Infinite-dimensional Geometry and Fluids: 23w5020

Martin Bauer (Florida State University)
Boris Khesin (University of Toronto)
Klas Modin (Chalmers University of Technology / University of Gothenburg)
Stephen Preston (Brooklyn College/CUNY Graduate Center)
Cornelia Vizman (West University of Timişoara)

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1 Overview of the Field

The overarching theme of the workshop was the use of differential geometric tools to model and analyze the governing equations of fluid dynamics. The idea of extending the terminology of manifolds to infinite dimensions can be traced back to the very roots of Riemannian geometry: in his Habilitationsschrift, Bernhard Riemann [31] discussed the possibility of extending his concepts from finite dimensions to the infinite-dimensional settings. Since then infinite-dimensional Riemannian geometry has matured into an active research area, where the interest in the field has been driven by a diverse set of applications in areas such as mathematical physics, hydrodynamics, computer vision, data science and medical imaging. Working in the infinite-dimensional setting comes with a variety of surprises and difficulties, as many results from finite-dimensional Riemannian geometry cease to exist. For example the well-known theorem of Hopf-Rinow [21], which connects metric completeness, geodesic completeness and existence of minimizers, has been shown to be wrong in a series of papers by Atkin [3, 4]. Understanding these purely infinite-dimensional phenomena has led to an active area of research, which comprised one of the fundamental topics of the proposed workshop.

Geometry of fluids.

The main motivation for this workshop is related to the fundamental role of infinite-dimensional Riemannian geometry in the governing equations of fluid mechanics. This approach, also called the Arnold approach to fluid dynamics, dates back to Arnold's seminal paper from 1966 [1], in which he recasts Euler's equations

$$\partial_t v + \nabla_v v = -\nabla p$$

for the motion of an inviscid incompressible fluid as the geodesic equation for the energy Riemannian metric on an infinite-dimensional Lie group, namely for the right-invariant L^2-metric on the group of volume-preserving diffeomorphisms. The geodesic property of the Euler equation is best seen in Lagrangian coordinates: for an incompressible flow $(t, x) \mapsto g(t, x)$ of fluid particles defined by its velocity field $v = \partial_t g \circ g^{-1}$, the Euler equation is equivalent to

$$\partial^2_{tt} g(t, x) = -(\nabla p)(t, g(t, x)).$$

The latter form represents a geodesic g on the group of volume-preserving diffeomorphisms $\mathrm{Diff}_\mu(M)$ regarded as a submanifold in $L^2(M, \mathbb{R}^n)$ (since the acceleration of the flow is opposite to the pressure gradient, hence always L^2-orthogonal to divergence-free fields). The motion of an ideal fluid filling an arbitrary

Riemannian manifold M is given by the same Euler equation, where $\nabla_v v$ stands now for the Riemannian covariant derivative of the field v in the direction of itself, and the geometric picture continues to hold.

The framework proposed by Arnold was much more general and applicable to geodesic flows on arbitrary (finite- or infinite-dimensional) Lie groups with respect to a suitable one-sided invariant Riemannian metric, and the resulting equations are referred to as Euler-Arnold equations. Over the past decades many other conservative dynamical systems in mathematical physics have been interpreted as Euler-Arnold equations, i.e., they have been found to describe geodesic flows on appropriate Lie groups. Examples include the Kirchhoff equation for a body in a fluid, the Hopf (or inviscid Burgers) equation, magnetohydrodynamics equations, Constantin-Lax-Majda-type equations, and the Korteweg-de Vries and Camassa–Holm equations. See Arnold-Khesin [2] for a survey.

As acknowledged by Arnold himself, his paper concentrated mostly on the geometrical ideas and not on the analytical difficulties that are inherent when infinite-dimensional manifolds are involved. In 1970, Ebin and Marsden [16] reconsidered this breakthrough geometric approach from the analytical point of view, looking at the Fréchet Lie group of smooth diffeomorphisms as an inverse limit of Hilbert manifolds, following some ideas of Omori [29]. The remarkable observation is that, in this framework, the Euler equation (a PDE) can be recast as an ODE (the geodesic equation) on these Hilbert manifolds. Furthermore, following their approach, if one can prove local existence and uniqueness of the geodesics (ODE), then the Euler equation is well-posed as well, and one is able to recover the well-posedness in the smooth (Fréchet) category using a no-loss-no-gain of regularity argument. In contrast to these results it was established by Himonas and Misiołek [19] that continuity of the data-to-solution map in Eulerian coordinates (i.e., of the Cauchy problem for the PDE) is the best one should expect. Similarly to Arnold's geometric picture, their analytic approach extends to many of the above mentioned PDEs; see for example Misiołek-Preston [26].

These well-posedness results are, however, only one example of analytic results that one can obtain from studying the geometric picture. In 2006 Ebin, Misiołek, and Preston [17] studied the Fredholm properties of the exponential map and proved that for two-dimensional fluids the Riemannian exponential operator is Fredholm. Furthermore, this exponential map turns out to be Fredholm quasiruled, i.e., it has a rigid global structure, as was established by Shnirelman [34] by using paradifferential calculus. These investigations also allowed them to establish a Morse Index theorem for 2D fluids and partial results for the group diameters and the Hopf–Rinow theorem. For 3D fluids it was shown by Misiołek and Preston [26] that the Fredholm property of the exponential map fails; it is however regained if one considers a slightly stronger metric of class H^s for $s > 0$. As a final example we would like to mention a Morse–Littauer theorem for 2D fluids, which was proved by Misiołek [24]: in any H^s neighbourhood of a conjugate point in $\mathrm{Diff}^s_\mu(M)$ for a two-dimensional M there always exist fluid configurations that can be reached from the initial configuration by two distinct geodesics in the same amount of time. These studies of the exponential maps shed new light on other properties of the hydrodynamical Euler equation and are again applicable to many other equations of interest in mathematical physics.

Fluids and optimal transport.

Otto's geometric picture [30] for the optimal transport problem directly links Arnold's approach for fluid dynamics and the field of optimal mass transport. Namely, given a Riemannian manifold M, equip the group $\mathrm{Diff}(M)$ of all diffeomorphisms of M with a "flat" Riemannian L^2-metric

$$\langle \partial_t g, \partial_t g \rangle_g := \int_M (v, v)\, g_*\mu\,.$$

for $g(t) \in \mathrm{Diff}(M)$, a vector field $v = \partial_t g \circ g^{-1}$, a Riemannian metric $(\cdot,\cdot)$, and a volume density μ on M. While this metric is not right invariant on the whole group $\mathrm{Diff}(M)$ of all diffeomorphisms, it is right invariant on the subgroup of volume-preserving ones, and it gives rise to the energy right-invariant L^2-metric there, thus recovering Arnold's geometric interpretation of the Euler equation.

To relate it to optimal mass transport, one regards the volume form μ as a reference density on M of unit total mass, and considers the projection $\pi : \mathrm{Diff}(M) \to \mathrm{Dens}(M)$ of diffeomorphisms onto the space of (normalized) smooth densities on M. The diffeomorphism group $\mathrm{Diff}(M)$ is fibered over $\mathrm{Dens}(M)$ by means of this projection π as follows: the fiber over a volume form $\tilde{\mu}$ consists of all diffeomorphisms g that push μ to $\tilde{\mu}$, $g_*\mu = \tilde{\mu}$.

Consider now the optimal mass transport problem: *find a map $g : M \to M$ that pushes the measure μ forward to another measure $\tilde{\mu}$ of the same total volume and attains the minimum of the L^2-cost func-*

tional $\int_M \mathrm{dist}^2(x, g(x))\mu$ *among all such maps.* We call the resulting distance on the space of densities the Kantorovich-Wasserstein distance. Brenier [10] established that the mass transport problem admits a unique solution for Borel maps and densities (defined up to measure-zero sets), called the optimal map $\tilde{g}$. Furthermore, a one-parameter family of maps $g(t)$ joining $g(0) = \mathrm{id}$ with the map $g(1) = \tilde{g}$, such that $g(t)$ pushes μ to $\mu(t) := g(t)_*\mu$ in an optimal way for every t, defines a geodesic $\mu(t)$ between μ and $\tilde{\mu}$ in the space of densities with respect to the Kantorovich-Wasserstein distance.

In the smooth setting, by results of Benamou-Brenier [8], the Kantorovich-Wasserstein distance is generated by a Riemannian metric on the space of smooth densities. According to Otto's seminal work, the bundle map $\pi : \mathrm{Diff}(M) \to \mathrm{Dens}(M)$ is a Riemannian submersion from the diffeomorphism group with the L^2-metric onto the density space with the Kantorovich-Wasserstein Riemannian metric. To summarize, one obtains a beautiful fiber bundle picture of the diffeomorphism group $\mathrm{Diff}(M)$ with a Riemannian metric, where the base $\mathrm{Dens}(M)$ (the horizontal geometry) is the main setting of optimal transport problems, while the (vertical) fiber $\mathrm{Diff}_\mu(M)$ carries the Euler equation of an ideal fluid, according to Arnold's interpretation.

2 Recent Developments and Open Problems

There are many long-standing open questions in hydrodynamics – turbulence, global well-posedness, and stability to mention a few. Infinite-dimensional geometry offers new ways to address many of these questions, already proven to be vital in the field of optimal transport, closely related to hydrodynamics. The overarching objective of the workshop was to drive these infinite-dimensional geometric techniques further, as well as exploring new, clever ways of using existing techniques. The specific objectives were:

- To extend exploring the underlying infinite-dimensional Riemannian and symplectic structures of fluid type equations from Euler's incompressible setting (as Arnold, Ebin and Marsden did) to include many other settings: compressible fluids, the nonlinear Schrödinger equation, geophysical and shallow water equations, Dirac equations, or equations of vortex sheets are such examples. This exploration has become possible only due to recent development of techniques such as infinite-dimensional Newton's equations, the study of Riemannian geometry of infinite-dimensional Lie groups and groupoids, as well as dual pairs in infinite-dimensional symplectic geometry.

- To establish even stronger links between optimal mass transport and hydrodynamics. This includes, but is not limited to, constructing proper analytical settings for infinite-dimensional reductions and submersions following the geometric approaches by Otto, Brenier, and Shnirelman.

- In the converse direction, to explore the subtleties of geometric analysis for infinite-dimensional manifolds and Lie groups of diffeomorphisms, as well as their quotients and embedding theorems, for which fluid-type equations are a source of peculiar properties and various counterexamples.

- To continue exploring the fruitful connections between information geometry and hydrodynamics, with a particular focus on infinite-dimensional Kähler geometry of the Madelung transform and possible applications to hydrodynamics – a quantum mechanics analogy related to bouncing droplets, pilot waves, and other effects.

- To share ideas and results about model equations for 3D fluid mechanics which have some common geometric features (including finite-dimensional approximations of the Lie algebra, one- and two-dimensional hyperbolic PDEs such as SQG and the De Gregorio equation, and geodesic equations in different Riemannian metrics) in order to further understand the fundamental principles underlying the major questions in the field.

These newly gained insights will not only have an impact on geometric, topological and analytic fluid dynamics, but will also be a new source for other applications of infinite-dimensional geometry, to such domains as mathematical shape analysis and geometric data science.

3 Presentation Highlights

Monday

The theme of Monday's talks was about topological and symplectic structures in Hamiltonian systems. Talks included explorations on metriplectic structures, gradient flows on the space of densities, and the difference between Hamiltonian vector fields and general divergence-free fields. We also had three "flash" talks from graduate students discussing their research in progress, as well as two tutorial sessions aimed at graduate students on both Euler-Arnold equations and optimal transport.

- Philip Morrison (University of Texas at Austin)

  On an inclusive curvature-like framework for describing dissipation: metriplectic 4-bracket dynamics

  Philip Morrison spoke about mathematical tools that generalize Poisson brackets which are used to give Hamiltonian descriptions of dynamical systems. Because such systems must be conservative, they cannot describe dissipative phenomena. Instead the dynamics can be described by a "metriplectic" bracket with four arguments rather than two. This generalization allows one to handle entropy production, and Prof. Morrison spoke about both finite-dimensional and infinite-dimensional examples, as in his recent paper with Updike [28].

- François-Xavier Vialard (Université Gustave Eiffel)

  On the global convergence of the Wasserstein gradient flow of Coulomb discrepancies.

  François-Xavier Vialard spoke about the gradient flow under the Wasserstein metric in optimal transport, and the issue of convergence of solutions to a minimizer. In some situations there is no local minimum, while in others one can prove linear convergence toward a solution and global regularity of the minimizer, as shown in the recent preprint of Boufadéne-Vialard [9].

- Albert Chern (University of California San Diego)

  Dynamics of fluid's cohomology

  Albert Chern spoke about an often-neglected issue in fluid mechanics: the effect of nontrivial cohomology on the equations of fluid mechanics. On a simply-connected domain, any divergence-free vector field can be written as the skew gradient of a stream function, but in nontrivial topology there may be extra harmonic terms. Prof. Chern demonstrated the effects of these harmonic terms, writing explicitly the equations for the coupling and the conservation laws that are generated, in addition to showing numerics of some interesting special solutions under nontrivial topology. The presentation was based on the recent paper [37].

- Ioana Ciuclea (West University of Timişoara)

  Flash talk: Coadjoint orbits of weighted nonlinear flags via dual pairs

  Ioana Ciuclea, a graduate student working with Cornelia Vizman, gave a short talk about her work on the group of diffeomorphisms that preserve a weighted nonlinear flag. In particular she spoke about aspects such as symplectic reduction, coadjoint orbits, and dual pairs. This was a report on her joint work with Prof. Vizman and Stefan Haller.

- Daniil Glukhovskiy (Stony Brook University)

  Flash talk: Pensive billiard system in vortex motion

  Daniil Glukhovskiy, a graduate student working with Theodore Drivas, spoke on his work on point vortices on surfaces as described in his paper [15]. On a plane a pair of opposite-circulation vortices will move in a straight line, and the question is what happens on more general domains, in particular domains in the plane with boundaries. This talk described some interesting results on the limit where two vortices approach each other: the dipole travels in a straight line until it reaches the boundary, at which point it splits into its two monopole components which move along the boundary with constant speeds in opposite directions.

- Sadashige Ishida (Institute of Science and Technology Austria)

  Flash talk: Exploration of an implicit representation for space curves

  Graduate student Sadashige Ishida gave the third flash talk, describing a new method of representing space curves in terms of level sets of two Clebsch variables rather than via explicit parameterization. This method avoids some issues related to singular points of curves, and allows one to consider a Marsden-Weinstein symplectic structure for such generalized curves.

- Cornelia Vizman (West University of Timişoara)

  Tutorial: Geometry of Euler-Arnold equations

  Workshop co-organizer Cornelia Vizman gave an introduction to the theory of Euler-Arnold equations. These arise as the equation in the Lie algebra that describes geodesics on a diffeomorphism group under a right-invariant metric, and typically they become partial differential equations that describe conservative continuum mechanical systems. Prof. Vizman explained in a talk directed to graduate students how to derive these equations, along with the general conservation law along a coadjoint orbit, and what this implies for reducing the degrees of freedom in general.

- Klas Modin (Chalmers University of Technology/University of Gothenburg)

  Tutorial: Wasserstein-Otto geometry

  Workshop co-organizer Klas Modin gave the second tutorial of the workshop, explaining aspects of optimal transport from the geometric point of view. In general the Wasserstein distance can be viewed, as expounded by Otto [30], as the quotient of a space of diffeomorphisms, where diffeomorphisms are related to each other if they generate the same density under pullback. The standard Riemannian metric on the space of all maps, where geodesics in the space of maps involve each particle moving along its own pointwise geodesic, generates the Wasserstein distance on the space of densities, and this quotient map is a Riemannian submersion. This fact has a variety of useful consequences, as expounded by Prof. Modin.

Tuesday

The main theme of Tuesday's session was the geometry of diffeomorphism groups, particularly those with right-invariant metrics, where the geodesic equation leads to an Euler-Arnold equation. Examples include the Euler equations for ideal fluids, the surface quasigeostrophic equation, the template matching equation in shape analysis, and the Hunter-Saxton equation. We also included three more flash talks from graduate students on similar topics.

- Gerard Misiołek (University of Notre Dame)

  On continuity properties of solution maps of the SQG family

  Gerard Misiołek gave an online talk in which he discussed the β-equation, a family of equations which interpolate between the surface quasiogeostrophic (SQG) equation and the 2D Euler equation (essentially involving the vorticity being given as some power of the Laplacian of a stream function). The SQG equation is considered a good two-dimensional model of the 3D Euler equation. In general all of these equations have similar properties, and for example Prof. Misiołek described how to extend an ill-posedness proof for the usual Euler equations to the more general family: the solution operator that takes an initial condition u_0 to a solution $u(t)$ is continuous, but cannot even be differentiable, much less smooth, expounding upon a paper with Truong Vu [25].

- Stephen Preston (Brooklyn College/CUNY Graduate Center)

  Liouville comparison theory for blowup of Euler-Arnold equations

  Stephen Preston described a new result [7] on using comparison theory to prove breakdown of smooth solutions to an Euler-Arnold equation arising from a Sobolev H^k metric on the diffeomorphism group of $\mathbf{R}^n$. In the Lagrangian point of view, this equation is an ODE, and the presentation described how comparison with the Liouville equation in a Banach space yields a general technique for proving breakdown, since the Liouville equation is an ODE in a Banach space with a simple explicit solution.

- Theodore Drivas (Stony Brook University)

  Irreversible features of the 2D Euler equations

  Theodore Drivas discussed some results from his paper [14] on the long-time behavior of two-dimensional ideal fluids. Although one expects ideal fluids to be theoretically reversible due to nondissipation, there are various ways in which a fluid becomes more complicated over time as particles twist around each other, in measurable ways. This paradoxical behavior is of primary importance in understanding the long-time behavior of fluids.

- Peter Michor (University of Vienna)

  Regularity and Completeness of half Lie groups

  Peter Michor discussed his recent paper [6] on the concept of "half Lie groups," a generalization of Lie groups which allows for the composition and inversion maps to not necessarily be smooth. This generalization is essential in practice, since for example in diffeomorphism groups of finite smoothness (C^k or H^s), the right translations will be smooth while the left translations will only be continuous, and these groups are the configuration spaces of fluids. Prof. Michor described a set of axioms for such spaces in general, and some results that hold in this level of generality.

- Alexander Shnirelman (Concordia University)

  Geometric structures on the group of volume preserving diffeomorphisms

  Alexander Shnirelman gave an online talk about the global topological properties of the group of volume-preserving diffeomorphisms. In two dimensions this group has a Fredholm exponential map [17], which implies that it behaves in many ways like a finite-dimensional manifold. Furthermore Prof. Shnirelman showed that it has the structure of a quasiruled map [34], which means that in some sense it can be approximated well by affine maps. He also discussed the connection between the intrinsic geometry of the space of volume-preserving maps and the extrinsic geometry (when viewed as a subspace of the space of all diffeomorphisms under the non-invariant metric). This is related to properties of weak solutions of the Euler equations, which can make sense in the L^2 closure of the volume-preserving diffeomorphisms despite not being elements of the group of smooth diffeomorphisms.

- Anton Izosimov (University of Arizona)

  Geometry of generalized fluid flows

  Anton Izosimov discussed a geometrization of generalized fluid flows, which are needed to solve the two-point minimization problem in the group of volume-preserving diffeomorphisms. Since minimizing geodesics may not exist between two volume-preserving diffeomorphisms, Brenier and Shnirelman introduced the concept of generalized flows, where fluid particles may split into clouds and later come back together. Prof. Izosimov described his paper with Boris Khesin [22] wherein a geometric structure is introduced on such flows: since they do not form a group, one must represent them as a groupoid. Some of the group-theoretic results of Arnold extend to this context, while others need to be modified, as described in this talk.

- Patrick Heslin (National University of Ireland)

  Geometry of the generalized SQG equations

  Patrick Heslin described the results he and his coauthors obtained in the recent paper [5] on the family of generalized SQG equations, as discussed by Gerard Misiołek earlier in the day. In particular he focused on the geometric aspects, including Fredholmness of the exponential map (which works for all parameters except in the actual SQG case). As the parameter in the family approaches the critical value corresponding to the SQG equation, the conjugate points along a geodesic can be seen to cluster until they collapse in the critical case, via explicit formulas.

- Levin Maier (University of Heidelberg)

  Flash talk: On Mañé's critical value for the Hunter-Saxton system

In the first flash talk of the day, graduate student Levin Maier discussed “magnetic geodesic flows,” an extension of geodesics that can be used to describe motion of charged particles in a magnetic field. Applying this to the Hunter-Saxton equation, a well-known and particularly simple infinite-dimensional Euler-Arnold equation, he described the geometry and the solution operator in this situation, showing in particular how to solve the two-point boundary problem.

- Luke Volk (University of Toronto)

  Flash talk: Simple Unbalanced Optimal Transport

  The second flash talk of the day featured graduate student Luke Volk describing his work with workshop co-organizers Boris Khesin and Klas Modin [23] on unbalanced optimal transport, which entails finding minimizing geodesics between two densities under a Wasserstein-type metric, but allowing for the second endpoint to vary (with a penalty for being far away from the desired endpoint). Some of the properties of this model can be best understood via finite-dimensional models.

- Archishman Saha (University of Ottawa)

  Flash talk: Symmetry and Reduction for Stochastic Differential Equations

  Graduate student Archishman Saha gave the final flash talk of the workshop, describing stochastic versions of Euler-Arnold equations. Because the geodesic equation is second-order on the diffeomorphism group, it is necessary to understand second-order stochastic differential equations (SDE) on manifolds in order to make sense of this. Mr. Saha described aspects of reduction and reconstruction in the second-order SDE models.

Wednesday

Wednesday’s session was focused on Hamiltonian aspects of the equations of 3D fluids, in particular the relationship with contact geometry and its uses in determining the topological structure of steady fluid flows, along with new Hamiltonian principles that can be used to derive equations of fluids with viscosity and heat.

- Eva Miranda (Universitat Politècnica de Catalunya)

  Navigating Uncharted Waters: Bridging Geometry and Fluid Dynamics

  Eva Miranda gave a far-reaching survey of the relationship between contact geometry and fluid dynamics. Beltrami fields (eigenfields of the curl operator) form a special class of steady solutions of the 3D Euler equation, and all such fields are the Reeb field of some contact form under a certain Riemannian metric. Hence the vast topological information we have about contact geometry can tell us about properties of steady fluids. Prof. Miranda surveyed a wide variety of results that she and her collaborators have obtained using this correspondence, some of which is summarized in her paper [11].

- Daniel Peralta-Salas (Instituto de Ciencias Matemáticas - Madrid)

  Obstructions to topological relaxation for generic magnetic fields

  Daniel Peralta-Salas spoke about a theorem he proved with Alberto Enciso [18] relating to magnetohydrostatic (MHS) equilibria, i.e., steady solutions of the equations of an ideal fluid with a coupled magnetic field. They specifically analyzed generic obstructions for a divergence-free vector field to be topologically equivalent to some MHS equilibrium, and showed that for any axisymmetric toroidal domain there is a locally generic set of divergence-free vector fields that are not topologically equivalent to any MHS equilibrium.

- François Gay-Balmaz (Ecole Normale Supérieure de Paris)

  Geometry and Numerics of Navier-Stokes-Fourier Fluids

  François Gay-Balmaz discussed a variational geometric setting for nonequilibrium thermodynamics, which extends the Hamilton principle and the geometric formulation of classical mechanics to include irreversible phenomena. This allows one to study the same geometric techniques used in ideal fluids for those that include irreversible processes such as viscosity and heat conduction. In addition he presented new work on discretizing this setting to yield structure preserving and thermodynamically consistent finite element schemes, particularly in the context of the Navier-Stokes-Fourier equations.

Thursday

In Thursday's session, we focused on analytical aspects of the Euler equations. In particular this includes the long-time behavior, such as the breakdown of smooth solutions of the Euler equations, one of the most famous open problems in the field. Long-time behavior of solutions also includes the counterintuitive phenomenon of inviscid solutions settling down to quasi-periodic solutions in a low-dimensional space (without viscosity, one would not expect such convergence), and several presenters discussed both special finite-dimensional families of solutions as well as quasi-periodic solutions in general.

- Javier Gómez-Serrano (Brown University)

  Self-Similar Blow up Profiles for Fluids via Physics-Informed Neural Networks

  Javier Gómez-Serrano described an exciting new field of research [36] aimed at finding smooth self-similar solutions for different equations in fluid dynamics. The approach is numerical and computational, using interval-based arithmetic in order to make the results rigorous. The primary innovation is in using physics-informed neural networks to find the solutions, and he showed that the framework is both robust and readily adaptable to several situations.

- Jiajie Chen (Courant Institute)

  Sharp functional inequalities related to singularity formation in incompressible fluids

  Jiajie Chen described his recent work with Thomas Hou [12] on the breakdown of both the Boussinesq equation and the 3D Euler equation, the latter of which has been a famous open problem, and the solution of which is of deep importance in fluid mechanics. In addition to the use of approximately self-similar smooth solutions which are shown to exist through numerics (and related to the work in the previous talk), sharp analytical inequalities in function spaces are crucial in the proof for estimating the nonlocal terms. He particularly explained those inequalities that are originally based in optimal transport and geometry.

- Milo Viviani (Scuola Normale Superiore Pisa)

  On the infinite-dimensional limit of steady states for the Euler–Zeitlin equations

  Milo Viviani discussed his work with Klas Modin [27] on the analysis of the long-time behavior of 2D fluids on a sphere. The Zeitlin model is a particularly successful finite-dimensional approximation of the infinite-dimensional geometry which uses the special structure of the sphere to obtain close analogues of the fluid structure in a Lie group of large but finite dimension. In particular the numerical simulations demonstrate the same behavior as those seen for 2D fluids: evolution towards simple quasi-periodic solutions, which likely represent the weak-* limits in Shnirelman's general theory.

- Gigliola Staffilani (Massachusetts Institute of Technology)

  Energy transfer for solutions to the nonlinear Schrödinger equation

  Gigliola Staffilani discussed a 2D version of the cubic defocusing nonlinear Schrödinger equation on the periodic (torus) domain. Energy transfer on different scales can be shown, but the dynamics of solutions differs in the case of the rational torus and the irrational torus.

- Jia Shi (Massachusetts Institute of Technology)

  On the analyticity of the Muskat equation

  Jia Shi described two results [32, 33] on the Muskat equation, which describes the interface of two liquids in a porous medium. The first result is that if a solution to the Muskat problem is sufficiently smooth (with the same viscosity and different densities), then it must be analytic except at a point where the fluids turn over. Her second result is analyticity also at the turnover points under some additional conditions.

- Francisco Javier Torres de Lizaur (University of Seville)

  Finite dimensional invariant manifolds of the Euler equation and their dynamics

Francisco Javier Torres de Lizaur described special solutions of the Euler equation of ideal fluids from his recent paper [13]. He showed that there are invariant manifolds of nontrivial, nonsteady solutions (generalizing simple cases such as steady flows where the invariant manifold is a point, or slightly more complicated cases such as the harmonic fields discussed earlier in the week by Albert Chern). If the Euler equation is on a manifold of sufficiently high dimension and the Riemannian metric can be specified arbitrarily, then the Euler equation on these invariant manifolds is general enough to capture all possible dynamical systems behavior.

- Peter Topalov (Northeastern University)

  Spatially quasi-periodic solutions of the Euler equation

  Peter Topalov described a framework for studying quasi-periodic maps and diffeomorphisms on $\mathbf{R}^n$. One application is proving that the Euler equation is locally well-posed in the space of quasi-periodic vector fields, which implies that the equation preserves the spatial quasi-periodicity of the initial data. In addition he showed some results on the analytic dependence of solutions on both time and the initial data, as detailed in his recent paper [35].

Friday

On the final day of the workshop, our speakers discussed some topics related to fluids that did not quite fit in earlier sessions.

- Sonja Hohloch (University of Antwerp)

  Hypersemitoric systems: Recent developments and advances

  Sonja Hohloch gave an overview of hypersemitoric systems based on her paper [20]. These are integrable Hamiltonian systems with two degrees of freedom on 4-dimensional compact symplectic manifolds, possibly with mild degeneracies, where one of the integrals gives rise to an effective Hamiltonian circle action. She gave an update on recent developments like important new examples, links with Hamiltonian S^1-actions, bifurcation theory, symplectic features like (non)displaceability of fibers, and steps towards a symplectic classification.

- Tsuyoshi Yoneda (Hitotsubashi University)

  Mathematical structure of perfect predictive reservoir computing for autoregressive type of time series data

  Tsuyoshi Yoneda discussed a particular form of machine learning called reservoir computing (RC), used for building prediction models for time-series data since it has low training cost, high speed, and high computational power. In particular he focused on some hidden structure in RC machine learning which leads to perfect prediction for autoregressive data as in his paper [38]. He argued that this structure is expected to be similar to the relationship between Lie algebras and Lie groups, which is used to derive the Euler-Arnold equations.

4 Scientific Progress and Outcomes of the Meeting

The workshop involved many high-quality presentations and discussion sessions, with enthusiastic participation of the audience. Moreover, particular highlights of the workshop were the tutorial sessions on the geometry of Euler-Arnold equations and the Wasserstein-Otto geometry for graduate students, as well as young mathematicians' flash talks. The small discussion rooms provided by BIRS were used extensively by groups of participants to brainstorm and work on their projects. We expect that many future papers and research programs have their beginnings rooted in this BIRS workshop. Participants also raised many interesting questions during and after the presentations that suggested new directions of research and different approaches to established topics, some of which were mentioned in this report. The workshop fostered interactions between several seemingly remote domains of hydrodynamics: geometric, numerical, and analytical. We anticipate that quite a few of these novel interactions and research questions will be influential in the ongoing development of these areas of infinite-dimensional geometry and fluid dynamics. Judging from the above observations and the informal feedback received by the organizers, the workshop was very successful at making new connections and fostering progress in these active research areas.

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