# Enumerative Geometry Beyond Spaces

Balázs Szendrői (University of Vienna),
Jim Bryan (UBC),
Kirsten Wickelgren (Duke University),
Maria Yakerson (Oxford University)

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## 1 Overview of the Field

The subject of enumerative geometry was classically concerned with counting geometric objects on a given geometric space, for example the number of lines on a cubic surface (answer: $27$), or the number of rational curves of some fixed degree $d$ on a general quintic threefold (for example, there are $2875$ lines). Deep connections with theoretical physics and symplectic geometry have led to enormous progress in the subject over the last three decades, which evolved far past its original goals. Sophisticated refinements of the numerical counts have replaced numerical invariants with invariants that take values in vector spaces, motivic classes, or even categories. Motivated in part by ideas from physics, interest in the individual invariants has given way to the study of partition functions, numerical invariants assembled into generating series of different kinds. Partition functions exhibit symmetries and structure that are not apparent in the individual invariants; for example, they sometimes exhibit modular properties which reflect underlying symmetries.

The general paradigm for getting enumerative invariants out of a space $X$ is to extract numbers (or vector spaces, or motivic classes, or categories) from a moduli space of objects $M_X$ defined on the original $X$. One of the lessons of the last three decades is that the notions of "space" can be replaced by some more general constructions in this process. On the one hand, one can replace the underlying space $X$ by a category, for example its derived category of coherent sheaves, together with some extra structure, and aim to extract invariants from this setup. This allows for derived symmetries of $X$ to come into play, which in some cases of interest form a substantially larger set than geometric symmetries, related perhaps by mirror symmetry to large monodromy groups. On the other hand, it is really the derived geometry of the moduli space $M_X$ that is important for extracting invariants. Standard constructions in enumerative geometry such as virtual classes have long been understood to be reflections of the derived structure of the moduli spaces, but only recently have techniques from derived and homotopical algebraic geometry been brought to bear on the issues. In particular, derived categories receive functors from (stable) homotopy theories in certain cases of interest.

In particular, Morel–Voevodsky's $\mathbb{A}^1$-homotopy theory naturally leads to enumerative geometry enriched in bilinear or quadratic forms. While it is mysterious why this is the case, its practical consequences are clear: such counts include information about the field of definition of the geometric objects being counted. For example, while the number of real lines on a real cubic surface may vary, $\mathbb{A}^1$-homotopy theory provides a well-defined bilinear form satisfying an invariance of number principle (answer: $15\langle 1\rangle + 12\langle -1\rangle$) which differentiates between the real lines, the complex lines, and when the tangent plane to the surface spins all the way around the line. Over other fields and even more general rings, there are highly functional $\mathbb{A}^1$-homotopy theories and associated categories of motives. This paradigm expands enumerative geometry over

the complex numbers or real numbers to enumerative results over other fields and number rings.

The primary aim of our workshop was to study emerging and exciting connections between enumerative geometry, the theory of derived categories, and derived and homotopical algebraic geometry. These areas of mathematics, each rich in their own right, are increasingly intersecting in ways that promise to unlock new approaches and insights. By fostering a deeper understanding of these connections, the workshop sought to advance the frontiers of knowledge and catalyze innovative research at the intersection of these disciplines.

To achieve this goal, we brought together a diverse group of participants, combining senior experts with extensive experience in these distinct yet related fields with younger mathematicians who are actively exploring them. This carefully curated mix of expertise and fresh perspectives created a dynamic and collaborative environment, where ideas could be freely exchanged and new collaborations could emerge. The hybrid format of the workshop allowed for broader participation, accommodating those who could not attend in person while still emphasizing the value of face-to-face interactions. By bridging gaps between different mathematical communities and career stages, the workshop provided a unique platform for deep discussions and cross-pollination of ideas, ensuring that both seasoned researchers and early-career participants could contribute to and benefit from the collective knowledge of the group.

## 2 Structure of the meeting

We ran the workshop in a hybrid format, balancing both in-person and remote participation to accommodate a diverse group of attendees. Admittedly, most interactions took place in person at the Banff International Research Station (BIRS), which remains an outstanding venue for fostering intellectual collaboration. The main backbone of our meeting consisted of a series of carefully curated talks delivered in a traditional lecture format. Of these, four were presented by speakers participating remotely, showcasing the flexibility and inclusiveness of the hybrid setup. A limited number of remote participants also joined in attending the talks, engaging with the content and contributing to the discussions. Despite the challenges inherent in remote interactions, meaningful engagements were achieved, particularly during the remote lectures. Participants noted that facilitating such interactions is often difficult, underscoring the exceptional technical quality and thoughtful design of the BIRS facility, which played a critical role in enabling a seamless experience. However, it was generally felt that having as many participants as possible attending in person remains the ideal scenario, as face-to-face communication allows for deeper connections, more dynamic exchanges of ideas, and the spontaneous discussions that are often key to scientific breakthroughs.

We actively encouraged the formation of mentoring relationships during our meeting, recognizing the importance of mentorship in fostering both personal and professional growth within the academic community. To facilitate this, we circulated a detailed call for all participants, inviting them to sign up either as mentors or mentees, depending on their experience level, career stage, and personal preference. The response to this initiative was overwhelmingly positive, demonstrating the enthusiasm of participants to engage in meaningful, supportive interactions. As a result, we successfully formed nine mentor-mentee pairs, each carefully matched based on shared interests and complementary goals. These pairs had opportunities to engage throughout the workshop, during breaks, collaborative sessions, and even informal gatherings, allowing them to build connections beyond structured activities. The feedback we received was encouraging, with participants expressing gratitude for the opportunity to engage in these relationships. Some specific outcomes and insights from these mentoring pairs are reported below, highlighting the success of this initiative and its potential to inspire future implementations in similar settings.

The meeting was honored to include a participant from Africa, Dr. Benjamin Kikwai of Machakos University, Kenya, a specialist in the field of tropical enumerative geometry. His participation added a unique perspective to the discussions, enriching the intellectual diversity of the workshop. However, it is important to recognize the significant challenges faced by researchers from the Global South in accessing opportunities like this. Many potential participants from these regions encounter substantial financial barriers, as local funding sources for travel and research expenses are often limited or unavailable. Despite these obstacles, we were fortunate to have Dr. Kikwai's travel expenses generously supported by the K-Theory Foundation. The foundation's support not only enabled Dr. Kikwai's participation but also highlighted the importance of fostering inclusivity in academic events by actively addressing financial disparities. We hope that this serves as an example for future efforts to create equitable opportunities for researchers from underrepresented regions

in the global academic community.

We were pleased to invite several promising PhD students to participate in the meeting alongside more established researchers, creating a vibrant and inclusive academic environment that bridged different career stages. The students brought fresh perspectives and enthusiasm to the discussions, while also benefiting significantly from the presence of senior colleagues. Many of these students reported having deeply inspiring one-on-one interactions with more experienced researchers, which they found both motivating and enlightening. These personal meetings offered them a rare opportunity to seek guidance on their academic journeys and to receive valuable advice on tackling some of the more technical and challenging aspects of the subject, such as motivic homotopy theory.

One of the PhD students shared particularly positive feedback, stating: "Due to the speakers' commitment to give accessible expositions, I was able to take something away from every single talk, and I never felt lost. The spirit of the workshop was one of appreciation and mutual interest, and I always felt seen and encouraged to ask questions. For me, this really makes this BIRS workshop stand out among other workshops and conferences I have attended." This sentiment encapsulates the inclusive and supportive atmosphere we aimed to cultivate throughout the event. The deliberate effort to ensure accessibility and approachability in the presentations, coupled with the welcoming environment, left a lasting impact on the students. We hope that these experiences will serve as a foundation for their continued growth in the field and reinforce their confidence as they tackle advanced topics and contribute to the broader academic community.

# 3 Presentation Highlights

## 3.1 Classical topics

Talks in this direction included a talk by Dhruv Ranganathan, who highlighted the theoretical development of enumerative geometry over the last 20 years, including tropical geometry, orbifolds, and logarithmic structures. Andrei Caldararu gave an update on how to use an old conjecture of Kontsevich on homological mirror symmetry to derive enumerative information on invariants associated to a Calabi-Yau category, which specialize to curve counts for the Fukaya category, and to the invariants obtained from a variations of Hodge structures for the family of derived categories. Joachim Jelisiejew's talk outlined the construction of an analogue of Hilbert schemes of points for Gorenstein algebras, giving a projective moduli space with some interesting degenerate objects on the boundary, discussing also enumerative aspects. Pierrick Bousseau defined generalized Block-Göttsche polynomials and Welschinger invariants beyond the toric context for rational surfaces, and proposed a surprising conjectural relation with refined Donaldson-Thomas invariants.

## 3.2 Motivic enhancements of enumerative counts

Foundational contributions to the subject were delivered by Maria Yakerson and Peter Haine, who explored motivic homotopy theory and its connections to algebraic geometry. Their talks laid the groundwork for understanding this framework and its broad implications across enumerative geometry.

Applications of the technology were presented in a talk by Marc Levine, that concerned itself with the following question: given a smooth quadric hypersurface over a field, when does the tangent bundle admit a nowhere zero section? Some conditions were presented whose proofs use applications of motivic homotopy theory to the question of splitting vector bundles and a computation of the $\mathbb{A}^1$-Euler characteristic of a smooth projective hypersurface, together with some classical results on quadratic forms.

Turning to computational aspects involving enumerative geometry, Ran Azouri explained how to extend some now classical computations of Donaldson-Thomas theory to the quadratic context.

Simon Pepin Lehalleur's talk concerned both joint work with Srinivas and Levine on the motivic Euler characteristic of hypersufaces as well as conductor formulas related to Axouri's talk. He furthermore discussed joint work in progress with Ran Azouri, Niels Feld, Yonathan Harpaz and Tasos Moulinos on motivic Euler characteristics of symmetric powers. Pajwani and Pal showed there is at most one power structure on the Grothendieck–Witt group compatible with the symmetric powers of varieties. Many mathematicians have contributed to these conjectures including Jesse Pajwani, Herman Rohrbach, Anna M. Viergever, Stephen McKean, and Dori Bejleri. Pepin Lehalleur with Taelman computed the motivic Euler characteristic of symmetric powers of smooth projective surfaces.

Ambrus Pal presented ideas about how to prove the birational invariance of the motivic Euler characteristic for certain Calabi-Yau varieties, and its computation via the theory of Galois invariants and unramified cohomology. He used the motivic Euler characteristic for its role in quadratic refinement of curve counting, as explored in his joint work with Jesse Pajwani on an arithmetic refinement of the Yau-Zaslow formula. They substitute the motivic Euler characteristic for the classical Euler characteristic in Beauville's argument.

Jake Solomon's presentation introduced a quadratically enriched count of rational curves on del Pezzo surfaces of degree $\geq 3$ over perfect fields, in collaboration with others. This count, taking values in the Grothendieck-Witt group of quadratic forms, generalizes classical invariants such as Gromov-Witten and Welschinger counts while ensuring compatibility with F. Morel's $\mathbb{A}^1$-homotopy degree.

Tom Bachmann discussed quadratically enriched residual intersections, beyond the case of excess intersection in a regular immersion. His talk bridged classical duality theories with modern motivic perspectives, connecting them to algebraic surgery computations of Euler classes in almost complete intersections. This work, in collaboration with Kirsten Wickelgren, contributes to understanding intersection theory over a non-algebraically closed field.

William Hornslien's talk, "Homotopy Classes of Endomorphisms of the Projective Line", explored a fundamental problem in algebraic topology by examining the $\mathbb{A}^1$-homotopy classes of endomorphisms of the projective line. In motivic homotopy theory, the projective line serves as an analogue of the sphere, making its homotopy classes a parallel to the fundamental group of the circle in classical topology. Building on Morel's foundational computation of this group through abstract methods, Hornslien presented a more accessible approach using elementary algebraic geometry. This work, conducted in collaboration with Viktor Balch Barth, Gereon Quick, and Glen Matthew Wilson, provides new insights into the interplay between algebraic geometry and motivic homotopy theory.

Andrés Jaramillo Puentes' talk studied enumerative geometry on toric varieties over arbitrary fields, proving a correspondence theorem between the quadratically enriched count of rational curves and a combinatorial tropical counterpart, and a wall-crossing formula for such invariants.

## 3.3 Extensions of intersection theory

Charanya Ravi and Elden Elmanto presented talks that illuminated recent advancements in intersection theory and its extensions. Charanya Ravi discussed joint work with Adeel Khan on the Virtual Kawasaki-Grothendieck-Riemann-Roch theorem, presenting two forms of the Grothendieck-Riemann-Roch theorem for derived algebraic stacks. The first form connects the lisse extended G-theory with Chow groups, generalizing to a higher equivariant context, while the second form relates the G-theory of a stack to the Chow group of its inertia stack for derived Deligne-Mumford stacks. These formulations culminate in a virtual Kawasaki-Riemann-Roch formula, showcasing the intricate interplay between G-theory and intersection theory in derived settings. Elden Elmanto revisited intersection theory on schemes, addressing Srinivas' conjecture on extending the Chow ring to singular varieties. Through joint work with Morrow, Elmanto introduced a novel motivic cohomology framework, solving the conjecture by demonstrating that the $(2n, n)$-line of this theory satisfies all expected properties, thus advancing the landscape of intersection theory beyond the classical confines of smooth varieties. Together, these talks emphasized the evolving mathematical frameworks for understanding and extending fundamental results in intersection theory.

## 3.4 Enumerative geometry in the Burnside ring

Candace Bethea and Thomas Brazelton presented significant contributions to the emerging field of equivariant enumerative geometry, emphasizing the role of symmetry in classical geometric questions. Candace Bethea discussed a joint project with Kirsten Wickelgren on defining a global degree in stable equivariant homotopy theory, relating it to a sum of local equivariant degrees. Their framework introduces the notion of orientation for equivariant maps of G-manifolds relative to an equivariant ring spectrum and demonstrates how this global degree generalizes G. Segal's classical equivariant degree for maps between representation spheres. Bethea illustrated the utility of local degrees in counting orbits of rational cubics through a G-invariant set of eight points on a suitable surface, highlighting the intersection of topology and geometry in equivariant settings. Thomas Brazelton further discussed the program of equivariant enumerative geometry, where symmetry informs classical questions like counting lines or circles under constraints. His main result,

the equivariant conservation of number, ensures that the orbits of solutions to equivariant problems are conserved, generalizing Schubert's classical principle. Using this theory, Brazelton computed $S_4$-orbits for the 27 lines on a symmetric cubic surface and connected these results to monodromy computations in symmetric enumerative problems. Together, their talks showcased the power of equivariant methods in addressing both classical and modern geometric problems.

### 3.5 Full list of speakers and titles

1. Dhruv Ranganathan: A story of degenerations in enumerative geometry (remote)
2. Andrei Caldararu: A gentle introduction to Categorical Enumerative Invariants
3. Vicky Hoskins: Moduli spaces with abelian motives
4. Pierrick Bousseau: Generalized Block-Göttsche polynomials and Welschinger invariants
5. Adeel Khan: Perverse microsheaves and cohomological DT theory
6. Peter Haine: Interactions between homotopy theory and algebraic geometry (remote)
7. Maria Yakerson: Glimpses of motivic homotopy theory
8. Marc Levine: An algebraic hairy ball theorem
9. Andrés Jaramillo Puentes: Tropical Methods in $\mathbb{A}^1$ enumerative geometry
10. Charanya Ravi: Virtual Kawasaki-Grothendieck-Riemann-Roch theorem
11. Joachim Jelisiejew: Analogue of Hilbert schemes of points for Gorenstein algebras (remote)
12. William Hornslien: Homotopy classes of endomorphisms of the projective line
13. Tom Bachmann: Quadratically enriched residual intersections
14. Simon Pepin Lehalleur: Quadratic Euler characteristics of singular varieties
15. Ran Azouri: On quadratic Donaldson-Thomas theory
16. Jake Solomon: Toward quadratically enriched Gromov-Witten theory
17. Candace Bethea: The degree in stable equivariant homotopy theory
18. Thomas Brazelton: Equivariant enumerative geometry
19. Ambrus Pal: Computing the motivic Euler characteristic (remote)
20. Elden Elmanto: Intersection theory on schemes

## 4 Scientific Progress and Outcomes

The principal aim of our meeting was to bridge distinct mathematical communities and enhance participants' understanding of enumerative geometry in its various contemporary contexts. By fostering interactions among researchers from diverse backgrounds, the workshop sought to illuminate ongoing work and uncover the rich tapestry of ideas shaping the field today. This cross-disciplinary engagement not only highlighted the multifaceted nature of enumerative geometry but also encouraged participants to explore its evolving connections with other areas of mathematics and theoretical physics.

Seasoned researchers who have worked in enumerative geometry for decades remarked that the meeting was a true eye-opener for them. They found the new directions discussed during the sessions to be not only intellectually stimulating but also profoundly exciting, beautiful, and inspirational. A particularly intriguing

concept that resonated with many participants was the idea of defining invariants that extend beyond traditional numerical values. For instance, the possibility of developing more intricate $K$-theoretic enumerative invariants, taking values in quadratic forms, opened up a novel realm of exploration and set the stage for groundbreaking advancements. This innovative perspective challenges long-held assumptions and invites a rethinking of fundamental approaches to enumerative problems, signaling a promising horizon for future research.

The feedback underscored the transformative impact of the meeting, as it inspired participants to embrace new methodologies and consider unexplored connections. By providing a platform for the exchange of pioneering ideas, the workshop succeeded in redefining how enumerative geometry is viewed and practiced within the mathematical community.

## 4.1 Feedback Received

- Andrei Caldararu shared that his discussions with Thomas Brazelton and Kirsten Wickelgren during the workshop proved to be highly stimulating and productive. These conversations revolved around some very classical topics, such as the intricate geometry of cubic surfaces, a subject that has fascinated mathematicians for generations due to its rich structure and numerous connections to other areas of geometry and algebra. Inspired by the fresh perspectives and insights exchanged during these interactions, Caldararu was motivated to revisit an old problem he had studied earlier in his career.

- Yun Shi shared that her discussions with her mentor, Jake Solomon, during the workshop were exceptionally productive and paved the way for exciting collaborative research. Their conversations focused on the intricate problem of counting objects within the derived category that remain invariant under autoequivalences, a topic at the forefront of contemporary mathematical inquiry. This area of study holds significant promise, as it bridges deep theoretical aspects of derived categories with practical computational techniques, offering new insights into the symmetries and structures inherent in complex geometric settings.

  Building on these discussions, Shi and Solomon are currently working to refine the precise formulation of this problem, aiming to ensure its mathematical rigor and to identify the broader implications of their findings. As part of this process, they have embarked on explicit computations for specific cases, including elliptic curves and $K3$ surfaces, which are known for their rich geometric and algebraic properties. These computations not only serve as concrete illustrations of the problem but also provide valuable test cases for their proposed framework.

  The collaborative efforts between Shi and Solomon exemplify the dynamic and forward-thinking spirit of the workshop, demonstrating how mentoring relationships and intellectual exchanges can lead to innovative research directions. Their ongoing work has the potential to open new pathways in the study of derived categories and autoequivalences, with implications that may extend well beyond the initial scope of their project.

- Martijn Kool reported having engaging and thought-provoking discussions with Benjamin Kikwai, focusing on the problem of counting divisors on singular toric threefolds. This topic, situated at the intersection of algebraic geometry and combinatorics, has long been a fertile area for exploration due to the intricate structures of toric varieties and their wide range of applications. Their discussions highlighted the potential of tropical geometry as a powerful tool for addressing these challenges, drawing inspiration from prior work such as [4].

  The use of tropical geometry in this context offers a novel and promising approach, as it enables a translation of complex algebraic problems into combinatorial frameworks that are often more accessible. Tropical methods allow for the study of degenerate limits of geometric objects, providing valuable insights that might otherwise remain obscured. Kool and Kikwai explored how these techniques could be adapted to the specific case of singular toric threefolds, with the aim of uncovering new patterns and invariants that govern the behavior of divisors in such spaces.

- Tom Bachmann shared that he had detailed and fruitful discussions with Elden Elmanto during the workshop, which intriguingly took place over the course of a long hike. These conversations delved

into the fascinating topic of analogs of factorization homology within the framework of motivic homotopy theory, a cutting-edge area of research that seeks to bridge ideas from algebraic topology and algebraic geometry. Factorization homology, originally developed in the context of topological field theories, provides powerful tools for studying structured invariants of manifolds and their associated categories. Adapting these ideas to motivic settings represents an exciting and ambitious challenge, with the potential to yield profound insights.

The informal yet intellectually stimulating setting of their hike allowed Bachmann and Elmanto to explore the conceptual underpinnings of their ideas in depth, brainstorm potential approaches, and identify key questions that could guide their work. Their discussions touched on both theoretical formulations and practical computations, aiming to extend the utility of motivic homotopy theory to incorporate the structures and techniques inspired by factorization homology.

As these conversations progressed, it became clear that this promising line of inquiry had the potential to evolve into a long-term collaborative project. Both researchers expressed enthusiasm about continuing to work together on this topic, drawing on their complementary expertise to tackle the complex and interdisciplinary challenges it presents. This collaboration, sparked by their engaging discussions at the workshop, exemplifies how such events can catalyze innovative research partnerships that extend well beyond their duration.

- Adeel Khan and Tom Bachmann engaged in deep and focused discussions on several approaches to tackling the Hopkins-Morel isomorphism in positive characteristic, a problem that stands as one of the key open challenges in stable motivic homotopy theory. This isomorphism, which in characteristic zero has been foundational for the development of the theory, remains elusive in the positive characteristic setting. Its resolution could potentially unlock new avenues of exploration, offering insights into the structure of motivic categories and their invariants.

  During their discussions, Khan and Bachmann explored a range of strategies to address this problem. They reviewed existing methods and examined how these might be adapted or extended to work in positive characteristic, where complications such as the failure of resolution of singularities create significant hurdles. The isomorphism is currently only known under the assumption of resolution of singularities or when inverting the prime $p$, leaving a substantial gap in understanding for more general cases.

  While their conversations led to some promising new ideas, they also underscored the intrinsic difficulty of the problem. The constraints imposed by positive characteristic introduce challenges that remain formidable, even with the latest tools and techniques available in motivic homotopy theory. Despite the lack of an immediate breakthrough, the dialogue between Khan and Bachmann was invaluable in refining the questions and clarifying the barriers that must be overcome. Their work represents an important step in the ongoing effort to address this fundamental issue, and their discussions highlighted the collaborative spirit and persistence required to make progress in such challenging areas of mathematics.

- Yun Shi and Adeel Khan engaged in thoughtful and exploratory discussions about a potential program aimed at adapting Khan's derived microlocalization machinery to the motivic framework of Grothendieck rings of varieties. Derived microlocalization, which has been a powerful tool in derived algebraic geometry, offers a sophisticated approach to studying the microlocal behavior of sheaves and their categories. By extending this machinery to the motivic setting, they aim to uncover new structural insights into Grothendieck rings, which serve as a cornerstone in enumerative geometry and related fields.

  The proposed adaptation would offer an alternative and potentially transformative perspective on motivic Donaldson-Thomas (DT) invariants, which are fundamental tools for counting objects in derived categories and understanding their enumerative significance. Motivic DT invariants generalize classical DT invariants by incorporating richer structures from motivic theory, allowing for deeper connections between geometry and algebra. By leveraging microlocalization techniques, Shi and Khan hope to provide a more refined understanding of these invariants, including their relationships with other key concepts in motivic and derived geometry.

Their discussions laid the groundwork for what could become a long-term collaborative project, requiring significant theoretical development and computational ingenuity. The idea of linking microlocalization and motivic DT invariants represents an exciting frontier in modern geometry, with the potential to open new pathways in the study of motivic structures and their applications. This collaboration exemplifies the innovative spirit of the workshop, where cutting-edge ideas are born from the exchange of perspectives and expertise.

- Lukas Bertsch, one of the attending PhD students, benefited significantly from the workshop, gaining valuable advice on several aspects of modern enumerative geometry. These insights arose particularly through his interactions with Martijn Kool, his designated mentor, whose guidance helped clarify challenging concepts and provided new perspectives on Bertsch's research questions. Kool's extensive expertise and thoughtful feedback encouraged Bertsch to explore more advanced methodologies and approaches, equipping him with tools to deepen his understanding of the field.

  In addition to these mentoring interactions, the collaborative and stimulating environment of the workshop sparked new ideas for Bertsch's research. One notable outcome was the prospect of a potential collaboration with Candace Bethea, another participant whose research interests align closely with his own. Their discussions laid the groundwork for exploring shared questions in enumerative geometry, opening up opportunities for joint work that could significantly enhance the scope and impact of his doctoral studies.

  These experiences highlight the profound value of mentorship and collaboration fostered during the workshop. For young researchers like Bertsch, such opportunities are pivotal, providing not only technical guidance but also the confidence and connections needed to thrive in the mathematical community. The exchange of ideas and formation of potential collaborations exemplify the workshop's success in supporting the next generation of researchers while advancing the broader goals of the field.

- During the workshop, an exciting possibility for fostering a deeper mentoring and support relationship involving Benjamin Kikwai and his colleagues in Kenya emerged through discussions with Balázs Szendrői, Martijn Kool, and András Jaramillo Puentes. These conversations revolved around building stronger academic connections and creating opportunities for collaborative growth that could bridge institutions across continents. The potential relationship aims to not only support Dr. Kikwai's research but also to enhance the academic environment for his colleagues and students in Kenya.

  One proposed idea involved reciprocal visits, with Kenyan researchers traveling to Europe to engage with institutions and collaborators, gaining access to additional resources, research networks, and advanced training opportunities. These visits would provide invaluable exposure to cutting-edge developments in the field and facilitate meaningful connections with European scholars. Similarly, European researchers expressed interest in visiting Kenya to participate in workshops, give lectures, and work directly with Kenyan academics. These visits strengthen local research capacities and provide mentorship that could benefit the wider academic community in the region. The first such visit will take place in September 2025 with Andrés Jaramillo Puentes visiting the University of Nairobi for the 2025 Nairobi Algebraic Geometry workshop, organised among others by Kikwai and Szendrői.

  Such an initiative holds significant promise for advancing the research capabilities of underrepresented regions while fostering a more inclusive and collaborative global academic environment. By creating a sustained exchange of ideas, knowledge, and resources, this relationship could serve as a model for other international partnerships. These discussions exemplify the workshop's commitment to fostering meaningful connections and supporting underrepresented researchers, extending its impact far beyond the immediate scope of the meeting.

- In addition to fostering new connections and collaborations, the meeting also provided a valuable opportunity for established research partnerships to reconvene and engage in in-depth discussions in person. For many researchers, this in-person interaction offered a level of immediacy and depth that is difficult to replicate in remote formats, allowing for the exchange of ideas in a dynamic and collaborative setting.

  One such collaboration involved Jake Solomon, Marc Levine, and Kirsten Wickelgren, who discussed their ongoing project aimed at finding a quadratic enrichment of the open WDVV equations. This

work builds on their previous research efforts, as documented in [1, 2, 3], and represents a significant step forward in enriching the mathematical structures underlying enumerative geometry and theoretical physics. Their discussions during the meeting helped refine key aspects of their approach and provided a clear direction for the next stages of their research.

Similarly, Maria Yakerson, Tom Bachmann, Elden Elmanto, and Adeel Khan took advantage of the workshop to reconnect and discuss their current projects as well as their broader aspirations for future research. This rare chance to align their visions and coordinate efforts in person helped to strengthen their ongoing collaborations and inspired them to consider new directions for joint work.

Jim Bryan and Thomas Brazelton continued to make progress on their collaborative research. In a stimulating conversation with Martijn Kool, they identified a novel connection between their project and recent developments in Donaldson-Thomas theory. This unexpected link not only deepened their understanding of their current work but also opened up new possibilities that could eventually lead to another paper. Such serendipitous discoveries exemplify the importance of bringing researchers together in a supportive and interactive environment.

Additionally, Jim Bryan and Mark Shoemaker discussed their ongoing efforts to explore a version of the crepant resolution conjecture related to the morphism $\mathrm{Hilb}^n(\mathbb{C}^3) \to \mathrm{Sym}^n(\mathbb{C}^3)$.

Their collaboration seeks to shed light on fundamental questions in the interplay between algebraic geometry and mathematical physics, and their discussions at the meeting helped clarify key challenges and refine their approach.

These interactions illustrate how the workshop not only facilitated the initiation of new collaborations but also provided an essential platform for existing partnerships to thrive. By enabling researchers to meet, discuss, and exchange ideas in person, the meeting fostered an environment that is critical for advancing both ongoing and future research endeavors.

# 5 Summary of the Report

The workshop, Enumerative Geometry Beyond Spaces, successfully brought together a diverse group of researchers to explore the evolving frontiers of enumerative geometry. This meeting highlighted emerging connections between classical enumerative geometry, derived categories, motivic homotopy theory, and tropical geometry, fostering interdisciplinary dialogue and collaboration.

Participants engaged in thought-provoking discussions on contemporary topics such as motivic enhancements of invariants, equivariant enumerative geometry, and new computational techniques. Keynote talks and informal exchanges illuminated ongoing research and uncovered novel directions, including the quadratic enrichment of enumerative counts and their applications to Donaldson-Thomas theory and Gromov-Witten invariants.

The event was especially impactful for early-career researchers, who benefited from mentorship opportunities and collaborative sessions. Several promising PhD students reported inspiring interactions with senior colleagues, sparking new ideas and potential partnerships. The inclusive hybrid format also enabled broader participation, allowing contributions from remote attendees while emphasizing the value of in-person connections.

Significant outcomes included the identification of innovative research questions, the strengthening of existing collaborations, and the initiation of new partnerships. Notably, discussions during the workshop inspired fresh approaches to foundational problems such as the Hopkins-Morel isomorphism in positive characteristic and the adaptation of derived microlocalization to motivic frameworks.

As the field continues to expand, the workshop underscored the importance of fostering global inclusivity and supporting researchers from underrepresented regions. Efforts to build sustained collaborations, such as those involving African mathematicians, exemplify the meeting's commitment to equitable academic opportunities.

Looking forward, participants expressed enthusiasm for future events to continue advancing the field. Themes such as tropical geometry, motivic cohomology, and their computational applications present exciting prospects for further exploration. The workshop served as a catalyst for groundbreaking research and demonstrated the transformative power of collaboration in mathematics.

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