Higher Segal Spaces and their Applications to Algebraic K-Theory, Hall Algebras, and Combinatorics

Julie Bergner (University of Virginia),
Joachim Kock (Universitat Autònoma de Barcelona),
Maru Sarazola (University of Minnesota)

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1 Overview of the Field

Segal spaces are used in homotopy theory, algebraic geometry, logic, and many other fields of modern mathematics, and in particular play an important role in many approaches to higher category theory [47], [43], [1]. While ordinary categories can be described by a binary associative composition law, higher categories must describe compositions and associativity that might be defined or hold up to (higher) homotopy. Segal spaces model such weak categories that have spaces of both objects and morphisms, and encode the homotopical ambiguity in the composition as higher simplices in a simplicial object. For this reason, the approach is well-suited to homotopy theory, not only for describing topological examples, such as cobordism categories, but also for investigating them using homotopy-theoretic tools such as model categories.

Generalizing this approach, 2-Segal spaces encode a still weaker structure, where composition can be multi-valued, or sometimes not defined at all, is still associative in a suitable sense. Alternatively, such a structure can be regarded as encoding decomposition rather than composition. Higher Segal spaces were first discovered by Dyckerhoff and Kapranov [23], motivated by an ample range of applications in homological algebra, representation theory, and geometry. Shortly after, the equivalent notion of decomposition space was discovered in combinatorics by Gálvez-Carillo, Kock, and Tonks [29]. It is characteristic for the richness of the theory that although the two notions are equivalent, the formulations, settings, and directions of development were rather disjoint. Over the past years, as the theory develops, the various viewpoints have spread, it is quite a lively area of research, especially in view of the diverse background of people using and developing the theory.

2 Recent Developments and Open Problems

The first connection between higher Segal spaces and algebraic K-theory is an early result, proven independently by Dyckerhoff and Kapranov and by Gálvez-Carrillo, Kock, and Tonks, that Waldhausen's $S_\bullet$-construction, when applied to an exact category, is a 2-Segal space. This result has been generalized in multiple directions. First, Bergner, Osorno, Ozornova, Rovelli, and Scheimbauer expanded the input of the $S_\bullet$-construction to produce an equivalence of homotopy theories between 2-Segal spaces and augmented stable double Segal spaces [3]. In work in progress, Bergner, Shapiro and Zakharevich show that the latter structures are very closely related to the CGW-categories of Campbell and Zakharevich [7], also developed as a very general input framework for different flavors of algebraic K-theory. In a different direction, Poguntke produces $2k$-Segal spaces for all k from an exact category [45]; when $k = 2$ his construction is related to the Real algebraic K-theory spectrum. Finally, Dyckerhoff's categorified Dold-Kan correspondence [20] is an even further extension of Poguntke's construction for stable $(\infty, 1)$-categories. Taken together, these results strongly suggest that higher Segal spaces encode refined information about algebraic K-theory spectra.

Dyckerhoff and Kapranov's original motivation for introducing 2-Segal spaces was to provide a unifying framework for understanding various Hall algebra constructions that appear in the literature, such as classical, motivic, and derived Hall algebras. They show that every sufficiently finitary 2-Segal space has a universal Hall algebra from which other Hall algebras can be obtained by applying a transfer theory. Studying the universal Hall algebra directly reveals features shared in common by all variants of Hall algebras. For example, Walde [55] and Young [58] have shown how to construct universal modules over the Hall algebra using relative 2-Segal spaces.

The often subtle bialgebraic structures on Hall algebras can also be understood through the universal Hall algebra, as shown by Dyckerhoff [19] and Penney [44]. Moreover, this new perspective has lead to the development of new Hall algebra constructions, such as Walde's Hall monoidal categories and Pogunkte's equivariant motivic Hall algebras [46]. At the same time, Dyckerhoff and Kapranov outlined many other uses of 2-Segal spaces; In particular, they used the triangulation interpretation of 2-Segal spaces to give a purely topological model for the Fukaya category of an oriented surface in terms of triangulations and labellings in certain triangulated categories [24]. They exploit the 2-Segal axioms to show that these invariants are are independent of choice of triangulation.

While 2-Segal spaces generalize ordinary Segal spaces, there are also higher k-Segal spaces for $k > 2$, which were also introduced by Dyckerhoff and Kapranov. Their basic theory was worked out by Poguntke [45] and Walde [56]. Just as 2-Segal spaces relate to triangulations of plane polygons, higher Segal spaces relate to cyclic polytopes, and to the orientals of Street [53].

Gálvez-Carrillo, Kock, and Tonks' development of decomposition spaces was motivated by incidence algebras in enumerative combinatorics, which since the work of Rota [48] has been an important construction on posets. Their starting point was the observation that many coalgebras and bialgebras in combinatorics (see for example Schmitt [50]) are not the incidence co- or bialgebra of any poset. They showed that in fact the construction is possible for a much larger class of simplicial sets and simplicial spaces than (nerves of) posets, namely what they called *decomposition spaces* [29]. Examples of this situation are the Faà di Bruno bialgebra [42], the Butcher-Connes-Kreimer Hopf algebra, the chromatic Hopf algebra of graphs, as well as Hall algebras. The theory also comprises a lift from the vector-space level to the objective level of slice categories and linear functors.

They went on to show that the Möbius inversion formula holds at this level [30] and describe the universal Möbius function [31], which induces every other Möbius function by pullback along a CULF functor. Many other classes of combinatorial bialgebras were shown to arise as the incidence bialgebra of a monoidal decomposition space [32], [10], [15], covering by now all the series of examples of Schmitt [50]. Carlier has also proven a substantial generalization of Rota's formula for higher adjunctions of decomposition spaces [9]. Carlier and Kock [12] showed that a version of Takeuchi's antipode formula holds for every monoidal decomposition space.

The algebraic intepretation of 2-Segal and higher Segal spaces has also been the subject of much recent investigation. Walde has shown that 2-Segal spaces are equivalent to invertible homotopy operads [54], while Stern has shown that cyclic 2-Segal spaces correspond to certain Calabi-Yau algebras [52]. Gal and Gal have given an algebraic description of higher Segal spaces as well [28]. It is expected that these different ways to think about higher Segal objects will lead to further applications of the theory.

3 Motivation for the workshop

In 2017, Julie Bergner and Mark Penney took the initiative of organizing the first international workshop on the topic, to be held at CMO/BIRS in Oaxaca. Unfortunately the workshop had to be cancelled due to the pandemic, and the workshop finally took place, now at BIRS in Banff, with a modified team of organizers (with Joachim Kock and Maru Sarazola substituting Mark Penney).

The goal was to bring together researchers working with higher Segal spaces in disparate areas such as algebraic K-Theory, Hall algebras, and combinatorics, but ended up involving also applications in group theory, mathematical physics, and symplectic geometry. For the participants, we invited most of the researchers who had already worked in and written papers on higher Segal spaces, including recent PhDs. Secondly we were keen on getting all students in the field to participate. Finally, as a result of interaction with many colleagues at workshops and conferences, we had become aware of many researchers who were curious about higher Segal spaces and saw potential in them for application to their undertakings in various fields, and expressed eagerness to learn more. The involvement of these participants was significant, both for the aim of the workshop to disseminate the topic and for the spirit of welcoming also non-experts. Although most participants were selected according to these criteria, we were also happy that several other people wrote to us expressing their interest in participating, and that we were able to accommodate them, either on-site or remotely. While the age distribution of the participants was ideal by design, we were also very happy to achieve a good gender balance and a reasonable geographical spread of origins of participants.

4 Presentation Highlights

In view of the diversity of the participants, it was decided that most of the talks should be introductory talks and survey talks, so as to give all participants an overview of the different aspects and directions of applications. The speakers were enthusiastic about giving such service talks. The remaining talks were more specialized talks, mostly given by young researchers. There was also room in the schedule for open problem sessions and discussion sessions, as well as further talks scheduled along the way, on topics deemed to deserve further treatment.

We now describe the talks that were given.

- Walker Stern: Introduction: The 2-Segal space perspective

  We began on Monday morning with expository talks by Walker Stern, introducing the 2-Segal space perspective, and Philip Hackney, introducing the decomposition space point of view.

  Walker Stern's talk was aimed at explaining and motivating the 2-Segal conditions, focusing on the underlying intuition. He first discussed higher associativity, and showed how contemplating the associativity of partially- and multiply-defined multiplications or compositions leads naturally to the 2-Segal conditions. He then explained how these conditions may be reinterpreted geometrically in terms of a kind of state sum. He made both of these perspectives precise in a variety of ways, and indicated some of the many directions in which they may be extended. Throughout, he emphasized the elementary nature of these intuitions, focusing on simplicial sets, 1-categories, and pictorial and diagrammatic arguments.

- Philip Hackney: Introduction: The decomposition space perspective

  Philip Hackney's talk was a gentle introduction to the decomposition-space viewpoint. Its starting point was the (active, inert) factorization system on the category Δ, and its basic properties. In particularly, the pushout property is the main ingredient in the definition of decomposition spaces: they are simplicial spaces sending active-inert pushouts in Δ to (homotopy) pullbacks. The notions of upper and lower 2-Segal spaces were also introduced. The main theorem that was presented was the fact that a decomposition space is a simplicial space that is both upper and lower 2-Segal. The ingredients of the proof were outlined, including the pasting law for pullbacks and the more economical axioms that one can reduce to. The second ingredient was the retract stability of pullbacks, used to handle the bottom degeneracy maps, corresponding to the fact that every 2-Segal space is unital [26]. Next, it was shown that 1-Segal spaces are 2-Segal, and the décalage, or path space, criteria were established.

- Andrew Tonks: Introduction: Incidence algebras

  On Monday afternoon, Andrew Tonks gave an introductory talk on incidence algebras and coalgebras, which were primary motivations for the decomposition space perspective [29]. He started by introducing the incidence coalgebra of a poset, as used in combinatorics, and then proceeded to increasingly more general inputs for this construction, leading to the incidence coalgebra of a decomposition space. Along the way, he introduced the key notion of CULF maps ("conservative" and "unique lifting of factorizations") and examples arising from décalage.

- Imma Gálvez-Carrillo: Transformations of decomposition spaces

  Andrew's talk was followed by a more specialized talk by Imma Gálvez-Carrillo, reporting on joint work with Kock and Tonks on decomposition spaces of symmetric functions. While symmetric functions in themselves are important objects appearing in many areas of mathematics, a secondary outcome of this work is to develop more machinery for working with decomposition spaces. In particular, a key ingredient in the application of decomposition spaces to symmetric functions is how to model the classical base changes between the various bases. To this end the transformations of decomposition spaces required are not just the CULF maps, but also a class of maps called IKEO ("inner Kan" and "equivalence on objects").

- Viktoriya Ozornova: Introduction: The S-construction, part 1

  The second day began with back-to-back expository talks on the relationship between 2-Segal spaces and Waldhausen's $S_\bullet$-construction from algebraic K-theory [57]. The first talk by Viktoriya Ozornova recalled the classical $S_\bullet$ construction due to Waldhausen. Aside from allowing for classical algebraic inputs such as the categories of (projective) modules over a ring, this framework can be used to compute the algebraic K-theory of more general inputs that carry homotopical information, encoded through a class of weak equivalences. It was then shown how this construction gives rise to the examples of 2-Segal spaces, as identified both by Dyckerhoff ad Kapranov and by Gálvez-Carrillo, Kock, and Tonks.

- Martina Rovelli: Introduction: The S-construction, part 2

  Martina Rovelli then gave a follow-up talk in which she identified an appropriate extra structure for a double Segal space, precisely the structure of an augmented stable double Segal space, for which a variant of Waldhausens $S_\bullet$-construction makes sense and outputs a 2-Segal space. She also described an inverse construction, called the *path construction*, which outputs an augmented stable double category for every 2-Segal space. In total, the two constructions give an equivalence of homotopy theories between 2-Segal spaces and stable augmented double Segal spaces, as proved by Bergner, Osorno, Ozornova, Rovelli, and Scheimbauer [2], [3], [4]. In this way, the theory of 2-Segal spaces have led to quite a far-reaching generalization of the Waldhausen $S_\bullet$-construction.

- Tanner Carawan: Left and right 2-Segal spaces

  Following the theme of connections with algebraic K-theory, the next three talks that day were shorter talks building on this relationship in various ways. First, Tanner Carawan gave a talk based on his recently posted preprint [8] about the weaker structure that arises from applying the $S_\bullet$-construction to a category with cofibrations, which was the main focus in Waldhausen's original work. In the course of this talk, it was determined that what he was calling "left" and "right" 2-Segal spaces agreed with the notions of "upper" and "lower" 2-Segal spaces originally introduced by Poguntke in [45] and defined in Philip Hackney's talk.

- Brandon Shapiro: CGW categories

  Second, Brandon Shapiro gave a talk about CGW-categories, as defined by Campbell and Zakharevich [7], which have the advantage that they are combinatorial in nature, as opposed to homotopical or algebraic, and thus have fewer difficult coherence issues to work with. Instead of revolving around the notion of splitting short exact sequences, or (co)fiber sequences, as the classical frameworks for algebraic K-theory do, this framework instead splits relations encoded in squares. As such, it can put four-term relations such as the inclusion-exclusion of sets in a K-theoretical footing. Shapiro introduced the key ideas and techniques for working with CGW-categories, focusing on the central example of finite sets and showing how the CGW formalism can be used to produce various classical homological algebra results, the foundations for which appear in [49].

- Carmen Rovi: Cutting and pasting of manifolds and K-theory

  Finally, Carmen Rovi gave a talk about the K-theory of manifolds, which arises from a general notion of squares K-theory introduced by Campbell, Kuijper, Merling, and Zakharevich in [6]. There are different relations between n-dimensional manifolds that one may want to encode in a K-theory spectrum. One such relation is given by the cut and paste operation, where one cuts a manifold along a codimension 1 submanifold, and glues the resulting two pieces back together to obtain a new manifold. On the other hand, one may consider the relation of cobordism, in which two manifolds are equivalent if their disjoint union forms the boundary of an $(n+1)$-dimensional manifold. In addition to giving an introduction to these techniques from her work with Hoekzema, Merling, Murray and Semikina in [37] and with Hoekzema and Semikina in [38], Rovi's talk served to raise the question of how squares K-theory fits into the picture of 2-Segal spaces.

- Matthew Young: Introduction: 2-Segal spaces and Hall algebras

  Addressing another main application of 2-Segal spaces, Matthew Young gave an introductory talk on the relationship between Hall algebras and 2-Segal spaces, as introduced by Dyckerhoff and Kapranov and Gálvez-Carrillo, Kock and Tonks, respectively. He began by building on the talks of Ozornova and Rovelli and recalling that the 2-Segal spaces of primary interest arise as the Waldhausen $\mathcal{S}_\bullet$-construction of proto-exact categories. He then showed that various linearizations of these 2-Segal spaces recover previously known Hall algebras. He also discussed relative variants of 2-Segal spaces and their role in the representation theory of Hall algebras.

- Tobias Dyckerhoff: Introduction: Higher Segal spaces

  On Wednesday, Toby Dyckerhoff began the day with a survey of the theory of higher Segal spaces. Starting with cyclic polytopes and their relation to Street's orientals [53], he explained how their triangulations give rise to the formulation of the higher Segal conditions. He then explained how various results, such as generalized path space criteria, nicely follow from the geometric combinatorics of these polytopes. Finally, he provided more abstract characterizations of the higher Segal conditions in terms of excision, due to Walde, and various examples, such as higher-dimensional variants of the Waldhausen $S_\bullet$-construction. While 2-Segal spaces quickly found numerous applications in various areas of mathematics, the k-Segal spaces for $k > 2$ have been more mysterious, and this talk played an important role in getting more people involved in these developments.

- Jonte Gödicke: Segal conditions and dualizability

  Toby's talk was followed by two two shorter talks on related subjects. First, Jonte Gödicke gave a talk on Segal conditions and dualizability. Taking its motivation from the Cobordism Hypothesis, he described characterizations of dualizability of higher categories and rigidity, giving a criterion for rigidity for certain 2-Segal spaces. This work grew out directly from concerns in topological field theories and quantum algebra.

- Justin Lynd: Partial groups and higher-than-2-Segal conditions

  Second, Justin Lynd provided examples of higher Segal spaces coming from the p-local theory of finite groups, reporting on joint work with Hackney [36]. This very new potential application of higher Segal spaces starts from the observation that the partial groups of Chermak [**?**], which recently has been used to solve longstanding conjectures in the homotopy theory of finite groups, are not examples of the partial monoids of Segal [51], themselves examples of 2-Segal spaces [2]. Rather, Chermak's partial groups are instead higher Segal spaces. This work is particularly of interest due to the scarcity of concrete examples of higher Segal spaces.

- Hiro Lee Tanaka: Introduction: 2-Segal spaces and Fukaya categories

  A final expository talk was given on Thursday morning by Hiro Lee Tanaka on the connection between 2-Segal spaces and Fukaya categories. He began with a brief introduction to symplectic manifolds and Weinstein sectors, and moved on to the notion of (partially) wrapped Fukaya categories. The main goal of the subject is to do some computations in dimension 2 and explain Kontsevich's cosheaf theorem, and how it can be interpreted in terms of 2-Segal conditions.

- Ivan Contreras: Frobenius objects in 2-Span, 2-Segal sets and the symplectic category

  Hiro's talk was followed by another application of 2-Segal spaces to symplectic geometry, given by Ivan Contreras. In this work, he described certain constructions in symplectic geometry in terms of

Frobenius objects in a 2-category of spans, which in turn can be understood in the 2-Segal framework, as shown by Contreras, Mehta, and Stern [18]. Special interest in these developments come from the fact that they are directly motivated by mathematical physics, an application areas that was not even envisioned in the original description of the workshop.

- Mikhail Gorsky: Operations on exact categories, and associated Hall algebras
- Peter Samuelson: Skein algebra of a surface
- Teresa Hoekstra Mendoza: Discrete 2-Segal spaces from graphs and trees

  The remaining shorter talks that day followed up on various themes from the workshop. Mikhail Gorsky described operations on exact categories and the associated Hall algebras [27], building on Matthew Young's introduction. Peter Samuelson then introduced the skein algebra of a surface, again presenting ideas whose connection with 2-Segal spaces we would like to understand better. Finally, Teresa Hoekstra-Mendoza discussed 2-Segal sets that arise from graphs and trees, which are discrete examples of more general decomposition spaces that were motivated by combinatorics and were discussed in Andrew Tonks' talk.

- Open problems session

  Thursday concluded with an open problems session. Four participants had offered to share problems that they wanted to pose, and they were given time to present their ideas, followed by a discussion period for each.

  – First, Matthew Young described the $R_\bullet$-construction and raised the question of whether the natural map from it to the $S_\bullet$-construction is relatively 2-Segal. This question would have implications for Real algebraic K-theory.
  – Second, Mikhail Gorsky asked the broad question of which constructions in the theory of Hall algebras can be upgraded to the context of 2-Segal spaces, or possibly to augmented stable double Segal spaces.
  – Third, Philip Hackney asked the question of whether the active-inert factorization system, used in the definition of decomposition space, can be generalized to other indexing categories besides Δ.
  – Finally, Toby Dyckerhoff introduced the notion of 2-simplicial homotopy theory, where the ordinary category Δ is replaced by a suitable 2-categorical generalization, and discussed its applicability to the $S_\bullet$-construction.

- Walker Stern: Cyclic 2-Segal spaces

  On Friday, we concluded with two shorter talks whose topics were determined earlier in the week, based on the participants interest and questions after the introductory talks. The first was by Walker Stern, describing 2-Segal spaces that have an additional cyclic structure. Such cyclic 2-Segal spaces can be understood via their enhanced algebraic structure, and this talk shed more light on some of the features that came up in Ivan Contreras' talk.

- Philip Hackney: Free decomposition spaces

  The final talk was by Philip Hackney and introduced the notion of free decomposition spaces, as recently defined in a preprints of Hackney and Kock [35].

The workshop concluded with a discussion session on how to continue the progress from the workshop. Participants discussed the publication of a proceedings volume for the workshop, which was met with much enthusiasm, as well as possibilities for further workshops, conferences, and opportunities for collaboration. It was proposed to keep a list of problems that arose from the workshop, and some participants formed working groups to continue to more formally discuss some of these problems.

Following the tradition for week-long events, the workshop included a planned space for social activities on Wednesday, and participants used the opportunity to explore Banff in small groups. This afternoon, together with all the shared spaces during meals and breaks were particularly enriching given the nature of the workshop, since many of the participants were coming from different backgrounds and did not know each other previously.

5 Scientific Progress Made

We are aware of some follow-up research that began at the workshop and/or has since been developed to its conclusion. We list a few of them here.

- Following the discussion during his talk, Tanner Carawan gave a proof that his definition of left and right 2-Segal spaces coincide with what previously appeared in the literature as lower and upper 2-Segal spaces [45], respectively. The proof now appears in the appendix of his recent preprint [8]. In the main body of the preprint, Carawan discusses whether several objects involved in the $S_\bullet$-construction (namely, the simplicial objects $S_\bullet\mathcal{C}$ and its restriction to weak equivalences $wS_\bullet\mathcal{C}$, and the spaces $|iS_\bullet\mathcal{C}|$ and $|wS_\bullet\mathcal{C}|$) are 2-Segal.

- After Carmen Rovi's presentation of squares K-theory as introduced in [6], Maxine Calle and Maru Sarazola investigated this double-categorical framework, and more precisely, its relation to 2-Segal objects, as well as its connection to augmented stable double Segal-objects.

  Originally, the K-theory of a squares category is given by a double-nerve construction, also called the $T_\bullet$ or Thomason's construction. Calle and Sarazola's forthcoming preprint gives an $S_\bullet$-construction for squares categories, together with sufficient conditions for the $S_\bullet$-construction to coincide with the $T_\bullet$-construction, providing a new model for the K-theory of a squares category.

  Inspired by the definitions from [2], they also introduce an additional stability condition for squares categories and show that every such squares category produces a stable pointed double category in the sense of [2]. Mirroring the setting for proto-exact categories, they show how the $S_\bullet$-construction of a stable category with squares produces 2-Segal objects in the category Cat of (small) categories and functors, but generally not 2-Segal spaces.

- Martina Rovelli and Toby Dyckerhoff had extensive discussions during the workshop about extending the relationship between augmented stable double Segal spaces and 2-Segal spaces to a suitable comparison between higher n-fold categories and higher Segal spaces.

- As a direct outcome of Philip Hackney's proposed open problem, Jonte Gödicke, Philip Hackney, Felix Nass and Walker Stern began a collaboration aiming to develop the theory of 2-Segal conditions in more general context where there is an active-inert factorization system, more precisely, in the setting of algebraic patterns of Chu and Haugseng [17].

- A long-standing open problem was explained by Andy Tonks during discussion sessions, namely that IKEO-CULF spans should model all algebra homomorphisms. As an outcome of this discussion, it appears that this problem has now been settled in the positive by Jonte Gödicke, using the theory developed by Walker Stern [52].

- The examples that Justin Lynd presented of higher Segal spaces are generating much interest, beyond the original audience from the workshop. At a subsequent research program, many participants from a diverse range of mathematical backgrounds were interested in hearing about both the abstract structure that he is developing with Hackney, as well as the specific examples.

- Julie Bergner and Ayelet Lindenstrauss have continued discussions about how to relate 2-Segal spaces to topological Hochschild homology (THH), and have discussed with another possible collaborator, J.D. Quigley. The close relationship between THH and algebraic K-theory suggests that there should be some meaningful connection here, but it likely requires some additional structure, such as being cyclic. Inna Zakharevich is also optimistic that some of her work with collaborators on trace methods and THH may likely be able to shed light on this question from another point of view.

6 Outcome of the Meeting

Overall the workshop went very well in a enthusiastic and friendly atmosphere, helped by the nice surroundings and facilities. The remote participation generally worked well during the talks, but naturally was difficult to maintain outside the scheduled talk slots. The time in between talks was essential to the interactions. The fact that all on-site participants could have all meals together contributed significantly to this collegial atmosphere, and we regret that the remote participants could not participate in those parts of the workshop.

The majority of the speakers agreed to record their presentations so they could be made available to the community through the BIRS workshop website. The quality of the recordings is very good, and for the remote participants in various time zones it was important that the recordings became available so promptly, for the ability to catch up.

We think the friendly relaxed atmosphere was also helped by the many introductory talks, aimed at conveying essential ideas rather than overwhelming technical details. The service talks were of high quality and were very well received, by experts and newcomers alike. It was readily agreed that the introductory presentations should form the basis for a proceedings volume of expository nature, so as to constitute a reference and a resource for people in application areas wanting to get deeper into the theory, and to give a picture of the various aspects of the theory.

The following is a list of participants who have committed to contributing expository or research articles to a proceedings volume, together with tentative titles. This volume will be edited by the workshop organizers, and has already received approval to be published in the AMS *Contemporary Mathematics* series:

1. Walker Stern: The 2-Segal space perspective: associativity and triangulations
2. Philip Hackney: The decomposition space perspective
3. Viktoriya Ozornova : 2-Segal spaces and the S-construction
4. Martina Rovelli: The Waldhausen construction as an equivalence between stable augmented double Segal spaces and 2-Segal spaces
5. Benjamin Cooper and Matthew Young: Hall algebras via 2-Segal spaces
6. Tobias Dyckerhoff: Higher Segal spaces
7. Maru Sarazola, Brandon Shapiro, and Inna Zakharevich: A course on CGW-categories through the lens of homology
8. Hiro Lee Tanaka: 2-Segal spaces and Fukaya categories
9. Julie Bergner and Walker Stern: Cyclic 2-Segal spaces
10. Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks: Decomposition spaces in combinatorics

Work is already well underway with many of these contributions. From what we have seen so far, the contributions will maintain the high quality level from the talks, and they will all be refereed to high standard. The timeline envisioned for the volume is an August deadline for submission of all contributions, a refereeing period of six months, so as to be able to hand over final "camera-ready" papers and frontmatter to the AMS productions team by March or April 2025. The current estimate is that the volume will fill about 300 pages.

References

[1] Julia Bergner. Three models for the homotopy theory of homotopy theories. *Topology* **46** (2007), 397–436.

[2] Julia E. Bergner, Angélica M. Osorno, Viktoriya Ozornova, Martina Rovelli, and Claudia I. Scheimbauer, 2-Segal sets and the Waldhausen construction, *Topology and its Applications* **235** (2018) 445–484.

[3] Julia E. Bergner, Angélica M. Osorno, Viktoriya Ozornova, Martina Rovelli, and Claudia I. Scheimbauer, 2-Segal objects and the Waldhausen construction, *Algebraic and Geometric Topology* **21** (2021) 1267–1326.

[4] Julia E. Bergner, Angélica M. Osorno, Viktoriya Ozornova, Martina Rovelli, and Claudia I. Scheimbauer, Comparison of Waldhausen constructions, *Annals of K-Theory* **6** (2021) 97–136.

[5] Julia E. Bergner, Angélica M. Osorno, Viktoriya Ozornova, Martina Rovelli, and Claudia I. Scheimbauer, The edgewise subdivision criterion for 2-Segal objects, *Proceedings of the American Mathematical Society*, **148** (2020), No. 1 71–82.

[6] Jonathan Campbell, Josefien Kuijper, Mona Merling, and Inna Zakharevich. Algebraic K-theory for squares categories, preprint available at arXiv2310.02852.

[7] Jonathan Campbell and Inna Zakharevich, Dévissage and localization for the Grothendieck spectrum of varieties, *Advances in Mathematics* **411** (2022), Paper No. 108710, 80 pp.

[8] Tanner Nathan Carawan, 2-Segal maps associated to a category with cofibrations, preprint available at arXiv:2405.11561.

[9] Louis Carlier, Incidence bicomodules, Möbius inversion and a Rota formula for infinity adjunctions, *Algebraic and Geometric Topology* **20** (2020), 169–213.

[10] Louis Carlier, Hereditary species as monoidal decomposition spaces, comodule bialgebras, and operadic categories. *International Mathematics Research Notices* **8** (2022) 5745–5780.

[11] Louis Carlier, Möbius functions of directed restriction species and free operads, via the generalized Rota formula, *Mediterranean Journal of Mathematics* **18** (2021), No.4 Paper No. 170, 20pp.

[12] Louis Carlier and Joachim Kock, Antipodes of monoidal decomposition spaces, *Communications in Contemporary Mathematics* **22** (2020), 1850081, 15.

[13] Alex Cebrian, A simplicial groupoid for plethysm, *Algebraic and Geometric Topology* **21** (2021), No. 1 421–445.

[14] Alex Cebrian, Plethysms and operads, *Collectanea Mathematica* **75** (2024), No. 1 247–303.

[15] Alex Cebrian and Wilson Forero, Directed hereditary species and decomposition spaces, preprint available at arXiv:2211.07721.

[16] Andrew Chermak, Fusion systems and localities, *Acta Mathematica* **211** (2013), 47–139.

[17] Hongyi Chu and Rune Haugseng, Homotopy-coherent algebra via Segal conditions, *Advances in Mathematics* **385** (2021), 107733.

[18] Ivan Contreras, Rajan Amit Mehta and Walker H. Stern, Frobenius and commutative pseudomonoids in the bicategory of spans, preprint available at arXiv:2311.15342.

[19] Tobias Dyckerhoff, Higher categorical aspects of Hall algebras, in *Advanced Course on (Re)emerging Methods in Commutative Algebra and Representation Theory*, *Quaderns* **70**, CRM, Barcelona, 2015.

[20] Tobias Dyckerhoff, A categorified Dold-Kan correspondence, *Selecta Mathematica. New Series* **27** (2021), Paper No. 14, 35pp.

[21] Tobias Dyckerhoff, Gustavo Jasso and Tashi Walde, Simplicial structures in higher Auslander-Reiten theory, *Advances in Mathematics*, **355** (2019), 106762, 73pp.

[22] Tobias Dyckerhoff and Mikhail Kapranov. Crossed simplicial groups and structured surfaces, *Contemporary Mathematics*, **643** (2015) 37–110.

[23] Tobias Dyckerhoff and Mikhail Kapranov, Higher Segal spaces, *Lecture Notes in Mathematics* **2244**. Springer, Cham, 2019.

[24] Tobias Dyckerhoff and Mikhail Kapranov, Triangulated surfaces in triangulated categories, *Journal of the European Mathematical Society* **20** (2018), 1473–1524.

[25] Matt Feller, Quasi-2-Segal sets, *Tunisian Journal of Mathematics* **5** (2023), No. 2 327–367.

[26] Matthew Feller, Richard Garner, Joachim Kock, May U. Proulx, and Mark Weber, Every 2-segal space is unital, *Communications in Contemporary Mathematics* **23** (2021), 2050055.

[27] Xin Fang and Mikhail Gorsky, Exact structures and degeneration of Hall algebras, preprint available at arXiv:2005.12130.

[28] Adam Gal and Elena Gal, Higher Segal spaces and lax A_∞-algebras, preprint available at arXiv:1905.03376.

[29] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks, Decomposition spaces, incidence algebras and Möbius inversion I: Basic theory *Advances in Mathematics* **331** (2018), 952–1015.

[30] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks, Decomposition spaces, incidence algebras and Möbius inversion II: Completeness, length filtration, and finiteness, *Advances in Mathematics* **333** (2018), 1242–1292.

[31] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. Decomposition spaces, incidence algebras and Möbius inversion III: The decomposition space of Möbius intervals, *Advances in Mathematics* **334** (2018), 544–584.

[32] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks, Decomposition spaces and restriction species, *International Mathematics Research Notices* **2020** (2020), 7558–7616.

[33] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. Homotopy linear algebra, *Proceedings of the Royal Society of Edinburgh Section A* **148** (2018), No. 2, 293–225.

[34] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. Groupoids and Faà di Bruno formulae for Green functions and bialgebras of trees, *Advances in Mathematics* **254** (2014), 79–117.

[35] Philip Hackney and Joachim Kock, Free decomposition spaces, to appear in *Collectanea Mathematica*, preprint available at arXiv:2210.11192.

[36] Philip Hackney and Justin Lynd, Partial groups as symmetric simplicial sets, preprint available at arXiv:2310.01513.

[37] Renee S. Hoekzema, Mona Merling, Laura Murray, Carmen Rovi, and Julia Semikina, Cut and paste invariants of manifolds via algebraic K-theory, *Topology and its Applications* **316** (2022).

[38] Renee Hoekzema, Carmen Rovi, and Julia Semikina, A K-theory spectrum for cobordism cut and paste groups, preprint available at arXiv:2210.00682.

[39] Joachim Kock, From Möbius inversion to renormalisation, *Communications in Number Theory and Physics* **14** (2020), No. 1, 171–198.

[40] Joachim Kock, The incidence comodule bialgebra of the Baez-Dolan construction, *Advances in Mathematics* **383** (2021), Paper No. 107693, 79pp.

[41] Joachim Kock and David Spivak, Decomposition-space slices are toposes, *Proceedings of the American Mathematical Society* **148** (2020), No. 6, 2317–2329.

[42] Joachim Kock and Mark Weber, Faà di Bruno for operads and internal algebras, *Journal of the London Mathematical Society* (2) **99** (2019), No. 3, 919–944.

[43] André Joyal and Myles Tierney, Quasi-categories vs Segal spaces, in *Categories in Algebra, Geometry and Mathematical Physics*, *Contemporary Mathematics* **431** (2007) pages 277–326.

[44] Mark Penney, The universal Hall bialgebra of a double 2-Segal space, preprint available at arXiv:1711.10194.

[45] Thomas Poguntke, Higher Segal structures in algebraic K-theory, preprint available at arXiv:1709.06510.

[46] Thomas Poguntke, Equivariant motivic Hall algebras, preprint available at arXiv:1808.04165.

[47] Charles Rezk, A model for the homotopy theory of homotopy theory, *Transactions of the American Mathematical Society* **353** (2001), 973–1007.

[48] Gian-Carlo Rota, On the foundations of combinatorial theory. I. Theory of Möbius functions, *Zeitschift für Wahrscheinlichkeitstheorie und Verwändte Gebiete* **2** (1964), 340–368.

[49] Maru Sarazola, Brandon Shapiro, A Gillet–Waldhausen Theorem for chain complexes of sets, preprint available at arXiv:2107.07701.

[50] William R. Schmitt, Hopf algebras of combinatorial structure, *Canadian Journal of Mathematics* **45** (1993), 412–428.

[51] Graeme Segal, Configuration spaces and iterated loop-spaces, *Inventiones Mathematica* **21** (1973), 213–221.

[52] Walker H. Stern, 2-Segal objects and algebras in spans, *Journal of Homotopy and Related Structures* **16** (2021), 297–361.

[53] Ross Street, The algebra of oriented simplexes, *Journal of Pure and Applied Algebra* **49** (1987), 283–335.

[54] Tashi Walde, 2-Segal spaces as invertible infinity-operads, *Algebraic and Geometric Topology* **21** (2021), 211–246.

[55] Tashi Walde, Hall monoidal categories and categorical modules, preprint available at arXiv:1611.08241.

[56] Tashi Walde, Higher Segal spaces via higher excision, *Theory and Applications of Categories* **35** (2020), 1048–1086.

[57] Friedhelm Waldhausen, Algebraic K-theory of spaces, *Algebraic and Geometric Topology, Lecture Notes in Mathematics* **1126**, Springer, 1996.

[58] Matthew B. Young, Relative 2-Segal spaces, *Algebraic and Geometric Topology* **18** (2018), 975–1039.