Advancing stability through rigorous computation

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1 Overview of the Field

Many patterns in the natural world correspond to traveling or standing waves. Understanding nonlinear waves is important to comprehending the emergence and organization of these patterns. Determining which patterns dominate the system in the long time limit involves answering questions about the stability of traveling wave solutions of the system.

Traditionally, mathematical stability analysis has relied on a suite of well-established methodologies, such as the GrillakisShatahStrauss criterion [6, 7], which have successfully addressed a myriad of challenges and will remain invaluable. However many of these techniques are primarily applicable to the case of solitary waves, and are not directly applicable to the study of the stability of periodic waves. In the solitary wave case it is generally straightforward to compute the essential spectrum, and instability is determined by the locations of the discrete eigenvalues. In the case of periodic waves, however, there is only essential spectrum and determining its location is a highly non-trivial problem. Aside from exactly integrable equations like the Korteweg-de Vries [1, 2, 15] and cubic nonlinear Schrödinger equations the only cases in which we can explicitly determine the spectrum of these non-self-adjoint differential operators are in some perturbative limit, such as the limit of small amplitudes [**?**, 5, 8, 9, 12] or modulationally in a neighborhood of the origin in the spectral plane [3, 4, 10, 11].

However, the field is ripe for fresh perspectives and innovative insights. The past two decades have seen a growing interest in computer-assisted proofs (CAPs) based on a posteriori validation. These methods have a long history, going back to the work of Lanford, Eckmann, Koch, and Wittwer in the 1980s on the Feigenbaum conjectures [**?**, **?**, **?**]. Important examples of results proven with CAP include Tucker's proof of the existence of the

Lorenz attractor (14th of Smale's problem) in the late 1990's [?, ?], and Jaquette's resolution of Wright's and Jones' conjectures about the Wright's equation [?, ?, ?]. Note that functional analytic methods of CAPs for studying periodic orbits of differential equations date back to the work of Cesari on Galerkin projections for periodic solutions [?, ?]. Many other examples of recent and ongoing results involving the use of CAP can be seen in [?] and in Vol. #74 of the classical series, Proceedings of Symposia in Applied Mathematics, published by the American Mathematical Society focused on "Rigorous Numerics in Dynamics" edited by Jan Bouwe van den Berg and Jean-Philippe Lessard [?]. Yet more results are available from the virtual CRM CAMP in Analysis seminar series hosted by the Centre de Recherches Mathématiques (CRM) at the University of Montreal [?]. Examples of using CAP to establish rigorous results regarding stability of traveling waves are found in [?, ?, ?, ?].

2 Recent Developments and Open Problems

The workshop focused on the question of the stability of exact periodic traveling wave solutions. This focus was an attempt to address a gap in the literature: while there are several techniques available to prove that a solution is unstable, including modulational stability calculations based on a rigorous theory of long-wavelength perturbations [?, ?, ?, ?, ?] and calculations based on the collision of eigenvalues of opposing Krein signature [?, ?, ?]. However there did not really exist any general framework for proving even spectral stability of periodic traveling wave solutions, except for the case of exactly integrable equations where the inverse scattering transform provides a fairly explicit representation formula for the solution [?, ?, ?]. The (linearized) stability question for Hamiltonian equations is fairly subtle: due to the Hamiltonian symmetry the spectrum has the well-known fourfold Hamiltonian invariance: If λ is in the spectrum then so are $-\lambda, -\bar{\lambda}, \bar{\lambda}$. Because of this (spectral) stability is only possible if the spectrum lies entirely on the imaginary axis, and one faces the difficult challenge of determining if a solution that appears to be stable really is so, or there are unstable eigenvalues with small real part lying close to the imaginary axis.

3 Presentation Highlights

This was a focused research group, and so we did not have formal talks or presentations per se, but the AV infrastructure was great and made it much easier to share (for instance) the results of computations with the group.

4 Scientific Progress Made

The group focused on two problems, representing two major issues that can arise in the analysis of the stability of periodic traveling waves. The first problem that we considered was developing techniques to show that the spectrum of linearization of the generalized KdV equation

$$u_t + u^p u_x + u_{xxx} = 0$$

about a suitable traveling wave solution has spectrum consisting of the entire imaginary axis. We developed a framework for establishing spectral stability using techniques of rigorous numerical computation for the KdV and a number of other structural similar equations including the Benjamin-Bona-Mahony and Kawahara equations. Roughly the scheme is thus: neighborhoods of $\lambda = 0$ and $\lambda = \infty$ in the spectral plane are handled primarily through classical analytic techniques, with certain necessary conditions checked via rigorous computation. The neighborhood of $\lambda = 0$ is particularly tricky since the imaginary axis has spectral multiplicity three, and there is a collision of three eigenvalues, but we have developed techniques to deal with these issues. The intermediate region is handled through rigorous computation. After getting a numerical approximation to the eigenvalues in the intermediate region we use the Hamiltonian symmetry together with a uniqueness argument to prove that the eigenvalues that lie near to the axis must in fact lie exactly on the axis.

The second project considers the stability of periodic traveling waves of equations of Camassa-Holm type, in particular the b-Camassa-Holm equation family [13, 14]

$$u_t - u_{txx} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}.$$

Here the focus is on stability to co-periodic perturbations. This allows one to use the Grillakis-Shatah-Strauss theory, making the analysis somewhat simpler, but there are several complications present in this problem that do not occur for the generalized KdV. One thing, for instance, that is simpler in equations of KdV type is that the highest order derivative occurs linearly. This means that the linearized operator is diagonally dominant in Fourier space, a fact that underlies several key arguments about the large λ behavior of the problem. This is not true of the b-Camassa-Holm equation, where even for large λ the spectral problem is highly non-trivial.

Nevertheless we were able to use the Grillakis-Shatah-Strauss theory together with the Picard-Fuchs relations for period integrals on a Riemann surface to reduce the spectral stability of the periodic solutions to questions about the sign of a polynomial in two elliptic-type integrals, a question that is well-suited to rigorous numerical computations. There are still a couple of issues to be overcome near the boundary of the region in parameter space where periodic solutions exist, connected with the vanishing of the discriminant and the singular nature of the peaked-wave solutions, but we were able to solve some of those issues during the meeting. As a consequence, we are now confident that the challenges remaining for the parameter values near the boundaries will be overcome.

5 Outcome of the Meeting

We are very enthusiastic about the payoff from the focused research group meeting. Certainly one positive outcome of the focused research group meeting will be a greater awareness of the power of methods of validated computation in the nonlinear waves community. While these techniques are used in the community they are not yet as common or widespread as we believe that they have the potential to be. We hope that both projects will help to spread the word that validated computations make possible proofs that are just not feasible to achieve with "classical" techniques. Additionally we believe that both projects have introduced new ideas into the field, and the techniques that we have used here will make it possible to analyze the stability of many more nonlinear dispersive Hamiltonian equations.

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