine the smallest constant s such that $\mathcal{W}^{\max}(C) \subseteq s \cdot \mathcal{W}^{\min}(C)$. For particular choices of C, e.g., when C is the matrix diamond, this question is closely connected to studying the joint measurability of measurements in quantum information. By exploiting connections to extreme points of matrix convex sets, this talk presented a variety of results and conjectures related to inclusion constants for these settings of interest.

This talk was based on joint work with Andreas Bluhm, Igor Klep, Victor Magron, and Ion Nechita.

- **Scott McCullough: Krein–Milman and more for partially convex free sets.**

  Prof. McCullough's talk gave an introduction to partial or $\Gamma-$convexity for non-commutative sets, which is a significant generalization of operator/ matrix convexity theory. The talk featured recent results in partial matrix convexity, including a Krein–Milman type theorem in the style of Davidson and Kennedy, and categorical duality in the spirit of Webster and Winkler. This talk was based on joint work with Igor Klep, Mike Jury, Mark Mancuso, James Pascoe and Tea Štrekelj.

- **Matyas Domokos: Noncommutative invariant theory.**

  While Derksen's talk addressed applications of classical commutative invariant theory in ring theory, this talk considered noncommutative invariants arising from group actions on relatively free rings. An overview of results was given, ranging from invariants in free algebra to invariants in the ring of generic matrices and related objects. The talk highlighted various finiteness conditions (one-sided noetherianity, presence of polynomial identities, etc.) that are fundamentally required for deriving a comprehensive framework of noncommutative invariants.

3 Open Problems, Recent Developments and Scientific Progress

A 90 minute open problem session was held in the evening of Tuesday May 06. Several researchers presented some interesting open problems and ideas. Here is a brief description of the open problems discussed:

- **Paul Muhly**: Basic examples of free functions in free analysis are noncommutative polynomials. When restricted to tuples of $n \times n$ matrices, they generate the so-called algebra of generic matrices GM_n, an eminent object in the theory of polynomial identities. We can endow GM_n with a norm (for a given polynomial f, take the supremum of norms of the evaluations of f on all tuples of $n \times n$ matrices that are row contractions). To progress the analysis of functions in several $n \times n$ matrix variables, it is crucial to understand the following.

  Problem: Identify the C*-envelope of GM_n.

  The C^*-envelope is the smallest C^*-algebra that contains a (completely isometric) copy of the operator algebra. In some sense, it measures the complexity of the operator algebra. Note that the C*-envelope of a slightly larger algebra (GM_n extended with traces of elements in GM_n) has been successfully identified by Griesenauer–Muhly–Solel [15].

- **Meric Augat**: Let F be the free skew field, the universal skew field of fractions of the free skew field. Since there are no relations between the generators of F, its structure is very rigid. For example, Schofield (1985) showed that given $r, s \in F$, either r and s commute, or they do not satisfy a common rational relation (i.e., they are free). Free analytic functions are likewise generated by relation-free elements, which inspires the following.

Problem: Let r and s be free analytic functions. If r and s do not commute, is it true that they do not satisfy a common (noncommutative analytic) relation?

The analog of this problem for formal power series is likewise open.

- **Zinovy Reichstein**: Let S be a generating set of the algebra of $n \times n$ matrices. Let $\ell(S)$ be the smallest number k such that the products of length at most k in elements of S span the algebra of $n \times n$ matrices. This number is also called the length of the algebra (of $n \times n$ matrices), and is a classical measure in the study of algebra growth. In 1984, Paz posed the following.

  Problem: Show that $\ell(S) \leq 2n - 2$.

  There are known instances of S with $\ell(S) > 2n - 3$. Furthermore, the problem has been resolved for $n \leq 6$, Pappacena [32] showed that $\ell(S) \in \mathcal{O}(n^{3/2})$, and Shitov [42] showed that $\ell(S) \in \mathcal{O}(n \log n)$.

- **Zinovy Reichstein**: Let R be the free algebra in m generators, and consider its n-dimensional representations. Two representations π and ρ of R are polynomially equivalent if there is an automorphism $f : R \to R$ such that $\rho = \pi \circ f$.

  Problem: Are any two irreducible n-dimensional representations of R polynomially equivalent? If so, are they always equivalent via tame automorphism of R?

  If $m \leq 2$, every automorphism of R is tame (a composition of affine and triangular automorphisms); this is no longer the case for $m \geq 3$. Furthermore, both parts of the problem above have a positive answer whenever $m \geq n + 1$, by Reichstein [34].

- **Jeet Sampat**: Davidson and Pitts [11] showed that for the NC Hardy algebra of uniformly bounded NC functions in the unit row-ball, there is a unique weak$-*$ representation, given by point evaluation, fibred over any interior point. This implies, in particular, that all weak$-*$ continuous finite–dimensional representations fibred at interior points are automatically completely contractive. This is useful for the classification of quotients of the NC Hardy algebra up to completely isometric isomorphism.

  Problem: Is there an analogue of this fact for the unit ball in the NC universe with respect to an arbitrary operator space structure on the NC universe? Three strategies were discussed:

  (i) Using Taylor–Taylor series and considering their convergence in the supremum norm.

  (ii) Applying realization theory and working with the Ball–Marx–Vinnikov transfer function realizations for the NC Hardy algebra of uniformly bounded NC functions in the operator space unit ball.

  (iii) A third "brute–force" approach based on the original methods of Davidson and Pitts.

  However, each of these strategies do not yield the desired result, as was shown in [38]. Therefore, either a modification of these strategies or a completely new approach is required.

- **James Pascoe**: The quantum annulus of type r is the set of operators with singular values between $1/r$ and r. Analytic functions on the classical annulus can be evaluated on operators in the quantum annulus via the holomorphic functional calculus. The spectral constant $K(r)$ is the optimal uniform bound on norms of evaluations on quantum annulus in terms of the supremum norms of functions.

  Problem: Show that $K(r) = 2$ for all $r > 0$.

  It is known that $2 \leq K(r) \leq 1+\sqrt{2}$ by Tsikalas [46] and Crouzeix-Greenbaum [9], and $\lim_{r\to\infty} K(r) = 2$ by Pascoe (2025).

- **Mike Jury** Let f be a free function in the noncommutative Hardy space (i.e., it is a square-summable power series in n variables). Let H_f be the corresponding Hankel matrix, indexed by noncommutative words whose (u, v) entry is the coefficient of u^*v in the series expansion of f. Since f is square-summable, H_f is a bounded operator; it is moreover of finite rank if and only if f is rational.

  Problem 1: Provide an effective estimate of $\|H_f\|$.

  Problem 2: Given a natural number k, estimate $\inf_{g\,:\,\mathrm{rk} H_g \leq k} \|H_f - H_g\|$.

  For $n = 1$, these two problems are resolved by Nehari's theorem and related results in complex analysis and operator theory. Such estimates are of interest in computer science (see, for example, [30].)

- **Orr Shalit**: The following is an analog of the "matrix cube problem" in semidefinite optimization, solved by Helton-Klep-McCullough-Schweighofer [18].

  Problem: Given d, what is the largest constant C, such that every d-tuple of operators of norm at most C can be dilated to a d-tuple of commuting unitaries?

  This problem naturally surfaces in free probability, random matrix theory, noncommutative tori, and quantum physics.

Recent Developments and Scientific Progress

Our conference participants and speakers, Daniel Smertnig and Jason Bell were interested in the ring-theoretic results and and open questions of Eli Shamovich and Jurij Volčič that arise in investigation of spectral data for noncommutative polynomials [41]. These questions can be expressed in terms of factorization phenomena in free algebra, which are within Daniel's and Jason's expertise. In particular, Daniel proved that the set of all factorization classes under an intertwining relation between free polynomials is finite (This was a question raised by Jurij). All the involved agreed that finer factorization methods can likely provide further advances on the spectral analysis of noncommutative polynomials, and intend to explore this venue.

During his talk on operator realizations, Meric Augat raised the question as to whether the ring of uniformly entire NC functions is a semifir. Shortly after the conference M. Augat, R. Martin and E. Shamovich proved that the rings of analytic NC functions given by free formal power series with "Cauchy–Hadamard" radius of convergence at least $r \in (0, +\infty]$ are semi-free ideal rings, $\mathcal{O}_d(r)$. In particular, the ring, $\mathcal{O}_d := \mathcal{O}_d(+\infty)$ of uniformly entire NC functions is a semifir. It is unknown whether the embedding $\mathcal{O}_d(s) \hookrightarrow \mathcal{O}_d(r)$ is totally inert, or at least honest, *i.e.* inner-rank preserving, for any $0 < r < s \leq +\infty$. In particular, considering the semifir, $\mathcal{O}_d = \mathcal{O}_d(+\infty)$ of all NC entire functions, it is unknown as to whether global NC meromorphic identities exist for all stably-finite algebras, $\mathcal{A}$, which admit a homomorphism $\phi : \mathcal{O}_d \to \mathcal{A}$.

That is, elements in the universal skew field of fractions of $\mathcal{M}_d$ can be identified with NC rational expressions composed with elements of $\mathcal{O}_d$. The interpretation of elements of $\mathcal{M}_d$ as "NC meromorphic functions" in the $d-$dimensional complex NC universe of $d-$tuples of complex square matrices of all finite sizes is problematic, as there could exist, in principle, a non-zero element of $\mathcal{M}_d$, which vanishes identically on its (necessarily Zariski-dense at sufficiently high levels) NC domain. Worse, its inverse is then defined essentially nowhere. It is hoped that analytic methods from free probability theory will be effective for solving this problem.

Scott McCullough gave a research talk on partial or $\Gamma-$matrix convexity, a significant generalization of matrix convexity theory which he is currently developing with Igor Klep, Tea Štrekelj and several other collaborators. This talk invoked many questions from the free analysis community, and initiated new efforts in this area.

On the topic of noncommutative invariant theory, there has been progress on finite generation and freeness of NC rational functions fixed under a finite group action. While the initial advances on this problem had a more analytic flavor [28], Harm Derksen suggested a module theory approach that he and Jurij Volčič are now pursuing to expand the findings of [28].

4 Outcomes of the Meeting

We believe that the workshop was quite successful. The plenary talks were largely targeted at an optimal level and provided a mathematical foundation and common language for workshop participants. This enabled our mathematically diverse group to interact fruitfully and discuss problems and ideas of mutual interest. There were several very well-received talks on a wide variety of topics in algebra and analysis.

Our 90 minute open problem session in the evening of Tuesday May 06 was also successful, with several problems of both analytic and algebraic flavors discussed and presented.

In particular, several open problems on noncommutative polynomials and rational functions originating from the free analysis community turned out to have a huge appeal to the algebra community, and new approaches to these problems have been initiated at the workshop.

An event for young researchers was held in the evening of Thursday May 08 in the BIRS lounge. Approximately 20 people were in attendance including many (if not all) of the young researchers in the workshop. This event began with the introduction of a panel of researchers at various career stages: Ken Davidson (U. Ottawa, Canada), Rob Martin (U. Manitoba, Canada), Jurij Volčič (U. Auckland, New Zealand), Eli Shamovich (Ben-Gurion U. of the Negev, Israel) as well as Francesca Arici (U. Leiden, the Netherlands), Meric Augat (James Madison U., U.S.A.) and Tea Štrekelj (U. Primorska, Slovenia). We then split into small groups to discuss academic life and career–related questions. Topics of discussion included:

- How to be an effective mentor/ advisor for graduate students.
- Postdoctoral opportunities in various countries.
- How to maintain a healthy work-life balance.
- How to develop useful collaborations.
- Applying for funding as an early-career principal investigator.
- How to be an effective and efficient researcher.

The event was very successful and ended up lasting 2 hours.

Inspired by the success of this BIRS workshop, in a meeting proposing sessions for the upcoming International Workshop in Operator Theory and its Applications (IWOTA) in 2026, a session on Free Analysis and NC Algebra will be organized by Be'eri Greenfeld and Eli Shamovich.

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