Random Perturbations of Landau Hamiltonians and Semiclassical Analysis: 25rit034

David Borthwick (Emory University)
Suresh Eswarathasan (Dalhousie University)
Peter D. Hislop (University of Kentucky)

May 26 2025 - May 30 2025

1 Overview of the Field

One of the major advances in condensed matter physics in the 20th century is the integer quantum Hall effect. This phenomena describes the transport properties of non-interacting electrons constrained to the plane in the presence of a constant, transverse magnetic field and a random potential. The fundamental mathematical model for a single electron is the random Landau Hamiltonian. The mathematical setup for it is as follows.

In $\mathbb{R}^2$, given a constant magnetic field parameter $B > 1$ define the vector potential

$$A(x, y) = \frac{B}{2}(y, -x).$$

The Landau Hamiltonian

$$H_0 := (-i\nabla - A)^2,$$

has point spectrum $(2\mathbb{N} - 1)B$, with each eigenvalue having infinite multiplicity.

We will consider the perturbed operator associated to a real scalar potential $V \in L^\infty(\mathbb{R}^2)$ with $|V| \leq 1$,

$$H_V := (-i\nabla - A)^2 + V. \tag{1}$$

The bounds on B and V imply that the perturbed spectrum is contained in disjoint bands associated to each Landau eigenvalue,

$$\sigma(H_V) \subset (2\mathbb{N} - 1)B + [-1, 1].$$

Random Landau Hamiltonians are families of ergodic, random Schrödinger operators $H_\omega := H_{V_\omega}$, where V_ω is a random potential of Anderson-type. These potentials are constructed from a real function $v \in C_0^K(\mathbb{R}^2)$, for large K, called a *single site potential*, and a family of independent, identically distributed random variables $\{\omega_j\}_{j\in\mathbb{Z}^2}$, by $V_\omega := \sum_{j\in\mathbb{Z}^2} \omega_j v(x - j)$. We assume that the common probability measure for ω_j has the form $\rho(\omega_0)\, d\omega_0$, with supp $\rho \subset [-1, 1]$. With these choices, the common spectrum Σ is contained in disjoints bands as above with probability one. we refer to $[(2n - 1)B - 1, (2n - 1)B + 1]$, $n \in \mathbb{N}$ as the n^{th}-Landau band.

These types of operators are examples of ergodic, random Schrödinger operator in dimension greater than 1 that exhibits both dynamical localization ("no transport") and dynamical delocalization ("nontrivial transport"). Band-edge localization, the existence of intervals of dense pure point spectrum with exponentially decaying eigenfunctions almost surely, was been proved by Combes-Hislop [3], Wang [14], and Germinet-Klopp [7]. Dynamical delocalization, the existence of nontrivial transport associated with an energy in each Landau band, was proved by Germinet, Klein, and Schenker [6].

2 Goals of Research in Teams 25rit034

Despite these advances, several technical difficulties prevent a deeper study of the localization-delocalization transition for random Landau Hamiltonians. In particular, the study of local eigenvalue statistics (LES) associated with these models is of interest in understanding the nature of the transport transition. LES are studied through the large L limit of rescaled eigenvalues in the neighborhood of a fixed energy $E_0 \in \Sigma$. For random Landau Schrödinger operators H_V, we first consider the restriction of V to cubes $\Lambda_L \subset \mathbb{R}^2$, denoted by V_L. The perturbation of H_0 by V_L is denoted by $H_V^L := H_0 + V_L$. The resulting self-adjoint Schrödinger operator H_V^L has discrete spectrum away from the Landau levels. The eigenvalues of H_V^L are random variables denoted by $\{E_j(\Lambda_L)\}_{j=1}^{\infty}$. The Wegner estimate (see Theorem 2 below) indicates that the average eigenvalue spacing is $|\Lambda_L|^{-1}$. For a fixed energy $E_0 \in \Sigma$, we define the random point measure $d\xi_L$ by

$$d\xi_L(s) = \sum_{j=0}^{\infty} \delta(|\Lambda_L|(E_j(\Lambda) - E_0) - s)\, ds. \tag{2}$$

Following the works of Minami [12], Dietlein-Elgart [4] and Germinet-Klopp [7], one expects that the LES is a Poisson point process in the localization regime. Due to the non-trivial transport, one expects a transition in the LES in the region of nontrivial transport.

1. **Semiclassical microlocal analysis:** The first goal is to simplify the results of Wang on the Wegner estimate for random Landau Hamiltonians in the Landau band edges. The main advance of [14] was the treatment of non-sign definite, single site potentials v excluded from the results of [3]. As in Wang's analysis, we consider the semiclassical regime of large small $h := B^{-1}$.

2. **Local eigenvalue statistics.** The second goal is the study of the local eigenvalue statistics (LES). The open conjecture about the limit $\lim_L \xi_L$ is that it exists and is a Poisson point process if E_0 is in the region of localization, and a point process associated with the Gaussian orthogonal ensemble if E_0 is in the delocalization regime. The second RIT goal is to prove this conjecture in the semiclassical regime of large small $h := B^{-1}$ for the random Landau Schrödinger operator.

The main tool of our investigations, as in [13, 14], is the use of microlocal methods to obtain detailed spectral information for these models in the semiclassical regime of large magnetic field B. It is important to note that these works are underpinned by the fundamental papers of Helffer-Sjöstrand [8] and Bellissard et al [2] in which the Grushin problem formalism is used to derive a key quantity, baptized as the "effective Hamiltonian" $Q_V(\mu)$. We will refer to a key theorem of theirs as the Bellissard-Helffer-Sjöstrand Theorem, which states the following:

Theorem 1 (Bellissard, Helffer-Sjöstrand)**.** *For B sufficiently large there a family of zeroth order pseudodifferential operators $Q_V(\mu)$ acting on $L^2(\mathbb{R})$, depending analytically on $\mu \in [-1, 1]$, such that*

$$(2n+1)B + \mu \in \sigma(H_V) \iff 0 \in \sigma(Q_V(\mu)).$$

For V as in (1), the family $Q_V(\mu) = \hat{V}(x; B^{-1}D_x) - \mu + R(B^{-1}, \mu)$ is defined for $B \geq B_0$ where B_0 depends only on $\sup_{\mathbb{R}^2} |D^\alpha V|$ for $0 \leq |\alpha| \leq m$, with m a dimensional constant.

Following the broad approach of [13], the PIs have performed a further analysis of $Q_V(\mu)$ and shown that one can extract crucial eigenvalue estimates necessary in order to study the nature of the LES.

3 Scientific Progress Made

We follow Germinet-Klopp [7] who gave the definitive results on LES in the localization regime for general Schrödoinger operators (assuming localzation bounds, and the Wegner and Minami estimates), and the original work of Minami [12] on lattice models. Working towards LES, one needs effective upper bounds on the probability that the local Schrödinger operator H_V^L has at least one eigenvalue (Wegner), respectively, two eigenvalues (Minami) in a given energy interval. These estimates are crucial for establishing the limit of

ξ_L and characterizing it as a Poisson point process. As mentioned, the Wegner estimate is also essential for multiscale analysis resulting on exponential bounds on the resolvent with probability one, and for studying the continuity properties of the density of states.

Our approach focuses on the effective Hamiltonian $Q_V(\mu)$. As above, we localize to a square Λ_L. Assuming that the single-site potential $v = v_0$ has support inside a unit square, we study the zeroth-order pseudodifferential operator $\widehat{V}_L$ corresponding to the restriction of the random potential $V_L(x,y) := \sum_{j\in\Lambda_L\cap\mathbb{Z}^2}\omega_j v(x-j_1,y-j_2)$. We define $\widehat{V}_L$ to be the Weyl quantization of V_L. Letting $v_j(x,y) := v(x-j_1,y-j_2)$, we obtain

$$\widehat{V}_L = \sum_{j\in\Lambda_L\cap\mathbb{Z}^2}\omega_j\widehat{v}_j.$$

The spectral analysis of $\widehat{V}_L$ leads, via Theorem 1, to estimates on the Landau Hamiltonian with the restricted potential $H_L := H_0 + V_L$. Hence we obtain Wegner and Minami estimates for H_L near the Landau band-edges from a study of $\widehat{V}_L$. The nonnegative, compact, zero-order pseudodifferential operator $\widehat{V}_L$ is a sum of similar operators. We proved that the spectrum of $\widehat{v_j}$ is independent of j. If $\{e_k\}$ is the set of eigenvalues of $\widehat{v_0}$, then the eigenvalues of $\widehat{v_j}$ are $\{\omega_j e_k\}$. Since the random variables are independent and identically distributed, the eigenvalues of operators $\widehat{v_j}$ are independent.

A tool which we use repeatedly, particularly in the proof of Theorem 3, is localization of the operators $\widehat{v}_j$ in phase-space: that is, microlocality. Although $v_jv_k = 0$ if $j\neq k$, the operators $\widehat{v_j}$ are not local. However, microlocality states that $\widehat{v}_j\circ\widehat{v}_{j'} = \mathcal{O}_{L^2}(h^\infty)$ for $j\neq j'$. It should be emphasized that these microlocal statements turn out to be sufficient substitutes for those coming from the strong locality properties of purely differential operators.

3.1 Wegner Estimate

There is a well-established approach to proving band-edge localization via multiscale analysis. One must obtain a sufficient estimation on the probability that H_V has at least one eigenvalue in a specific interval I. For sign-indefinite site potentials v, Wang gave such an estimation, however the estimate itself was insufficient in establishing the Lipschitz nature of the integrated-density-of-states measures. Considering our desire to obtain both Lipschitz continuity of the IDS measure and estimate the more restrictive event of H_V having at least two eigenvalues in I, it was important to fill in certain gaps in Wang's proof of the Wegner estimate. We state our final estimate below, proven during our residence at BIRS, albeit with a slightly different statement than in [14, Proposition 3.1]. We emphasize that our proof still hinges upon Theorem 1:

Theorem 2 (Wegner Estimate)**.** *Fix a Landau level n and assume $\mu_0\in[b_0,1]$ for $b_0>0$. Then, there exists h_0, depending only on v_0, and b_0, such that for $h\le h_0$ and $0<\delta<1$,*

$$\mathbb{P}\left[(2n+1)B+\mu\in\sigma(H_{V_\omega})\text{ for some }\mu\in[\mu_0e^{-\delta},\mu_0e^{\delta}]\right]\le CB|\Lambda|^2\delta,$$

where C depends only on v_0, g, and b_0. Furthermore, with h_0 possibly smaller, we have

$$\mathbb{P}\left[\#\left\{\sigma(H_{V_\omega})\cap(2n+1)B+[e^{-\delta},1]\right\}\ge 1\right]\le C|\Lambda|\delta.$$

This theorem is proven by a careful analysis of $\widehat{V}$ and a new bound on the number of eigenvalues of $\widehat{V}_L$ near in a fixed interval away from zero. This bound follows by a Hilbert-Schmidt on the integral kernel of $\widehat{V}$.

Although this estimate is sufficient to control the resolvent of H_L, it is not strong enough to establish the Lipschitz continuity of the integrated density of states due to the factor of $|\Lambda|^2$. A stronger estimate with $|\Lambda|$ was obtained for sign-definite potentials in [3]. We were unsuccessful during our stay at improving the factor of $|\Lambda|^2$ to $|\Lambda|$ in the first probability estimate. Such a linear factor would give us the first proof of the Lipschitz continuity of the IDS measure at the band edges [13] for sign-indefinite potentials. However, as one can see, the second estimate at the edge's limit does give the Lipschitz continuity for energies in that region. Note that our proof of a crucial eigenvalue counting estimate was simpler than that in [14, Lemma 3.1], relying solely on a simple calculation with Hilbert-Schmidt norms and the Weyl quantization formula.

Alternatively, again invoking Theorem 1 but this time using the more common route of estimating traces, we have a somewhat different Wegner estimate:

Theorem 3. *Under the same hypotheses and given parameter ranges as in Theorem 2, but with h_0 possibly smaller, we have*

$$\mathbb{P}\left[(2n+1)B+\mu \in \sigma(H_{V_\omega}) \textit{ for some } \mu \in [\mu_0 e^{-\delta}, \mu_0 e^{\delta}]\right] \leq C|\Lambda| \left(\delta + |\Lambda|^{\frac{1}{2}} B^{-\frac{1}{2}}\right).$$

While we do see an improvement in the overall power of $|\Lambda|$, the coupling of δ and $|\Lambda|^{\frac{1}{2}} B^{-\frac{1}{2}}$ forces other difficulties to occur en route to Lipschitz continuity. However, the robustness of the overall method towards band-edge localization makes the above estimate effective.

3.2 Minami Estimate

Obtaining a sufficient estimate for the probability of the event of at least two eigenvalues being in an interval is more difficult, particularly for random Schrödinger operators on $L^2(\mathbb{R}^d)$. The only result to date is that of Dietlein-Elgart [4] who prove a weak Minami estimate at the bottom of the deterministic spectrum. It is not clear that this proof can be extended to random Landau Hamiltonians. However, we have made progress in the semiclassical regime of large B by exploiting the simple structure of the effective Hamiltonian.

We recall that the principal part of the effective Hamiltonian $Q_V(\mu)$ is $\widehat{V}_\Lambda$. This operator is the sum of pseudodifferential operators $\sum_{j\in\Lambda\cap\mathbb{Z}^2} \widehat{v}_j$, where $v_j(x,\xi) = v(x-j_1, \xi - j_2)$, and v is the single site potential. Hence, we can study $\widehat{V}_\Lambda$ by analyzing the independent, identically distributed random compact operators $\omega_j \widehat{v}_j$.

We start with a simple lemma on the spectrum of a single site's quantization:

Lemma 1 ("mini Minami" estimate)**.** *Assume that $\sigma(\hat{v}_0)$ is simple and has gaps of size at least h between its consecutive eigenvalues. Then for all $h \leq h_0$,*

$$\mathbb{P}\left[\#\left\{\sigma(\omega\hat{v}) \cap [\mu_0 - \delta, \mu_0 + \delta]\right\} \geq 2\right] = 0$$

This lemma, which turns out to be a key reduction, leads to our desired:

Theorem 4 (Minami estimate)**.** *Under the same hypotheses and given parameter ranges as in Theorem 2, but with h_0 possibly smaller, we have*

$$\mathbb{P}\left[\left\{\mu \in [\mu_0 e^{-\delta}, \mu_0 e^{\delta}] \textit{ such that } (2n+1)B+\mu \in \sigma(H_{V_L})\right\} \geq 2\right] \leq C|\Lambda_L|^2 \left(\delta + |\Lambda_L|^{\frac{1}{2}} B^{-\frac{1}{2}}\right)^2.$$

The proof of this theorem, like that of the Wegner estimate, begins with an analysis of one operator $\widehat{v}_j$. For the Minami estimate, Lemma 1 that states that a single operator $\widehat{v}_j$ cannot produce two eigenvalues in the given energy interval. Consequently, at least two sites must contribute an eigenvalue, and because of independence, this occurs with probability given by the square of the Wegner estimate for any one operator. Since there are $|\Lambda_L|$ sites, the number of pairs is $\mathcal{O}(\Lambda|^2)$.

To our knowledge, this is the first Minami-type resut in the context of random Landau Hamiltonians and a major step towards the first LES results in this setting. We are currently en route towards completing the arc started in the works of Combes-Hislop [3] and Wang [14].

3.3 Numerics and examples

Along the way to establishing the previously described results, we naturally stumbled upon a few smaller yet noteworthy findings. First, in order to understand better the spectra of a single site v's quantization, namely $\hat{v}(x, B^{-1}D_x)$, we ran some experiments in MatLab using a simple example. However, the results themselves were not as illuminating as expected.

This eventually led us to being flexible with the support properties of v, and in turn to see that for $v = v(y^2+\eta^2)$, which are functions solely of the principal symbol of the 1D harmonic oscillator, the eigenvalues of the corresponding quantization $\hat{v}$ can be explicitly computed and they are all simple. Such a finding was done through some manipulations of the Mehler kernel, itself being the Schwartz kernel of the unitary propagator for the harmonic oscillator, see the work of Hörmander [10] and Lerner [11]. This property feeds into the hypothesis of our preliminary estimate Lemma 1, which itself leads to our desired Minami estimate but in the case of a specific sign-definite potential. Subsequently work along these lines has allowed us to verify the hypotheses of a spectral gap and simple spectrum for $\widehat{v}$ constructed with non-sign-definite single site potentials v.

4 Outcome of the Meeting

We are currently writing a paper on these results and working on establishing a theorem on the nature of the LES in the large B regime.

References

[1] M. Aizenman, S. Warzel, *Random operators*, Grad. Stud. Math. **168**, American Mathematical Society, Providence, RI, 2015.

[2] J. Bellissard, A. van Elst, and H. Schulz-Baldes, *The noncommutative geometry of the quantum Hall effect,* J. Math. Phys . **35** (1994), 53735451.

[3] J. M. Combes, P. D. Hislop, *Landau Hamiltonians with random potentials: localization and the density of states*, Comm. Math. Phys. **177** (1996), no. 3, 603629.

[4] A. Dietlein, A. Elgart, *Level spacing and Poisson statistics for continuum random Schrdinger operators.*, J. Eur. Math. Soc. (JEMS) **23** (2021), no. 4, 12571293.

[5] J.M. Combes, F. Germinet, A. Klein *Poisson statistics for eigenvalues of continuum random Schrödinger operators*, Anal. PDE 3, 4980 (2010); Erratum: Anal. PDE 7, 1235 1236 (2014)

[6] F. Germinet, A. Klein, J. H. Schenker, *Dynamical delocalization in random Landau Hamiltonians*, Ann. of Math. (2) **166** (2007), no. 1, 215244.

[7] F. Germinent, F. Klopp, *Spectral statistics for random Schrödinger operators in the localized regime*, J. Eur. Math. Soc. (**16**), 19672031 (2014).

[8] B. Helffer and J. Sjöstrand, J., *Équation de Schrödinger avec champ magnétique et équation de Harper*, *Lecture Notes in Physics* **345** (1989), 118–197.

[9] P. Hislop and M. Krishna, *Eigenvalue Statistics for Random Schrӓinger Operators with Non Rank One Perturbations*, Comm. Math. Phys., Volume **340**, pages 125143, (2015).

[10] L. Hörmander, *Symplectic classification of quadratic forms, and general Mehler formulas*, Math Zeit. (219), 413-449 (1995).

[11] N. Lerner, *Mehler's formula and functional calculus*, Science China: Mathematics (**62**), 1143-1166 (2019).

[12] N. Minami, *Local fluctuation of the spectrum of a multidimensional Anderson tight binding model*, Comm. Math. Phys. **177**, 709725 (1996).

[13] Wei-Min Wang, *Asymptotic Expansion for the Density of States of the Magnetic Schrodinger Operator with a Random Potential*, Commun. Math. Phys. **172**, 401-425 (1995).

[14] Wei-Min Wang, *Microlocalization, Percolation, and Anderson Localization for the Magnetic Schrodinger Operator with a Random Potential*, Journal of Functional Analysis **146**, 1-26 (1997).