Exponential Fields (24w5224)

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February 4 – 9, 2024.

1 Introductory Note:

Now that our above BIRS event has concluded, we are pleased to report below on our fruitful and productive (hybrid) workshop. We will thereby implicitly refer to any material (schedule, titles and abstracts, list of participants) already available on our workshop's website

$$https://www.birs.ca/events/2024/5-day-workshops/24w5224$$

2 Overview of the Field

The exponential function exp is one of the most important functions in the mathematical sciences. It is a transcendental function, which has important algebraic and analytic properties. Its behaviour on the real closed field of real numbers is very different from that on the algebraically closed field of complex numbers. In both cases, it is a group homomorphism from the additive onto the multiplicative group of the field (i.e. fulfills $exp(a+b) = exp(a)exp(b)$) but its kernel in the complex case is not trivial. Two major conjectures regarding the elementary theory of the real, respectively of the complex exponential function have been driving research in this area for the last few decades. A. Tarski's exponential function problem asks whether the elementary theory of the real numbers together with the real exponential function is decidable [22], whereas B. Zilber [24] proposed an axiomatization for the elementary theory of the complex numbers together with the complex exponential function. Both conjectures are intimately related to Schanuel's conjectures (see [17] and [24]) in transcendental number theory, and remain open until today.

Although exponentiation is often studied from an analytic point of view over the real or complex numbers, some of its properties are better understood in the more general setting of abstract fields of characteristic zero endowed with a homomorphism between the additive and multiplicative groups of a field. Such structures are called *exponential fields*. Exponential fields are both useful tools with established connections to the classical exponential function and objects of study in their own right. Recent work on exponential fields has concentrated in several different areas: the study of solutions to *polynomial-exponential equations*, various special cases of *conjectures in arithmetic geometry* on the algebraic behaviour of the complex exponential function, the abstract theory of exponential fields from a *model-theoretic* standpoint, and finally the *explicit constructions* of non-standard models of real and complex exponentiation using formal power series.

The purpose of our workshop was to bring together the (real and complex) researchers under one roof to explore the most recent progress and challenging open problems.

3 Presentation Highlights and Outcome of the Meeting

The first six talks of the workshop were survey lectures given by senior researchers on the six main themes we chose to highlight. These survey talks were designed to introduce participants to important goals and perspectives in each of these areas. This was important because the researchers at our conference think about the exponential function in a variety of different ways. We also asked survey speakers to discuss 1-2 guiding open problems during their lectures. The schedule was then organised around these six topics. We have uploaded the slides of the talks on the BIRS Workshop site under Description.

3.1 Models of Real Exponentiation, Fields of Generalized Power Series and Hardy Fields:

In this section, the focus is on ordered exponential fields, in particular models of real exponentiation as well as models of real exponentiation and restricted analytic functions. There are two important classes of objects that play parallel roles: Hardy fields on the functional side, and fields of generalized power series on the formal side.

The speakers of this section updated us on important recent progress and open questions in this area. An open question that emerged from the survey talk was whether the CH is required to prove that every maximal Hardy field is an omega-field, and indeed more generally, whether every ordered transexponential field is an omega field. A systematic study of the group of valuation preserving automorphisms of Hahn groups was presented, leading to a structure theorem decomposing the automorphism group into a semidirect product of two notable subgroups, the internal and external automorphisms. Many open questions regarding the normal subgroup of internal automorphisms were presented, and initiated further collaborations (see the section on outcome of the meeting below). The exponential is a formal power series that can be evaluated in various contexts. Relying on an axiomatic notion of sum of infinite families in certain algebras, a new correspondence between valuation preserving automorphisms and derivations on Hahn fields was also presented.

Survey talk by M. Matusinski, **senior talk** by D. Panazzolo, **early career researcher talks** by V. Bagayoko, N. Pynn-Coates and M. Serra.

3.2 Models of Complex Exponentiation, exponential-algebraic closedness, and Quasi-minimality

Zilber's exponential-algebraic closedness conjecture is motivated by model-theoretic questions on the structure of subsets of the complex numbers that are definable using polynomials and the exponential. It proposes sufficient conditions for solvability of systems of equations involving algebraic operations and the complex exponential map.

The speakers of this section updated us on important partial progress and remaining open questions. They reported on work in progress concerning the case of the conjecture in which the subvariety of the tangent bundle splits as a product of subvarieties, explaining how previously developed methods to approximate algebraic varieties can be used to tackle this problem. The strategy of passing to non-standard models of real exponentiation and restricted analytic functions, and looking at infinitesimal deformations for solving new cases of existential closedness is novel and seems very promising.

Survey talk by J. Kirby, **senior talk** by V. Mantova, **early career researcher talk** by F. Gallinaro.

3.3 Conway's Field of Surreal Numbers

Conway's field of surreal numbers **No** is a proper class endowed with an ordered field structure. It is a real closed field which contains the class of ordinals, and is an elementary extension of the field of real numbers. It also admits an exponential function defined by Gonshor, which makes it an elementary extension of the real exponential field. The field of surreal numbers is constructed by a recursive process, which endows the

surreals with a well-founded partial order, and the downward closed subclasses with respect to this partial order are called initial. The focus is on the initial substructures of **No**.

Survey talk by P. Ehrlich, **early career researcher talk** by E. Kaplan.

3.4 Pfaffian chains and o-minimality

Khovanski's work on Pfaffian chains has been fundamental in the breakthrough results in the last two decades concerning o-minimal expansions of the ordered field of real numbers. Motivated by recent work of our invited speakers listed below, a re-examination of the commonly used notion of pfaffian function is underway, comparing it to Khovanskiis original definition. Upon close inspection of Khovanskiis pfaffian functions, the notion of *nested pfaffian functions* has been coined, leading to renewed exploration of their class of germs at 0.

The workshop speakers of this section updated us on major recent developments on the Pila–Wilkie Theorem, a seminal result in o-minimality, which has many applications to diophantine geometry. The recent results concern effective, uniform versions of the Pila–Wilkie Theorem and its variants for sets definable using Pfaffian functions. These effective, uniform Pila–Wilkie bounds allow to derive several effective diophantine applications, including an effective, uniform Manin–Mumford statement for products of elliptic curves with complex multiplication. The use of the effective Pila-Wilkie Theorem in the setting of Pfaffian functions raises some new issues one needs to deal with, such as the definition of complexity of definable sets in terms of format and degree. Another recently raised issue is that solutions of differential equations *need not* be Pfaffian.

Survey talk by P.Speissegger, **senior talks** by J. Freitag, and M. Thomas, **early career researcher talk** by R. McCulloch.

3.5 Transexponential ordered fields

A celebrated open problem in this research area is whether there exists a transexponential o-minimal expansion of the ordered field of real numbers. In particular, in analogy to the growth dichotomy in the exponential case, it is reasonable to conjecture that an o-minimal expansion is either exponentially bounded or defines Kneser's transexponential solution to Abel's functional equation. Recently, it was shown that a class of transexponential expansions cannot be o-minimal.

Survey talk by C. Miller, **early career researcher talks** by S. Krapp.

3.6 Exponential algebra

Exponential polynomial rings over an exponential field have a special status within commutative algebra. Indeed, they are not Hilbertian, and some classical results, such as Hilberts Basis Theorem and Nullstellensatz fail.

The workshop speakers of this section updated us on a systematic investigation of exponential ideals (E-ideals) leads to a classification into three categories: prime E-ideals, E-ideals which are maximal as ideals, and E-ideals which are maximals *among* E-ideals. It turns out that these three notions are independent, unlike in the classical case. Radical E-ideal are currently under investigation, towards a characterization.

Survey talk by B. Zilber, **senior talks** by A. Fornasiero, F. Point and G. Terzo.

4 Early-Career Researchers

The structure of the schedule was largely focused on young participants. In particular, we organized the following special features during the first three days of the workshop in order to quickly bring less experienced participants up to speed and integrate them into the community:

4.1 Poster Session:

The following posters by PhD students have been exhibited in our poster event. We have uploaded the slides of these posters on the BIRS Workshop site.

1. A. Dmitrieva: Quasiminimality of a correspondence between two elliptic curves.
2. C. Eagles: Domination and Fibers.
3. C. Kesting: A dichotomy for T-convex fields with a monomial group.
4. L. Vogel: Purely transcendental extensions of formally real fields have the independence property.
5. L. Wirth: Archimedean Ordered Fields with the Independence Property.

4.2 Speed-Talks for Doctoral Students

We have given the PhD students the opportunity to present their dissertation research in a speed talk which lasted just a few minutes. All the students presented, and many of them spoke very positively afterward about having a relaxed and low-pressure way to introduce themselves to the community. The following presentations were given:

1. T. Borgard: Harmonic Functions Definable in O-minimal Structures.
2. J. Brown: Definable Quotient Spaces.
3. A. Dmitrieva: Quasiminimality and elliptic curves.
4. C. Eagles: Domination and Fibers.
5. C. Kesting: T-convex fields with a monomial group.
6. F. Vermeulen: Hensel minimality.
7. L. Vogel: Nonarchimedean Ordered Fields with the Independence Property.
8. L. Wirth: Archimedean Ordered Fields with the Independence Property.

4.3 Mentoring Session

In order to foster otherwise *unlikely intersections*, we included an evening dedicated to this event. The participants were divided in groups, each headed by a senior participant answering career related issues. We received enthusiastic feedback from both junior and senior participants, following the event.

This is the composition of the groups:

1. T. Scanlon: P. Andjar-Guerrero, N.Pynn-Coates
2. J. Freitag: R. McCulloch, J. Brown
3. C. Miller: A. Padgett, A. Dmitrieva
4. M. Thomas: F. Vermeulen, A. Block Gorman, S. Eterovic
5. D. Haskell: L. Vogel, A. De Mase

6. A. Fornasiero:, F. Gallinaro, R. Mennuni
7. G. Terzo:, L. Wirth, E. Kaplan
8. F. Point: C. Eagles, S. Van Hille
9. M. Serra: G. Fowler, C. Kesting
10. P. Speissegger: T. Borgard , J. Dobrowolski
11. V. Mantova: L. Krapp, V. Bagayoko

4.4 Social Interaction and Excursions

We asked for advice and proposed a few feasible destinations, as listed below. The excursions and social activities provided a less formal and beneficial environment for interaction between senior and junior participants.

1. Downtown bus to Sulphur Mountain Gondola / Banff Upper Hotsprings
2. 3 Nearby hikes/walks: Bow River Falls trail, Tunnel Mountain trail, Hoodoos Trail
3. Skiing: Some participants had cars and planned skiing trips.
4. Visual Arts Open Studio: The Banff Centre was hosting a Visual Arts Open Studio

5 Notes on the Physical and Virtual environments

We would like to share below a few general comments, in particular regarding the hybrid delivery format - what worked, what didn't and our suggestions for improvement:

5.1 Poster-session

Contributing a poster if a very constructive way of participation for young researchers, moreover it is often necessary condition to obtain student travel grants. We had our poster session in the large beautiful half circle room in the evening, and that was not good for two reasons: (i) the huge panels holding the posters cluttered the room and blocked the view and (ii) the electrical lights in that room are rather dim which made it hard to read the posters. We later discovered that the black(green)boards in both the bright lecture room and the bright seminar room are *magnetic* and large enough so that we could have posted the posters directly on them with little magnets! So, no need to rent poster holders and obstruct the magnificient mountain view!

5.2 Maintenance of computer equipment

We encountered the following problems and had some concerns:

1. We followed the system instructions on uploading the slides and it did not work. We were told that it is not possible, and would not be fixed quickly.
2. The microphones in the lecture room are placed in such a way that the virtual participant mostly cannot hear questions from the in-person participant.
3. It was frustrating that there was *only one screen* (where one could see the virtual speakers and their slides), but no screen where one could see the virtual participants. This was particularly disturbing when the virtual participants commented or asked questions.
4. An extra screen could also display the Zoom chat for in-person participants and speakers to see, which would make it easier for virtual attendees to communicate with in-person speakers and participants.

6 Objectives and Goals stated in our proposal

We were seeking to realise two main objectives, stated in our proposal for a 5-day workshop:

- to highlight recent developments on exponential fields arising from different perspectives,
- to facilitate collaborationsparticularly between early-career and experienced researchers that exploit synergies between new techniques.

As explained in the sections above, we tailored the schedule and the program accordingly. We are also particularly satisfied that a large number of participants were early career researchers, and belonging to minority groups, see updated list of participants below. Our strategy proved to be very fruitful, as testified by the selected individual feedback that we report in the next section.

7 Additional Outcome of the Meeting

Apart from the general progress and new directions (mentioned in the paragraph on presentation highlights), and the assessment of fulfilling our objectives, we would like to point out below some selected individual feedback that we got from some participants:

7.1 Progress on Joint Work

1. Kuhlmann (organizer), Bagayoko, Krapp, Serra (in-person speakers) and Panazzolo (online speaker) made significant progress on their joint work titled "Automorphisms and derivations on algebras endowed with formal infinite sums" while participating in the event. The preprint was posted shortly thereafter, see arXiv:2403.05827
2. Kaplan, Krapp and Serra continued their joint project "Automorphism groups of the surreal numbers", which had been initiated under the Ontario/Baden-Wrttemberg Faculty Mobility Program at McMaster University, Ontario, in September 2023. With all collaborators present at the BIRS event, these researchers from three different universities seized the rare opportunity to work face-to-face on their transcontinental project.
3. Dobrowolski, Gallinaro and Mennuni continued a joint work in progress on existentially closed difference valued fields.
4. Eterovic continued a collaboration with Vermeulen on p-adic exponentiation.

7.2 Initiating New Collaborations

1. Mennuni and Serra drafted a cooperation project on topological structures on automorphism groups of formal power series that will be initiated during a research stay of Serra in Pisa.
2. De Mase and Serra started a discussion on the valuation theoretic aspects of automorphism groups of formal power series, to be continued during De Mase visit to the University of Dortmund this spring.
3. Bradley-Williams, Kuhlmann and Kaplan initiated a research project dedicated to the notion of *definable spherical completeness*, they plan to continue this initial discussion by mutual visits in Germany during the upcoming academic year.

8 Participants (updated list)

8.1 In-Person

1. Padgett, Adele (McMaster University)
2. Eterovic, Sebastian (University of Leeds)
3. Kuhlmann, Salma (Universitt Konstanz)
4. Andujar Guerrero, Pablo (University of Leeds)
5. Bagayoko, Vincent (imj-prg)
6. Block Gorman, Alexi (The Ohio State University)
7. Borgard, Tyler (The Ohio State University)
8. Brown, Jordan (UC Berkeley)
9. De Mase, Anna (University of Campania)
10. Dmitrieva, Anna (University of East Anglia)
11. Dobrowolski, Jan (University of Manchester)
12. Eagles, Christine (University of Waterloo)
13. Fornasiero, Antongiulio (Universita degli studi di Firenze)
14. Fowler, Guy (Leibniz Universitt Hannover)
15. Freitag, James (University of Illinois at Chicago)
16. Gallinaro, Francesco (University of Freiburg)
17. Haskell, Deirdre (McMaster University)
18. Kaplan, Elliot (McMaster University)
19. Kesting, Christoph (McMaster University)
20. Krapp, Lothar Sebastian (University of Konstanz)
21. Mantova, Vincenzo (University of Leeds)
22. McCulloch, Raymond (University of Manchester)
23. Mennuni, Rosario (Universit di Pisa)
24. Miller, Chris (Ohio State)
25. Point, Francoise (Universite de Mons)
26. Pynn-Coates, Nigel (University of Vienna)
27. Scanlon, Thomas (University of California, Berkeley)
28. Serra, Michele (University of Dortmund)
29. Speissegger, Patrick (McMaster University)
30. Terzo, Giuseppina (Universit della Campania)
31. Thomas, Margaret (Purdue University)

32. Van Hille, Siegfried (McMaster University)
33. Vermeulen, Floris (KU Leuven)
34. Vogel, Lasse (University of Konstanz)
35. Wirth, Laura (University of Konstanz)

8.2 Virtual

1. D'Aquino, Paola (Universita della Campania)
2. Bradley-Williams, David (Czech Academy of Sciences)
3. Ehrlich, Philip (Ohio University)
4. Freni, Pietro (University of Leeds)
5. Jones, Gareth (University of Manchester)
6. Kirby, Jonathan (University of East Anglia)
7. Martinez, Nicolas (Universite de Bordeaux)
8. Matusinski, Mickal (Universite de Bordeaux)
9. Panazzolo, Daniel (Universit de Haute-Alsace)
10. Zilber, Boris (Oxford University)

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