Movement and Symmetry in Graphs

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1 The Context

In algebraic graph theory, combinatorial matrix theory, infection processes on graphs, and extremal combinatorics, the best modern results are often found using an interdisciplinary approach, leveraging tools and techniques from these other fields. The tools developed in solving these types of problems are often strong and transferable. Algebraic techniques, a deeper understanding of graph symmetries, probabilistic techniques and structural extremal results show great promise to develop a deep and general theory that encompasses many graph and hypergraph classes all at once.

This workshop highlighted recent results in these areas that connect to research topics and projects for the PIMS-funded Collaborative Research Group (CRG) on "Movement and symmetry in graphs". Drawing together researchers from around the world, the event served as a showcase for the work of this CRG. Although its funding period is over, the CRG continues to develop a network of excellence around these topics centred in the prairie provinces and with women and other underrepresented minority groups at its core, as leaders and mentors. The workshop served to create connections through which students and post-docs who are working in this field have begun to build collaborations and develop their career goals. It was a culminating event of the Collaborative Research Group, whose funding ended in the spring of 2024.

The work of the Collaborative Research Group, and therefore this workshop, was focused around four topics relating to movement and symmetry in graphs: (1) Algebraic Graph theory, (2) Combinatorial Matrix Theory, (3) Graph and Hypergraph Infection and Percolation, and (4) Extremal Combinatorics. These are elaborated upon below.

1.1 Algebraic Graph Theory

Algebraic graph theory is a growing field as researchers come to appreciate the powerful techniques that it provides. The workshop focused on two approaches in algebraic graph theory: representing the graph as a matrix and using matrix properties to understand the graph; and using the symmetries of the graphs to gain an understanding of it. These two approaches bring together many areas of mathematics. One of the challenges is that it requires a strong background in algebra, group theory and matrix theory, but the advantage is that the tools developed are strong and transferable.

In the first approach, the general problem is to study the relationships between algebraic and combinatorial parameters for graphs through a matrix representation. These results can be interpreted as results on designs and often have applications in extremal combinatorics. The eigenvalues of Cayley graphs can be determined using group theory and representation theory. Using group theory to understand graphs is the focus of this area.

The second approach involves understanding the symmetries of a graph, which can give insight into the graph's properties. Often the symmetry group of a graph can give a bird's eye view of the graph, so structures can be understood in a more general way. Symmetries allow us to apply some of the powerful tools of permutation group theory to the study of graphs and their properties.

1.2 Combinatorial Matrix Theory

One of the most famous problems in combinatorial matrix theory is the Inverse Eigenvalue Problem. The objective of this problem is to describe all possible eigenvalues of a given set of symmetric matrices with a fixed zero-nonzero pattern. This problem is notoriously difficult and a general result seems to be far out of reach, but there are simpler problems that could be considered. For example, a group based in Regina worked on a project to determine which multiplicities of eigenvalues can be obtained. One question that this work asked is for which graphs is there a matrix that has only two eigenvalues and these eigenvalues have the same multiplicity. The Regina group showed that this holds for the hypercube; amazingly this fact was later used by Huang to prove the famous sensitivity conjecture.

Problems in combinatorial matrix theory like this one often require tools and knowledge from different areas, such as graph theory, linear algebra, group theory and probability. This is where an inter-disciplinary approach is essential.

1.3 Graph and Hypergraph Infection and Percolation

There are many different processes that model the spread of an infection or the flow of information through a network, for example bootstrap percolation, zero forcing and various notions of hypergraph infection.

Graph infection and percolation (and similar problems) can be studied on Cayley graphs and on vertex-transitive graphs, as well as on hypergraphs that have strong symmetry properties. Infection and percolation provide excellent entry-level research problems (for undergraduate and Master's students in particular) that continue to inspire and attract researchers from underrepresented groups. There are also an abundance of deeper questions about the behaviour of various infection processes on infinite Cayley graphs in relation to properties of the underlying graph. In the context of random graphs, there is often a strong connection between critical probabilities for full infection under infection processes in a random regular graph and in a related infinite tree.

Hypergraphs can be frustratingly general objects to work with, and it is often difficult to extend results on graphs to the hypergraph setting. The workshop also presented recent work and open problems in this area.

In the area of bootstrap percolation, there has been some success achieving extremal results using linear algebraic and polynomial-method techniques. Applying these methods to a larger variety of graphs with symmetry, generalizing the approaches, and studying closely-related problems for weak-saturation of graphs are avenues for research that were included in presentations, demonstrating work at the intersection of algebraic combinatorics, infection processes and extremal combinatorics.

1.4 Extremal Combinatorics

Extremal properties of infection processes and structural properties of graphs can imply small percolating sets. Many results in this area have consisted of conditions on the minimum degree of a graph and the resulting extremal structures are often highly asymmetric. New results in this area may make use of further structural properties and develop an understanding of how global symmetry properties can influence the range of values for measures of these infection processes.

In the realm of extremal combinatorics, there is often a huge gap in the understanding of what can happen in graphs compared to what can happen in hypergraphs. One example of this are Turán numbers, the extremal numbers for edges in a (hyper)graph forbidding a fixed (hyper)graph. For graphs, Turán numbers are well-understood in terms of the chromatic number via the Erdős-Stone-Simonovitz theorem. For hypergraphs, much less is known and there are many small 3-uniform hypergraphs for which the asymptotic behaviour of the Turán number is not known precisely.

1.5 Summary of context

As a culminating event of our PIMS-funded Collaborative Research Group (CRG) on "Movement and symmetry in graphs", the objective of this workshop was to highlight and explore new research (both within the CRG and from other researchers) on problems central to the work of the CRG.

The workshop was focused around early-career researchers (including students and post-doctoral fellows). It gave them an opportunity to showcase their work in front of world-renowned leaders in the field, also allowed them to hear about new research by those same experts, and also gave them the opportunity to collaborate in small groups with senior researchers on problems central to the field. This established connections through which collaborations are underway that will assist these researchers to develop their career goals.

2 Structure of the Workshop

2.1 Outline

In advance of the workshop, we reached out to participants to ask if they would be willing to present open problems and/or lead working groups on open problems related to the main topics of the workshop.

Monday was reserved for introductory talks that provided background for the open problems that had been proposed for the working groups. These talks encouraged participants to move out of their comfort zones, and to consider working on problems that were not necessarily within their main area of expertise.

On both Monday and Tuesday evenings we held a series of 5- to 10-minute "lightning talks". We made every effort (we think successfully) to make these sessions light-hearted, low-pressure, and supportive. They were broadcast over zoom, but not recorded. These served a couple of purposes:

- on Monday evening, some additional open problems were presented for the consideration of participants; and
- for early-career participants in particular, this served as a bit of an ice-breaker, providing a less-formal context in which to present key ideas of their research to the group.

On Monday evening, we circulated a google form on which we listed all of the open problems that had been proposed for working groups, and asked participants to rank the top 3 working groups that they would be interested in joining during the week, in order of preference. On Tuesday morning, we determined the 6 proposals that had received the most interest from participants, and announced that these would be the basis for the working groups that would operate during the remainder of the week, beginnning that afternoon.

One of our online participants, Nathan Lindzay, coordinated a working group for the online participants that used gathertown and other resources to work on a problem he had chosen. This was made more challenging by the wide variety of time zones involved, and also by American Thanksgiving and other meetings and commitments that created time conflicts for participants.

After 7 additional 20-minute talks by ECRs (almost all postdoctoral fellows) showcasing their research during the rest of Tuesday morning and early afternoon, the working groups held their first meetings. Wednesday morning included more time for working groups as well as 3 more 20-minute talks by ECRs; the afternoon was free for excursions in and around Banff, and many took advantage of the beautiful snow-covered mountain hiking opportunities and warm temperatures, although low-hanging clouds impeded some views.

On Thursday morning working group leaders gave brief reports on their progress, and an opportunity was provided for participants to switch working groups if they wished to do so. There were 6 more 20-minute talks, and more time for working groups to discuss their problems.

For those who did not leave early, Friday morning held two more 20-minute talks, and more progress reports and working time for the working groups.

2.2 Participant demographics

After some late cancellations, there were 38 in-person participants at the workshop. An additional 21 people signed up as virtual participants, of whom 5 gave talks.

More than half of the in-person participants were female (20 of 38, 52.6%). One of the organisers had a 15-month-old baby, who accompanied her along with a caregiver. We will not attempt to assess most other EDI-categories as we are not necessarily aware of how participants self-identify. However, we can address current locations and career stages.

Given that many of our participants were ECRs, including students and postdoctoral fellows, current locations are somewhat transitory. However, the latest information we have is that 26 of our 38 in-person participants are currently based in Canada. Given our object of showcasing research and students and post-doctoral fellows from the prairie provinces, this over-representation was expected and appropriate. Another 5 are currently in the USA; 3 in Europe; 1 in Australia; 1 in New Zealand; 1 in India; and 1 in Brazil.

In-person participants consisted of:

- MSc students: 1; PhD students: 4
- Post-doctoral fellows: 12
- Assistant Professors: 4; Associate Professors: 5
- Professors: 12

2.3 Challenges and Opportunities of the Timing

The workshop took place November 24–29, 2024.

Challenge: American Thanksgiving. This was during the week of American Thanksgiving, which meant that a number of potential participants had other priorities that prevented their participation. It also hindered virtual participation during parts of the week, from Americans who chose to join us virtually.

Opportunity and Challenge: CMS Winter Meeting. The Canadian Math Society (CMS)'s annual winter meeting was originally scheduled to be held in Vancouver on December 6–9, 2024. Late in 2023 (after our workshop had been scheduled), Taylor Swift announced dates for her concert tour that included a stop in Vancouver on that same weekend. To avoid excessive difficulty in securing hotel rooms, the CMS winter meeting dates were moved to November 29–December 2.

Many of our participants wanted to attend the CMS meeting, and in fact many were invited to speak in a variety of scientific sessions at the meeting that were relevant to the topics of our workshop. These included two sessions on *Algebraic Graph Theory*, one on *Cayley Graphs*, and one on *The Theory of Pursuit-Evasion Games on Graphs*.

The adjacent dates and reasonable proximity of this BIRS workshop and the CMS meeting meant that it was convenient to combine the two events into a single trip. Particularly for researchers coming from abroad or from eastern North America, this opportunity made the long trip more worthwhile. However, many participants wanted to attend the public lecture in Vancouver on the evening of Friday, November 29, or simply to have time to settle in in Vancouver before scientific sessions began early on the morning of Saturday, November 30. The time required to reach Calgary, get through airport security, and travel to Vancouver resulted in a significant number of participants leaving Banff early Friday morning and missing the Friday events.

Challenge and Opportunity: Virtual Work. Although this was impacted in some ways by the timing of our workshop, it is more an effect of the changes Covid brought to the world. We are accustomed to being able to carry out work virtually from almost anywhere.

For a large number of participants, the week of the workshop was very near the end of the semester. This made some teaching-related commitments harder to work around, and some participants were running some or all of their classes remotely during the week. Others had meetings or other commitments that 20 years ago they would have declined or rescheduled due to being out of town, but now felt obligated to participate in virtually. This meant that the in-person participants were not always as fully "present" in the workshop as might have been the case in the past.

The flip side of this, of course, was the relative ease with which virtual participants were able to contribute to the workshop. Successfully including virtual participants from a wide variety of time zones, many of whom did not come at least in part due to other things they needed to be doing during the week, was a challenge that we did not really solve. Largely their participation consisted of providing some talks and virtually attending talks that fit into their schedules.

Challenge: Visas. This challenge did not relate to the time of year, but the speed at which the Canadian government currently processes visas was a problem for several of our participants that we think has changed over time. We had two late cancellations by post-doctoral fellows due to visa applications that were not approved within the available time. At least one of these had applied for their visa more than 5 months in advance of the workshop, but despite multiple follow-up attempts received no useful updates and was ultimately forced to cancel their plans.

3 Open Problems Proposed

We acknowledge with gratitude that much of the writing in this section was contributed by those who proposed the topics.

3.1 Simple Oriented Graphs

This problem was proposed by Chris Duffy. It was not chosen for a working group but may nonetheless be the subject of joint work by workshop participants or others in the future.

Let G be an oriented graph. A set $S \subseteq V$ is $\overrightarrow{P}_3$-convex when for every 2-dipath u, v, w: if $u, v \in S$, then $v \in S$. The convex hull of a set of vertices can be found via a bootstrap percolation process, which proceeds according to the following rule: a vertex is *on* in S_k when it is on in S_{k-1} or at least one out-neighbour and one in-neighbour are *on* in S_{k-1}. An oriented graph is called simple when the convex hull of any arc is the entire vertex set. In general, it is NP-complete to decide if a graph can be given an orientation to be simple. There is a known characterisation for when a 2-tree admits a simple orientation.

Open Problem 1. Find some classes of graphs for which we can nicely characterise which graphs in the family admit an orientation as a simple oriented graph.

Open Problem 2. Which Cayley digraphs are simple?

Open Problem 3. Find necessary and sufficient conditions for a planar graph to admit a simple orientation? Outerplanar?

Open Problem 4. Find an tourament T on $n < 80$ vertices such that if P is a simple oriented planar graph, then $P \to T$.

3.2 Transitive groups with large intersection density

This topic was proposed by Sarobidy Razafimahatratra, and was selected for a working group. The members of the group were Sarobidy Razafimahatratra, Raghu Pantangi, Roghayeh Maleki, Shonda Dueck, Ted Dobson, Francis Clavette, Xiaohong Zhang, Kyle Yip

Given a finite transitive group $G \leq \mathrm{Sym}(\Omega)$, a set $\mathcal{F} \subset G$ is *intersecting* if, for any $g, h \in G$, there exists $\omega \in \Omega$ such that $\omega^g = \omega^h$. The *intersection density* $\rho(G)$ is the maximum ratio of $\frac{|\mathcal{F}|}{|G_\omega|}$, where $\mathcal{F}$ runs through all intersecting sets of G and G_ω is the stabilizer of $\omega \in \Omega$ in G.

The problem of finding the intersection density of a finite transitive group $G \leq \mathrm{Sym}(\Omega)$ is equivalent to finding the size of the largest cocliques in the Cayley graph $\Gamma_G := \mathrm{Cay}(G, D_G)$, where D_G is the set of all derangements of G. The graph Γ_G is the so-called *derangement graph* of G.

In [6], it was proved that if $G \leq \mathrm{Sym}(\Omega)$ is a transitive group with $|\Omega| \geq 3$, then $\rho(G) \leq \frac{|\Omega|}{3}$. This upper bound is sharp since it is attained by the groups:

1. `TransitiveGroup(6,4)`,

2. `TransitiveGroup(18,142)`,

3. `TransitiveGroup(30,126)`, and

4. `TransitiveGroup(30,233)`.

In particular, the group `TransitiveGroup(6,4)` is permutation equivalent to $\mathrm{Alt}(4)$ acting on the 2-subsets of $\{1, 2, 3, 4\}$, which is the smallest transitive group with intersection density larger than 1. The derangement graphs of the transitive groups in (1)-(4) are all complete tripartite. Moreover, if G is one of the groups in (1)-(4), then there exists a complete block system or a system of imprimitivity $\mathcal{B}$ of G such that the induced action $\overline{G}$ of G on $\mathcal{B}$ is permutation equivalent to `TransitiveGroup(6,4)`.

The aim of this project is to answer the following questions and problems.

Open Problem 5 $(**)$**.** Find more examples of transitive groups whose derangement graphs are complete tripartite.

Open Problem 6. If $\rho(G) = \frac{|\Omega|}{3}$, then is Γ_G always complete tripartite?

Open Problem 7. If $G \leq \mathrm{Sym}(\Omega)$ is transitive such that Γ_G is complete tripartite, does G always "factor through" `TransitiveGroup(6,4)`?

In [5], it was shown that if $G \leq \mathrm{Sym}(\Omega)$ is innately transitive (i.e., G admits a transitive minimal normal subgroup) with $|\Omega| \geq 2$ and $\frac{|\Omega|}{4} < \rho(G) \leq \frac{|\Omega|}{3}$, then $|\Omega| = 3$. It is interesting to ask how this result generalizes to transitive groups, in general.

Open Problem 8. Is there a transitive group $G \leq \mathrm{Sym}(\Omega)$ such that $\frac{|\Omega|}{4} < \rho(G) < \frac{|\Omega|}{3}$?

3.3 Inverse eigenvalue problems

This topic was proposed by Shaun Fallat and Shahla Nasserasr, and was selected for a working group. Group Members were Shaun Fallat, Shahla Nasserasr, Ada Chan, Mahsa Shirazi, Johnna Parenteau, Hermie Monterde.

Many problems were identified by the leaders of this group. However, after their initial discussion, it was decided that the group would begin by considering the spectra of generalized Laplacians.

Let $S_L(G)$ be the set of $n \times n$ real symmetric matrices $A = [a_{i,j}]$ such that

$$a_{i,j} \begin{cases} < 0 & \text{if } \{i,j\} \in E(G), \\ = 0 & \text{if } \{i,j\} \notin E(G),\ i \neq j, \\ -\sum_{k:k\sim i} a_{i,k} & \text{if } i = j. \end{cases}$$

Analogous to the inverse eigenvalue problem for graphs (IEP-G), we are interested in an inverse eigenvalue problem among matrices in the class $S_L(G)$, and abbreviate this problem as *IEPL* (inverse eigenvalue problem for generalized Laplacian matrices associated with a graph G). To this end, we say a collection of real numbers $0 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$, is *Laplacian realizable* if there exists $L \in S_L(G)$ with $\sigma(L) = \{0, \lambda_2, \lambda_3, \ldots, \lambda_n\}$.

For a given connected graph G on n vertices let $L \in S_L(G)$ with $\sigma(L) = \{0^{(1)}, \lambda_2^{(m_2)}, \lambda_3^{(m_3)}, \ldots, \lambda_q^{(m_q)}\}$, where $\lambda_i^{(m_i)}$ means the eigenvalue λ_i of L has multiplicity m_i (in this case we have $1 + \sum m_i = n$). If further we assume that $0 < \lambda_2 < \lambda_3 < \cdots < \lambda_q$, then the *ordered multiplicity list for* L is defined to be $(m_1 := 1, m_2, \ldots, m_q)$.

Open Problem 9. Consider all possible ordered multiplicity lists over matrices in the set $S_L(G)$ and determine the smallest q possible for a realized Laplacian spectrum.

3.4 Second Common Neighbourhood Conjecture

This topic was proposed by JD Nir, and was selected for a working group. Working group members were JD Nir, Gabriel Verret, Andrii Arman, Candida Bowtell, Emily Heath, Jeanette Janssen, Karen Gunderson.

Definition 10. A set of vertices $S \subseteq V(G)$ is called *dominating* in G if for each $v \in V(G)$, $v \in S$ or $u \in S$ for some edge $(u, v) \in E(G)$.

Let $\partial(G)$ be the number of dominating sets in G.

Theorem 11. *If $S \subseteq V(G)$ is **not** a dominating set in G, then $\overline{S}$ **is** a dominating set in $\overline{G}$.*

Theorem 12 (Wagner, 2013)**.** *If G is a graph on n vertices then*

$$\partial(G) + \partial(\overline{G}) \geq 2^n.$$

This bound is tight for every n, and we can classify the extremal examples.

Theorem 13 (Keough-Shane, 2018)**.** *If G is a graph on n vertices then*

$$\partial(G) + \partial(\overline{G}) \leq 2^{n+1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lceil \frac{n}{2} \rceil - 1}.$$

Conjecture 14 (Keough-Shane, 2018)**.** *The extremal graph is the balanced complete bipartite graph, $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$, or its complement, and*

$$\partial(G) + \partial(\overline{G}) \leq 2(2^{\lfloor \frac{n}{2} \rfloor} - 1)(2^{\lceil \frac{n}{2} \rceil} - 1) + 2.$$

Definition 15. Given a (simple) graph G and a set $S \subseteq V(G)$, the *common neighborhood* of S is the set

$$\mathcal{C}(S) = \{v \in V(G) \mid \forall s \in S, v \sim s\}.$$

Definition 16. Given a (simple) graph G and a set $S \subseteq V(G)$, the *second common neighborhood* of S is the set

$$\mathcal{C}^2(S) = \mathcal{C}\big(\mathcal{C}(S)\big).$$

Define $\mathcal{S}$ to be the collection of nonempty sets of vertices with nonempty common neighborhoods:

$$\mathcal{S} = \{S \subseteq V(G) \mid S \neq \varnothing \quad \text{and} \quad \mathcal{C}(S) \neq \varnothing\}$$

Conjecture 17 (Second Common Neighborhood Conjecture)**.**

$$\sum_{S \in \mathcal{S}} |\mathcal{C}(S)| \leq \sum_{S \in \mathcal{S}} |\mathcal{C}^2(S)|$$

Theorem 18. *The second common neighborhood conjecture holds for complete bipartite graphs.*

Let $\text{NDom}(G) = \{S \subseteq V(G) \mid S \text{ does not dominate } G\}$.

$$\text{NDom}(\overline{G}) = \{S \subseteq V(G) \mid \mathcal{C}(S) \neq \varnothing\} = \mathcal{S} \cup \{\varnothing\}$$

We use this to show

$$n(\eth(\overline{G}) - 1) = \sum_{S \in \mathcal{S}} n \geq \sum_{S \in \mathcal{S}} |\mathcal{C}(S)| + |\mathcal{C}^2(S)|$$

because $\mathcal{C}(S)$ and $\mathcal{C}^2(S)$ are disjoint. Then if the second common neighborhood conjecture holds,

$$n(\eth(\overline{G}) - 1) \geq \sum_{S \in \mathcal{S}} |\mathcal{C}(S)| + |\mathcal{C}^2(S)| \geq 2 \sum_{S \in \mathcal{S}} |\mathcal{C}(S)|$$

Now

$$\sum_{S \in \mathcal{S}} |\mathcal{C}(S)| = \sum_{v \in V(G)} 2^{d_G(v)} - 1$$

as we can count ordered pairs (v,S) where $v \in \mathcal{C}(S)$ in two ways.

$$\begin{aligned} n(\eth(G)+\eth(\overline{G})-2) &\geq 2\left(\sum_{S\in\mathcal{S}_G}|\mathcal{C}_G(S)|+\sum_{S\in\mathcal{S}_{\overline{G}}}|\mathcal{C}_{\overline{G}}(S)|\right)\\ &=2\left(\sum_{v\in V(G)}2^{d_G(v)}-1+\sum_{v\in V(\overline{G})}2^{d_{\overline{G}}(v)}-1\right)\\ &=2\sum_{v\in V(G)}2^{d_G(v)}+2^{n-1-d_G(v)}-2\\ &\geq 2n\left(2^{\frac{n}{2}}+2^{\frac{n}{2}-1}-2\right)\end{aligned}$$

So

$$n(\eth(G)+\eth(\overline{G})-2)\geq 2n\left(2^{\frac{n}{2}}+2^{\frac{n}{2}-1}-2\right)$$

or

$$\eth(G)+\eth(\overline{G})\geq 2^{\frac{n}{2}+1}+2^{\frac{n}{2}}-2$$

Compare to:

$$\begin{aligned}\text{Theorem:}\quad &\eth(G)+\eth(\overline{G})\geq 2^{\frac{n}{2}}+2^{\frac{n}{2}-1}\\ \text{Conjecture:}\quad &\eth(G)+\eth(\overline{G})\geq 2^{\frac{n}{2}+2}-4\end{aligned}$$

Open Problem 19. Solve the second common neighbourhood conjecture for other interesting families of graphs.

3.5 Eigenvalues and eigenvectors of graphs

This topic was proposed by Krystal Guo, and was selected for a working group. The members of the working group were Krystal Guo, Karen Meagher, Himanshu Gupta, Bobby Miraftab, Jozsef Balogh, Harmony Zhan, and Soffía Árnadöttir.

Graphs with three distinct eigenvalues

If a connected graph has only 1 distinct eigenvalue, then it has to have no edges. If it has two distinct eigenvalues, then it must be a complete graph. If a connected graph has 3 distinct, then apparently many things can happen.

If it is regular, it has to be strongly regular. We can tell if a graph is regular from the spectrum; we can find the average degree and the graph is regular if and only if this number is an eigenvalue. No connected regular graph can be cospectral to a connected irregular graph.

Open Problem 20 (De Caen)**.** Do connected graphs with three distinct eigenvalues have at most three valencies?

There are only finitely many known examples of graphs with three distinct eigenvalues, but they are not known to be finite in number. If there are at most three valencies, the partition by valency must be equitable. It is also open to find a graph with three distinct eigenvalues where the partition by valency is not equitable, or show it is not possible.

We note that the connectedness is an important requirement. All complete bipartite graphs with m edges have the same eigenvalues and, thus, taking disjoint unions can produce many different degrees. If there are 4-distinct eigenvalues, then any number of different degrees is possible.

Let X be a connected graph where $A := A(X)$ has three distinct eigenvalues. Suppose the distinct eigenvalues are $\theta_0 > \theta_1 > \theta_2$.

By expanding the minimal polynomial, we would (eventually) get

$$(A-\theta_1 I)(A-\theta_2 I)=vv^T$$

where v is an eigenvector for θ_0.

$$d(u) + \theta_1\theta_2 = v(u)^2$$

where $d(u)$ denotes the degree of u. Since θ_1, θ_2 do not depend on the choice of vertex, the number of different degrees

Does the graph exist?

Van Dam, Koolen, Xia leave a morsel for us in [2]. They write

> A putative parameter set with four valencies (in fact, the one with the smallest number of vertices according to 15-year old, but unverified, computations) is the following one on 51 vertices and spectrum $\{30^{(1)}, 3^{(20)}, -3^{(30)}\}$. The computations show that a graph with this spectrum must have valencies 13, 18, 34, and 45, occurring 15, 5, 30, and 1 times, respectively. In fact, using the techniques of [9] it can be shown that in this particular case, the valency partition is also equitable, with quotient matrix
> $$\begin{pmatrix} 2 & 0 & 10 & 1 \\ 0 & 0 & 18 & 0 \\ 5 & 3 & 25 & 1 \\ 15 & 0 & 30 & 0 \end{pmatrix}.$$
> Quite a bit of this graph is therefore determined. Besides the trivial parts, one can show that the incidence structure between the five vertices of valency 18 and 30 vertices of valency 34 is a 2-$(5, 3, 9)$ design, and there is only one such design: three times the full design of all triples on five points. We leave it as a problem to the reader to finish the (de-)construction.

Pjotr Buys had the idea to expand also the minimal polynomial itself and get

$$2e(\Gamma(u)) = (\theta_0 + \theta_1 + \theta_2)d(u) + \theta_0\theta_1\theta_2$$

where $e(\Gamma(u)$ is the number of edges in the graph induced by the neighbours of u. Despite our best efforts, we did not find it and would be interested in methods of showing that it does not exist. It's possible that if there are 4 distinct degrees and only 3 distinct eigenvalues, then the partition cannot be equitable.

Graphs with some eigenvalues

With a student, V. Schmeits, Krystal Guo looked at a variant of perfect state transfer in discrete-time quantum walks in [3]. In the course of their research, they needed to find graphs with very special eigenvalues.

Let $\widetilde{A}$ be the normalized adjacency matrix; that is

$$\widetilde{A} = \Delta^{-\frac{1}{2}} A \Delta^{-\frac{1}{2}}.$$

Open Problem 21. Characterise all graphs whose normalized adjacency matrix has its distinct eigenvalues contained in the set $\{\pm k, \pm\frac{k}{2}, 0\}$ for some $k > 0$.

They found all examples in distance-regular graphs that appear in the tables of [1] and we determined all examples among bipartite incidence graphs of symmetric designs.

Spectrally central vertex of a graph

In [7], the authors define the notion of a central vertex in a graph, based on continuous-time quantum walks. For a graph X with spectral decomposition

$$A(X) =: A = \sum_{r=0}^{d} \theta_r E_r$$

the *vertex centrality* of a vertex v to be

$$\sum_{r=0}^{d} (e_v^T E_r \mathbf{1})^2$$

where $\mathbf{1}$ is the all ones vector. Then, the *central vertex* is the one maximizing centrality.

One can show that the centrality measures are the diagonal entries of the matrix

$$M = \sum_{r=0}^{d} E_r J E_r.$$

This is the orthogonal projection of J the all ones matrix into the Bose-Mesner algebra of the graph.

If a graph is regular, then J is in Bose-Mesner algebra so projects onto itself; that is $J = \sum_{r=0}^{d} E_r J E_r$. Thus all vertices have the same centrality.

Isolated vertices have huge centrality. Better to consider connected graphs.

Note that the average mixing matrix $\widehat{M}$ is the matrix which sends a vector v to the diagonal of $\sum_{r=0}^{d} E_r \widehat{D(v)} E_r$ where $\widehat{D(v)}$ is v placed on the diagonal of a diagonal matrix. So $\widehat{M} e_v$ will give the diagonal of the matrix $\sum_{r=0}^{d} E_r D_v E_r$ (which we must sum the entries of).

1. If a graph is not regular, it is still possible for all centrality measures to be equal? That is, is it possible that the orthogonal projection of J has constant diagonal? This would be more interesting as a question about the projections.

2. When does it coincide with other spectral centers?

   - The average mixing matrix is $\widehat{M} = \sum_r E_r \circ E_r$. We can rank the vertices based on the diagonal entries of $\widehat{M}$.
   - We can rank the vertices based on the the entries of the Perron vector.
   - Maximum degree.
   - Every tree has a central vertex or edge, which are fixed by every automorphism. When is the central vertex also "central" with respect to another ordering?

Local complementation and eigenvalues

The motivation comes from quantum stabilizer codes used for quantum error correction. Local complementation in graphs corresponds with local Clifford equivalence in graph states (stabilizer states with an underlying graph structure).

Let G be a graph and v a vertex of G. The *local complementation* of G at v is the graph G^v obtained by complementing the neighbourhood of v. Let S be a matrix where rows and columns are indexed by the vertices of G:

$$S(G)_{v,w} = \begin{cases} 0, & \text{if } v = w; \\ 1, & \text{if } v \sim w; \\ -1, & \text{otherwise.} \end{cases}$$

If G_1 is obtain by G_2 by local complementation at v, let X be the matrix with 1s everywhere except for the principal submatrix corresponding to $\Gamma(v)$, where it is -1. Then $S(G_1) = X \circ S(G_2)$.

This operation does not preserve the eigenvalues of S, but perhaps we can pose the question of what this does to the spectrum and whether or not equivalence classes of graphs under local complementation share spectral properties with respect to the Seidel matrix.

3.6 Eternal Domination Problems

This topic was proposed by Gary MacGillivray, and was selected for a working group. The members of the working group were MacKenzie Carr, Nancy Clarke, Gary MacGillivray, and Joy Morris.

A dominating set for a graph is a set of D of vertices such that every vertex not in D is adjacent to a vertex in D. The domination number of a graph is the minimum cardinality of a dominating set for the graph. Lured by the prospect of eternal world domination leading to plenty of NSERC funding, this group worked on several problems related to domination.

An eternal dominating collection for a graph is a collection of dominating sets all having the same cardinality, such that every vertex of the graph not contained in a given dominating set D in the collection, is contained in another dominating set in the collection that can be reached from D by moving some subset of the vertices of D along edges of the graph. This concept can also be defined in game-theoretic terms. The eternal domination number for a graph is the minimum cardinality of the dominating sets in an eternal dominating collection.

A paired dominating set for a graph is a dominating set that induces a matching in the graph. An eternal paired dominating collection, as well as the paired domination number and the eternal paired domination number, can be defined analogously to an eternal dominating number, the domination number, and the eternal domination number.

This group considered two problems:

Open Problem 22. How does the eternal paired domination number compare to other domination parameters, in the case of interval graphs?

and

Open Problem 23. How does the eternal domination number compare to the domination number for Cayley graphs on dihedral groups?

3.7 Walking to MDS codes

This topic was proposed by Brett Stevens, and was selected for a working group. The working group members were Brett Stevens, Peter Dukes, Robert Bailey, and Alice Lacaze-Masmonteil.

Background and introduction

Definition 24. Let b be a positive integer. A code $\mathcal{C}_{\mathbb{F}_q^b}$ is said to be an $\mathbb{F}_q$**-linear code** of length n over $\mathbb{F}_q^b$ if it is a linear subspace of the vector space $\mathbb{F}_q^{nb}$. Equivalently it is an $\mathbb{F}_q$-linear code over $\mathbb{F}_q^b$ if the code $\mathcal{C}_{\mathbb{F}_q}$ is a linear code of length nb over $\mathbb{F}_q$.

Notice that both $\mathcal{C}_{\mathbb{F}_q}$ and $\mathcal{C}_{\mathbb{F}_q^b}$ refer to the same set of codewords, but over the alphabets $\mathbb{F}_q$ and $\mathbb{F}_q^b$, respectively. Therefore, the codewords of $\mathcal{C}_{\mathbb{F}_q^b}$ of length n over $\mathbb{F}_q^b$ can also be viewed as codewords of length nb over $\mathbb{F}_q$. It is worth pointing out that the code symbols of $\mathcal{C}_{\mathbb{F}_q^b}$ can be regarded as elements in the field $\mathbb{F}_{q^b}$. However, linearity over this field is not assumed.

One common construction of $\mathbb{F}_q$-linear codes is from cyclotomy. Let $p = rb + 1$ be a prime and let G be the subgroup of order r in the multiplicative group, $\mathbb{F}_p^*$. Define a near resolution of $\mathbb{F}_p$ to be $\{0\}$ together with all the cosets of G. Developing these in the additive group of $\mathbb{F}_p$ gives a Near Resolvable Design. For any field $\mathbb{F}_q$, define a $p \times p(b+1)$ block matrix where the rows are indexed by the elements of $\mathbb{F}_p$ and the columns of the ith block are the incidence vectors of the sets in the ith near resolution class, including the set of size one. Then delete one row and every column which has a 1 in the deleted row. This gives a $p - 1 \times pb$ block matrix with blocks of b columns each which generates a $\mathbb{F}_q$-linear code with $n = p$, and q^{rb} codewords. If every $p - 1 \times p - 1$ submatrix resulting from any r of the blocks is full rank then the code $\mathcal{C}_{\mathbb{F}_q^b}$ is Maximum Distance Separable over $\mathbb{F}_q^b$, that is it Singlton Defect 0: $|\mathcal{C}_{\mathbb{F}_q^b}| = (q^b)^{n-d+1}$

For example if $p = 5$ and $r = 2$ then the Near Resolvable Design has resolution classes

$$\begin{aligned} R_0 &= \{0\}, \{1,4\}, \{2,3\} \\ R_1 &= \{1\}, \{2,0\}, \{3,4\} \\ R_2 &= \{2\}, \{3,1\}, \{4,0\} \\ R_3 &= \{3\}, \{4,2\}, \{0,1\} \\ R_4 &= \{4\}, \{0,3\}, \{1,2\} \end{aligned}$$

and first yields matrix

$$M = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right)$$

After removing the first row and all columns with a 1 in the first row we have

$$M = \left(\begin{array}{cc|cc|cc|cc|cc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

The proof that this is MDS uses the fact that the union of any two resolution classes is a Hamilton path (indeed when $r = 2$ then this construction gives the one-point deletion of a perfect 1-factorization) and deleting the first row is equivalent to deleting the point 0 and this the union is two disjoint paths. traversing each path from one end to the other gives a series of row operations which, one by one, reduce the number of 1s per row to exactly one. This is a permutation matrix and thus has full rank. Xu et al. proved that for $r = 2$ then the code being least density and MDS is equivalent to the existence of a perfect 1-factorization. We note that this path traversal proof means that these codes are MDS over $\mathbb{F}_q^b$ for any field $\mathbb{F}_q$

When $r > 2$ very little is known. Louidor and Roth prove several things [4]: When $r = 3$ and 2 is primitive in $\mathbb{F}_p$, then the code is MDS over $\mathbb{F}_2^b$; When $r = 4$ ($b \neq 3$) and 2 is primitive in $\mathbb{F}_p$ then the code is MDS over $\mathbb{F}_2^b$; For any r if q is primitive in $\mathbb{F}_p$ and q is sufficiently large then the code is MDS over $\mathbb{F}_q^b$. Because each result only applies to the codes over specific fields, the proofs cannot be of the "path-traversing/row-reduction" type.

Open Problem 25. Is there a "path-traversing/row-reduction" type proof for any $p = rb + 1$ when p is prime and $r > 2$.

Open Problem 26. What is the proper generalization of perfect 1-factorizations to r-uniform hypergraphs. If we want a perfect 1-factorization to still correspond to MDS $\mathbb{F}_q$-linear codes then it will require a property of the union of r factors.

3.8 Online working group – Cliques of the Birkhoff Polytope

This problem was proposed by Nathan Lindzay and was worked on by online participants.

Let $\mathcal{C} \subseteq S_n$ be the set of all permutations of $\{1, 2, \ldots, n\}$ that have precisely one cycle. Let $\mathrm{Cay}(S_n, \mathcal{C})$ be the (normal) Cayley graph of the symmetric group S_n generated by $\mathcal{C}$. The *1-skeleton* of a polytope P is the graph $G = (V, E)$ whose vertices V are the vertices of P and whose edges E are the edges of P. The Cayley graph above is isomorphic to the 1-skeleton of the *Birkhoff polytope*, i.e., the convex hull of $n \times n$ permutation matrices.

Open Problem 27. Give good lower and upper bounds on the size of a maximum clique of $\mathrm{Cay}(S_n, \mathcal{C})$.

For example, a lousy lower bound is given by a latin square, i.e., the cyclic group $\mathbb{Z}_n$ when $n = p$. An exponential upper bound of $O(c^n)$ for some $c > 1$ is known combining previous work with the clique-coclique bound (e.g., `https://arxiv.org/abs/1702.05773`). The truth should be in the middle. Purely Fourier-analyical/spectral methods don't have a prayer here I believe.

An analogous problem can be formulated for perfect matchings of K_{2n} where one takes the polytope to be the convex hull of all characteristic vectors of perfect matchings of K_{2n}. Its 1-skeleton lives in an association scheme. Showing that the maximum cliques of this 1-skeleton are small would have algorithmic applications.

Here are some relevant papers:

- `https://arxiv.org/abs/2212.12655`
- `https://www.math.ucla.edu/~pak/papers/bir.pdf`
- `https://arxiv.org/abs/1702.05773`
- `https://people.cs.uchicago.edu/~lenacore/pdfs/birkhoff.pdf`

Much more is known about the independence number. Good lower and upper bounds are known, but the bounds are not tight (even asymptotically).

Open Problem 28. Give a tight bound on the independence number of $\mathrm{Cay}(S_n, \mathcal{C})$ as $n \to \infty$.

Fourier-analytical/spectral methods might have a chance here.

3.9 Other open problems proposed

We did not record all of the open problems that were mentioned but not selected for working groups, but will touch on some of them briefly.

Jozsef Balogh presented a talk on Monday with open problems about sunflowers in set systems with small VC-dimension. Gabriel Verret and Bobby Miraftab gave a joint talk in which they presented problems on locally finite graphs with eigenvectors of finite support.

Chris Duffy gave a lightning talk in which he presented the topic of distance 2 convexity in oriented graphs. Peter Dukes presented problems related to balancing graphs, matrices, and polynomials in another lightning talk.

Ted Dobson gave a lightning talk in which he mentioned a new technique he has been working on that he hopes will lead to progress on several problems including the Cayley Isomorphism problem. He invited participants to connect with him if they were interested in participating in this work.

4 Working Group Progress and Other Outcomes

Given the limited space available to us, we will be brief in outlining preliminary progress by the working groups. We do want to note that feedback received by the organisers about the working groups was uniformly enthusiastic. Organisers participated in many different working groups, so witnessed much of this positive energy and success ourselves. Participants enjoyed the time spent in working groups, felt the topics were well-formulated and interesting to work on collaboratively, and appreciated the time that working group leaders had put into preparing for the groups. While some groups and some proposed problems achieved more obvious rapid progress than others, everyone seemed to feel good about what they had learned and achieved in their working groups. Indeed, the one mild critique we heard was that some participants would have appreciated more time in the schedule for the working groups. We were enormously pleased with the success of this aspect of the workshop.

Recognising that sometimes working groups do not go well and are not fruitful, we felt that the time we had chosen to allocate to working groups was judicious. We expect that most of the collaborations initiated during the working group time will continue, and hope that many may lead to research outcomes. We believe that even if tangible research outcomes are not produced, the connections that the working groups (and other workshop activities) established between the postdoctoral fellows and other ECRs, and the world expert senior researchers, will have a significant and positive impact on the careers of the ECRs.

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