Spaces of manifolds: algebraic and geometric approaches

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1 Overview

This BIRS workshop brought together researchers with a variety of backgrounds and from different generations, working on *spaces of manifolds*. It took place during an exciting era in the field, which is still ongoing: old and new ideas—both on the more geometric side, as well as in the related field of higher algebra; especially in the study of K- and L-theory—are merging and the field is moving forward rapidly. Some of the resulting advances were the topics of the talks and discussions during the workshop. Before explaining this in more detail, we survey the state of the art of the field until about a decade ago.

2 Overview of the field

2.1 The question

One of the foundational questions of topology is: can compact smooth d-manifolds be classified, or—more ambitiously—can smooth compact d-manifold bundles be classified? Through the lens of algebraic topology, this is asking for an understanding of the homotopy type of the *moduli space of compact smooth d-manifolds* Man_d. Since the 1950s, the study of Man_d and related variants has continuously driven innovation in several mathematical fields, and even led to the development of new ones. For example, Thom's work on cobordism [Tho54] motivated the development of *stable homotopy theory*; ideas surrounding the s-cobordism theorem of Barden–Mazur–Stallings [Ker65] and the surgery theory of Browder–Sullivan–Novikov–Wall [Wal99] fostered advances in *algebraic K-theory and L-theory of rings*; and Waldhausen's work on pseudoisotopy theory [Wal78] led to the invention of algebraic K-theory of *ring spectra* and the rise of *higher algebra*—algebra over the sphere spectrum $\mathbb{S}$ instead of over the integers $\mathbb{Z}$ (see e.g. the work of Lurie [Lur17]).

Recent years have seem dramatic progress in the understanding of the moduli space of manifolds Man_d from a variety of different angles: new geometric insights as well as significant advances in higher algebra (in particular the study of algebraic K- and L-theory of ring spectra) have led to major applications to Man_d. The potential of such applications, however, is still yet to be fully understood. This workshop fostered applications of this kind, and also encouraged reversing the flow of information: breakthroughs in the geometric study of manifolds can for instance inspire new directions in the study of higher algebraic structures.

2.2 Classical approach

To put the aforementioned breakthroughs in context, we survey the classical approach to the study of the moduli space of manifolds Man_d.

Originally, manifolds were studied by comparison to topological spaces, or rather their homotopy types: given a topological space X, is it homotopy equivalent to a smooth d-manifold and if so, in how many ways? These questions informed some of the development of topology during the 20th century, and in high dimensions, the results proven as part of *surgery theory* culminated in a satisfying answer in terms of *algebraic L-theory* and *stable homotopy theory* [Wal99]. From the perspective of spaces of manifolds, surgery theory provides a description of the difference between the *moduli space of block-manifolds* $\widetilde{\mathsf{Man}}_d$ and the *moduli space of topological spaces* Top. The space $\widetilde{\mathsf{Man}}_d$ is a simplification of Man_d, but for the original question of classifying manifolds *up to diffeomorphism*, this simplification is harmless: $\widetilde{\mathsf{Man}}_d$ and Man_d have the same path-components. However, to achieve the more ambitious goal of classifying smooth d-manifold bundles, a full understanding of Man_d is needed.

Hence as a next step one tries to understand the difference between $\widetilde{\mathsf{Man}}_d$ and Man_d; this is the subject of *pseudoisotopy theory*. This "difference" is again encoded in the homotopy type of a space, and classical work of Cerf [Cer70] and Hatcher–Wagoner [HW73] determines its fundamental groups at various basepoints in terms of *algebraic K-theory* of the group rings $\mathbb{Z}[\pi_1(M)]$ of the fundamental groups $\pi_1(M)$ of manifolds M. Waldhausen [Wal78] then had the fundamental insight that in order to understand the actual homotopy type of this "difference" as opposed to just its fundamental group, one must replace the integers $\mathbb{Z}$ by the sphere spectrum $\mathbb{S}$, the group $\pi_1(M)$ by the loop space ΩM, and the ring $\mathbb{Z}[\pi_1(M)]$ by a *ring spectrum* $\mathbb{S}[\Omega M]$. His work, with later contributions by Weiss–Williams [WW88], resulted in a complete description of the space encoding the difference between $\widetilde{\mathsf{Man}}_d$ and Man_d in a range depending on d in algebraic K-theoretical terms, namely as the infinite loop space of a spectrum closely related to the algebraic K-theory $K(\mathbb{S}[\Omega M])$ of $\mathbb{S}[\Omega M]$. In Rognes' words [Rog18]: "this is one of the main reasons to be interested in the algebraic K-theory of ring spectra".

One of the limitations of this approach is that it requires an understanding of the K- and L-theory of $\mathbb{S}[\Omega M]$ and $\mathbb{Z}[\pi_1(M)]$, which is generally very difficult. The more crucial limitation, however, lies in the above *range depending on* d. Although Igusa's work on parametrised Morse theory [Igu88] shows that this range grows at least linearly with the dimension d, it is finite, so the above approach captures for instance at most finitely many of the homotopy groups of Man_d—a pessimist might say 0%.

3 Recent developments and talk highlights

We will now summarise some of the recent breakthroughs in overcoming both of the above limitations in our understanding of Man_d, and closely related other advances. Moreover, we will summarise the talks during the workshop which were related to these topics.

3.1 Cobordism categories and parametrised surgery theory

Madsen–Weiss' [MW07] celebrated solution of the Mumford conjecture on the cohomology of the moduli space of Riemann surfaces suggested a new method to access Man_d: instead of comparing Man_d to the moduli spaces of block-manifolds $\widetilde{\mathsf{Man}}_d$ and topological spaces Top, one compares it to the classifying space $B\mathrm{Cob}_d$ of the *d-dimensional cobordism category*. The homotopy type of the latter was determined in terms of stable homotopy theory by Galatius–Madsen–Tillmann–Weiss [GTMW09], and using this Galatius–Randal-Williams [GRW14, GRW17] proved that in even dimensions $d = 2n$ the homology of the components of Man_d have a complete description in stable homotopy theoretical terms *after* stabilising Man_{2n} by taking connected sums with $S^n \times S^n$. Their method of proof is often referred to as *parametrised surgery theory*.

3.1.1 Related talks during the workshop

In odd dimensions $d = 2n + 1$, a result of similar strength has not yet been established and even the statement of such an odd-dimensional analogue is still unclear. **Steinebrunner** pointed out in his talk that in the case $d = 3$, this can be approached using an higher-algebraic generalisation of the notion of a *modular* *operad*. Closely related, he spoke about his solution together with Boyd and Bregman [BBS24] of a conjecture of Kontsevich saying that the moduli space of manifolds diffeomorphic to a connected 3-manifold with nonempty boundary has the homotopy type of a finite CW-complex. **Randal-Williams** gave a talk on a result joint with Galatius [GR23] which shows that the space of homeomorphisms of a contractible d-manifold M relative to its boundary is for $d \geq 6$ contractible, which generalises the case $M = D^d$ of a closed disc, known as the Alexander trick. The proof involves ideas close in spirit to parametrised surgery theory. They also offered an alternative proof based on *embedding calculus*, which is the topic we will discuss next.

3.2 Embedding calculus

To use the results mentioned in Section 3.1 to obtain information about Man_d, one needs to understand how the homotopy type of Man_d is affected by various forms of stabilisation, e.g. by attaching $S^n \times S^n$ if $d = 2n$. The difference between the stabilised and the unstabilised variant can often be described in terms of embedding spaces, and Weiss [Wei21] realised that one can combine this with Goodwillie's multiple disjunction lemma [Goo90], conveniently packaged in the form of *embedding calculus* [Wei99], to analyse this difference. In the past, embedding calculus has been successful in the study of embedding spaces $\mathrm{Emb}(M, N)$ between manifolds of handle-codimension at least 3, but Weiss' insight combined with the programme outlined in Section 3.1 has opened the way to also apply it to access the homotopy type of Man_d, in principle *without being constrained by a range*. This strategy and variants of it have led to a variety of new results on Man_d, which were out of reach until recently (see e.g. [Kup19, KRW20, Kra22, KR21, BKK24]).

3.2.1 Related talks during the workshop

Embedding calculus featured in a number of talks during the workshop. **Boavida de Brito** described joint work with Weiss [BdBW24] which involved embedding calculus and an analogue of the torus trick for homeomorphisms of tori to show that the space of topological embeddings between Euclidean spaces is equivalent to the space of derived between the corresponding little discs operads if the codimension is at least 3. **Malin** spoke about the variant of embedding calculus resulting from replacing the category of spaces by that of spectra, and its relation to Koszul duality and Goodwillie calculus [Mal24]. **Muoz-Echniz** gave a talk about his work [Mu23] which combines Goodwillie's multiple disjunction lemma and the resulting bound on the concordance embeddings stable range from [GKK22] with an analogue for embedding spaces of Weiss–Williams' partial description of the difference between spaces of diffeomorphisms and block diffeomorphisms in terms of algebraic K-theory [WW01], to analyse the homotopy type of embedding spaces in a range. **Naef** spoke in his talk about work with Safranov [NS24] in which they exhibited a relation between string topology, embedding calculus, and the trace of the Whitehead torsion. **Kosanovic** explained a construction of classes in the space of embeddings of 1-manifolds into higher manifolds whose nontriviality can be detected using embedding calculus [Kos24]. **Arone**'s talk was in the context of functor calculus, which is related to embedding calculus. He explained his computation with Barthel, Heard and Sanders [ABHS24] of the Balmer spectrum for the category of n-excisive functors from spaces to spectra.

3.3 Configuration space integrals and graph complexes

Inspired by ideas from mathematical physics and low-dimensional topology, Kontsevich [Kon94] suggested a new source of characteristic classes of manifold bundles. These are defined in terms of integrals over configuration spaces and take value in the homology of combinatorial *graph complexes*. Watanabe [Wat09] used them to construct novel provably non-trivial families of smooth bundles with fibre a closed d-disc, far outside the range accessible to the classical approach to Man_d. Unlike most methods in geometric topology these ideas are insensitive to the dimension, which allowed him to also disprove the 4-dimensional Smale conjecture [Wat18]. Beyond Watanabe's work, the application and study of graph complexes (especially in combination with embedding calculus as mentioned above) has seen several recent advances (e.g. [FTW17, Wil15, ALV07, BM20]), most of them yet to be applied to the study of Man_d.

3.3.1 Related talks during the workshop

Watanabe spoke about an extension of his work, joint with Botvinnik, leading to a chain map from a larger part of Kontsevich's graph complex to the singular chain complex of the classifying space of the diffeomorphism group of even-dimensional discs, likely to lead to more nontrivial elements in the homology of this classifying space. **Stoll** explained his computation of the stable cohomology of block diffeomorphisms of certain products of spheres, in terms of a Lie graph complex, which is closely related to his earlier work on homotopy automorphisms [Sto24] and Berglund–Madsen's work before that [BM20]. His result relies on an algebraic rational model for the homotopy type of classifying spaces of homotopy automorphisms of Poincaré duality spaces due to Berglund and Zeman [BZ22]. **Berglund** gave a talk on this algebraic model, and he also explained a new source of characteristic classes constructed in terms of the algebraic models.

3.4 New foundations for Hermitian K-theory and L-theory

Recently, a team of nine mathematicians made significant advancements in the foundations and computations of Hermitian K-theory—a unification of algebraic K- and L-theory [CDH+23, CDH+20a, CDH+20b]. Their extension of classical methods from ordinary algebra to higher algebra allows for computations relevant to the study of Man_d that were previously out of reach, and it has the potential to put the classical approach to Man_d outlined in Section 2 and the more recent bordism-theoretic perspective from Section 2.2 into a common framework.

3.4.1 Related talks during the workshop

The workshop featured talks by three of the nine pioneers. **Steimle** spoke about how classical Nil-Nil theorems in algebraic K- and L-theory can be united to results in Hermitian K-theory in the setting of their above mentioned framework. **Land** talked about an application of their framework to the study of the behaviour of the signature in fibre bundles. In a different but related direction, **Hebestreit** gave a talk on theory of *homology manifolds* and explained an inconsistency between two foundational results in this area that he recently discovered together with Land, Weiss, and Winges [HLWW24].

3.5 Highly connected manifolds via stable homotopy theory

Coming from a different angle, Burklund, Hahn, and Senger [BHS23] recently managed to resolve a long-standing open question in Wall's classification of highly-connected manifolds from the 1960's [Wal62]. This question was reformulated in terms of stable homotopy theory by Stolz [Sto85], which they solved using a new perspective on Adams spectral sequence calculations via the theory of *synthetic spectra* [Pst23].

3.5.1 Related talks during the workshop

Senger gave a summary of some of these results and further advances in the study of highly-connected manifolds via stable homotopy theory joint with Burklund, Hahn, and Zhang [BHS23, BS20, BHS20].

3.6 New developments in algebraic K-theory

The study of algebraic K-theory has seen major breakthroughs in the past years, independent of the connections to geometric topology. Prominent examples include Land–Tamme's work on the behaviour of algebraic K-theory under pullbacks [LT19] or Nikolaus–Scholze's new perspective on trace methods [NS18]. These developments are likely to yield a better understanding of the K-theory of spherical group rings in the coming years, and thus eventually of the part of Man_d captured by the method discussed in Section 2.2. Moreover, recent developments in equivariant K-theory, in particular the interpretation of G-spectra as *spectral Mackey functors* [GM, BO15, Bar17, BGS20, MMO, GMMO23] have opened the way for generalisations of the classical approach to the moduli space of manifold Man_d to its equivariant generalisations Man_d^G for manifolds with a group action [MM19, MM].

3.6.1 Related talks during the workshop

During the workshop, **Malkiewich** spoke about one of the crucial ingredients in generalising the classical approach from Man_d to Man_d^G, an equivariant analogue of the *stable parametrised h-cobordism theorem* of Waldhausen–Jahren–Rognes [WJR13], joint with Goodwillie, Igusa, and Merling [GIMM23]. **Rovi** and **Semikina** spoke about a different relation between manifolds and algebraic K-theory. They explained a construction of a cut-and-paste K-theory of manifolds closely related to the classical notion of scissors congruence, alongside with several results on this object and connection to cobordism categories as featuring in Section 3.1 [HRS22, HMM+]. **Abouzaid** talked about joint work with Courte, Guillermou, and Kragh [ACGK] which involves relations between symplectic topology and algebraic K-theory as featuring in geometric topology.

4 Open problems brought up during the workshop

An important component of the workshop was an open problem session. The following list consists of the presented open problems; we hope that they will inspire further advances on the topics of this workshop.

4.1 Oscar Randal-Williams: Miller–Morita–Mumford classes

Given an oriented closed d-dimensional manifold M, an oriented fibre bundle $M \to E \xrightarrow{\pi} B$ with classifying map $T_\pi \colon E \to B\mathrm{SO}(d)$ for the vertical tangent bundle, and a characteristic class $c \in H^*(B\mathrm{SO}(d))$, one can form the generalised Miller–Morita–Mumford class

$$\int_\pi c(T_\pi) := \kappa_c \in H^{|c|-d}(B).$$

In even dimensions, after stabilising by copies of $S^n \times S^n$, the subalgebra of the rational cohomology of $B\mathrm{Diff}_\partial(M)$ generated by these classes is free polynomial algebra on κ_c for certain monomials c [GRW14]. Unstably, however, these satisfy many relations.

Question 4.1. *Do relations among the generalised MMM-classes κ_c's on a given M yield interesting (matric) Massey products on $H^*(B\mathrm{Diff}^+(M))$?*

Example 4.2. *For the mapping class group Γ_4 of a surface Σ_4 of genus 4, Tommasi computed the cohomology group $H^5(B\mathrm{Diff}^+(\Sigma_4);\mathbb{Q}) \cong \mathbb{Q}$ [Tom05]. Does this arise as a (matric) Massey product? This would give meaning to some odd degree cohomology classes on moduli spaces of curves.*

Question 4.3. *For the 4-dimensional smooth manifold $M = \mathbb{C}P^2$, Baraglia proved that there is a relation $p_1(T_\pi) = e(T_\pi) + \pi^*(\kappa_{e^2})$ in $H^4(B\mathrm{Diff}^+(M);\mathbb{Q})$ [Bar23]. The right side only depends on underlying fibration, so for smooth fibre bundles the first Pontryagin class only depends on the underlying fibration, not the smooth structure. Is this true topologically, or smoothly without using gauge theory? Or for the exotic topological manifold $*\mathbb{C}P^2$?*

4.2 Fabian Hebestreit: unimodular symmetric forms over $\mathbb{S}$

Using Waldhausen's matrix model [Wal85], the A-theory spectrum $A(*)$ can be described as $\mathbb{Z}\times B\mathrm{GL}_\infty(\mathbb{S})^+$. In [WW14], Weiss and Williams introduced a mixture $LA^v(*)$ of visible L-theory and A-theory (see also [CDH+20a]).

Question 4.4. *Is there a similar matrix model description for $LA^v(*)$, e.g. involving orthogonal groups?*

Another description of $A(*)$ is as group completion of a finitely generated projective modules over $\mathbb{S}$, via a cofinality statement. We know that $LA^v(*)$ is the group completion of visible unimodular forms over $\mathbb{S}$, but not enough to understand the required cofinality theorem.

Question 4.5. *Can we classify indefinite (visibly) unimodular symmetric forms on $\mathbb{S}^{\oplus g}$?*

This is not possible without "indefinite," as over the integers. A fact by Weiss–Williams is that you can lift forms over $\mathbb{Z}$ to $\mathbb{S}$, but for these lifts the relations usually fail, e.g.

$$E_8 \oplus \langle -1\rangle \cong 8\langle 1\rangle \oplus \langle -1\rangle,$$

holds over $\mathbb{Z}$ but not for lifts over $\mathbb{S}$ [WW14, Theorem 4.3], c.f. [CDH+20a, 4.6.4].

Question 4.6. *Is there an Hasse–Minkowski principle for indefinite (visibly) unimodular symmetric forms?*

4.3 Cary Malkiewich: Spaces of equivariant PL or topological h-cobordisms

In the talk, I discussed the spaces of equivariant or isovariant smooth h-cobordisms. You can also ask about the PL or topological settings. In the setting the following is known:

Theorem 4.7 ([BQ75, Rot78]). *In dimensions ≥ 6, isovariant h-cobordisms over M up to diffeomorphisms are given by $\oplus_{(H)}\mathrm{Wh}(M_H/WH)$, in smooth, PL, and topological settings. Here M_H is the stratum with isotropy H and $WH = NH/H$.*

Theorem 4.8 ([Ste88]). *Isovariant topological h-cobordisms over M up to homomorphisms inject into $\mathrm{Wh}_G^{\mathrm{TOP}}(M)$, but this does not split.*

Question 4.9. *Is the splitting true or false in the PL category? Either on π_0 or on the level of moduli spaces?*

In the topological case, Steinberger gave a 5-term exact sequence describing the isovariant topological h-cobordisms.

Question 4.10. *Is there a space-level Steinberger exact sequence?*

4.4 Jan Steinebrunner: Cyclic E_n-operads

More refined than operads are cyclic operads, in the sense that we can obtain the former from the latter by forgetting structure.

Question 4.11. *For which n is the E_n-operad equivalent to some cyclic operad, or cyclic ∞-operad?*

This is known for $1, 3, \infty$, possibly obstructed in even dimensions. This is equivalent to the following question, involving the ∞-categories $\mathsf{Disc}_n^{\mathrm{fr}}$ of finite disjoint unions of framed n-discs and framing-preserving embeddings between them and S of spaces.

Question 4.12. *Is there a functor $(\mathsf{Disc}_n^{\mathrm{fr}})^{\mathrm{op}} \to \mathsf{S}$ such that $F(\sqcup_2 D^n) \simeq *$ and $\mathrm{Emb}^{\mathrm{fr}}(\sqcup_k D^n, D^n) \times F(\sqcup_2 D^n) \xrightarrow{\sim} F(\sqcup_{k+1} D^n)$? Note that the domain is equivalent to $\mathrm{Emb}^{\mathrm{fr}}(\sqcup_k D^n, D^n)$, determining the homotopy type of $F(\sqcup_{k+1} D^n)$.*

Example 4.13. *For $n = 3$, you can use $F(\sqcup_k D^3) = \mathrm{Emb}^{\mathrm{fr}}(\sqcup_k D^3, SU_2)/SU_2$. This suggests $n = 7$ may also be special.*

4.5 Nils Prigge: Diameters and presheaves

Let Disc_d be the topological category or ∞-category of finite disjoint unions of d-discs and embeddings between these, and $\mathsf{PSh}(\mathsf{Disc}_d)$ the category of space-valued presheaves on this. A general theme of recent work is: given d-dimensional closed manifold M, how much information is retained by the presheaf $\mathrm{Emb}(-, M) \in \mathsf{PSh}(\mathsf{Disc}_d)$?

Theorem 4.14 ([CF79]). *For $d \geq 5$ there exists an $\epsilon > 0$ depending on Riemannian metrics on closed M and N, so that if $f\colon M^d \to N^d$ is a map with $\mathrm{diam}(f^{-1}(p)) < \epsilon$ then f is homotopic to a homeomorphism.*

Question 4.15. *Can one build a homeomorphism $\phi\colon M \to N$ from an equivalence $\mathrm{Emb}(-, M) \to \mathrm{Emb}(-, N)$, by gaining enough control to apply Chapman–Ferry's theorem?*

4.6 Manuel Krannich: Automorphisms of E_d

Let $\mathrm{Aut}(E_d)$ denote the derived automorphisms of the little d-discs operad in spaces. Little is known about the homotopy type of this group.

Question 4.16. *What is $\pi_0(\mathrm{Aut}(E_d))$? Is the map $\pi_0(\mathrm{Aut}(E_d)) \to \{\pm 1\}$ given by degree of the map $E_d(2) \to E_d(2)$ an isomorphism?*

Question 4.17. *Are the homotopy groups of $\mathrm{Aut}(E_d)$ countable groups?*

If the answer were yes, then $B\mathrm{Aut}^{\mathrm{id}}(E_d) \to B\mathrm{Aut}^{\mathrm{id}}(E_d^{\mathbb{Q}})$ would be a rationalisation, see [KK22]. The latter is amenable to computations via graph complexes, see [FTW17].

4.7 Pedro Boavida-de Brito: smoothing theory in dimension 4

Question 4.18. *Does smoothing theory work for 1-manifolds in 4-manifold? More precisely, is there a cartesian square as follows:*

$$\begin{array}{ccc} \mathrm{Emb}^s(M,M) & \longrightarrow & \mathrm{Emb}^t(M,N) \\ \downarrow & & \downarrow \\ \mathrm{Imm}^s(M,N) & \longrightarrow & \mathrm{Imm}^t(M,N) \end{array}$$

If this is true then, then $\mathrm{Emb}^t(\mathbb{R}^1, \mathbb{R}^4) \simeq \mathrm{Map}^h(E_1, E_4)$.

Question 4.19. *What is the connectivity of $- \times \mathrm{id}_{\mathbb{R}} \colon \mathrm{Top}(3) \to \mathrm{Top}_1(4)$ (the homeomorphisms of $\mathbb{R}^4$ which fixes $\mathbb{R}^1$ pointwise)?*

Question 4.20. *What are the automorphisms of the Goodwillie tower $P_n\mathrm{id}(\vee_m S^1)$ of a wedge of circles?*

String links (with m strands, up to concordance) seem to act on this Goodwillie tower in an interesting way. Indeed, by Biederman–Dwyer [BD10], $\pi_1 P_n\mathrm{id}(\vee_m S^1)$ is identified with the lower central series quotient of the free group $F_m/\Gamma_{n+1}F_m$. And the said action lifts the well known "Artin representation" of string links (with m strands, up to concordance) on the lower central series quotients of the free group F_m. In particular, all Milnor invariants factor through it.

4.8 Markus Land: homology manifolds

Question 4.21. *For all n sufficiently large, does there exist a PD-complex X which is n-connected but which is not equivalent to a homology manifold?*

The motivation is to understand the philosophical difference between homology manifolds and Poincaré complexes. One may think it is related to giving a PD diagonal, and this would say the difference is still quite large.

4.9 Victor Turchin: 2-knots in dimension 4

Question 4.22. *Are there smooth embeddings $S^2 \to S^4$ which are PL-isotopic to the standard inclusion but not smoothly non-isotopic to the standard inclusion?*

There is a slogan that in dimension 4, PL and smooth are the same but this is not such a case. The above question amounts to computing $\pi_2\mathrm{PL}_2(4)$. For comparison, $\pi_0\mathrm{Emb}(\mathrm{S}^3, \mathrm{S}^6) = \pi_3\mathrm{PL}_3(6)/\pi_3\mathrm{O}(3) \cong \mathbb{Z}$ generated by the Haefliger trefoil.

4.10 Connor Malin: Blow-ups

In embedding calculus and orthogonal calculus, the representable functors $\mathrm{Emb}(-, \mathrm{M})$ and $\mathrm{LinInj}(\mathrm{V}, -)$ play essential roles. In a precise way, the Taylor towers of these functors are controlled by the spaces $\mathrm{Emb}(\bigsqcup_{\mathrm{i}} \mathbb{R}^{\mathrm{n}}, \mathrm{M})$ and $\mathrm{LinInj}(\mathbb{R}^{\mathrm{i}}, \mathrm{V})$, as i varies, together with their actions by the categories of disc embeddings and vector space injections, respectively. The former has the homotopy type of framed configuration space, and the existence of the Fulton-MacPherson models of configuration spaces simplifies many technical arguments in embedding calculus. These models are given by the oriented blowup of $M^i = \mathrm{Map}(\mathrm{i}, \mathrm{M})$ at the non-injective maps. The existence of an analogous model of $\mathrm{LinInj}(\mathbb{R}^{\mathrm{i}}, \mathrm{V})$ is both an interesting geometric question and has applications to orthogonal calculus:

Question 4.23. *Does the oriented blow-up of the space of linear maps $\mathbb{R}^n \to V$ at the subspace of non-injective linear maps have a description functorial in linear embeddings $V \hookrightarrow W$?*

Here oriented blow-up is a bordification, a manifold with boundary whose interior is homotopic to the complement.

4.11 Ian Hambleton: Groups of inertial h-cobordisms

There is a group $\mathcal{H}(M^n)$ of h-cobordisms $W\colon M \rightsquigarrow M$ up to diffeomorphism relative to boundary, with composition given by concatenation. In [Kre01], Kreck computed its underlying set in the case of 1-connected 4-manifolds in terms of automorphisms of the intersection form.

Question 4.24. *What is the space-level version of Kreck's theorem?*

There is a map $\mathcal{S}(M \times I; \partial) \to \mathcal{H}(M)$ from the structure space.

Question 4.25. *Can we understand the map from the perspective of the answer of the previous question?*

Example 4.26. *If $\pi_1(M)$ has finite odd order then there is an exact sequence [HK04, Theorem B]*

$$0 \longrightarrow \mathcal{S}(M \times I; \partial) \longrightarrow \mathcal{H}(M) \longrightarrow \mathrm{Isom}(\pi_1, \pi_2, k, s_M) \longrightarrow 1.$$

4.12 Alexander Berglund: Characteristic classes and cohomology of arithmetic groups

Let X be a simply connected finite CW-complex and let $\mathrm{Baut}(X)$ denote the classifying space of the topological monoid of self-homotopy equivalences of X, aka the classifying space for fibrations with fiber X.

Theorem 4.27 (Berglund–Zeman [BZ22])**.** *The space $\mathrm{Baut}(X)$ admits a normal covering space $\mathrm{Baut}_u(X)$ with deck transformation group Γ_X such that*

(i) Γ_X is an arithmetic subgroup of a reductive algebraic group G over $\mathbb{Q}$, and

(ii) $\mathrm{Baut}_u(X)$ is Γ_X-equivariantly rationally equivalent to the geometric realization of a nilpotent dg Lie algebra $\mathfrak{u}$ of algebraic representations of G.

In particular, this implies that there is an isomorphism of graded algebras

$$H^*(\mathrm{Baut}(X); \mathbb{Q}) \cong H^*\big(\Gamma_X, H^*_{CE}(\mathfrak{u})\big). \tag{1}$$

The isomorphism (1) provides a link between characteristic classes of fibrations and cohomology of arithmetic groups, which should be investigated further.

Question 4.28. *What arithmetic groups can be realized as Γ_X for some X?*

The group Γ_X is the image $\mathrm{im}(h)$ of the representation in *semisimple homology*,

$$h\colon \mathrm{aut}(X) \to \mathrm{GL}\big(H_*(X; \mathbb{Q})^{ss}\big),$$

where the semisimple homology, $H_*(X;\mathbb{Q})^{ss}$, is the sum of the composition factors of $H_*(X;\mathbb{Q})$ viewed as a representation of the algebraic group $\pi_0\mathrm{aut}(X_\mathbb{Q})$. It is often equal to $H_*(X;\mathbb{Q})$. The space $\mathrm{Baut}_u(X)$ is the classifying space of the "Torelli monoid" $\ker(h)$. In view of Sullivan's result [Sul77, Theorem (10.3)(iv)], we expect that all arithmetic subgroups of reductive groups can be realized up to commensurability, but the question is what groups within a given commensurability class can be realized. For example, $\mathrm{GL}_n(\mathbb{Z})$ can be realized as Γ_X for $X = (S^3)^n$, but what about finite-index subgroups of $\mathrm{GL}_n(\mathbb{Z})$, e.g., the principal congruence subgroup of level N?

Question 4.29. *What is the geometric meaning of characteristic classes of fibrations constructed using* (1) *and known sources of cohomology classes of arithmetic groups such as automorphic forms?*

For example, for odd d, the rational cohomology of $\mathrm{Baut}(S^d \times S^d)$ can be expressed in terms of cuspidal modular forms of certain congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ by using (1) and the Eichler–Shimura isomorphism (see [BZ22]), but the geometric meaning of the characteristic classes associated to modular forms in this way is somewhat convoluted.

Question 4.30. *Suppose that X is a simply connected smooth compact manifold. Is there an analog of Theorem 4.27 for $B\mathrm{Diff}(X)$?*

4.13 Alexander Berglund: Poincaré duality fibrations and graph homology

Consider an oriented fibration $E \xrightarrow{\pi} B$ with fiber a simply connected Poincaré duality complex X of dimension m. In my talk, I sketched the construction of maps

$$\int_\pi^\alpha : H^*(E;\mathbb{Q})^{\otimes s} \to H^{*-m(1-n)-|\alpha|}(B;\mathbb{Q}),$$

associated to homology classes α in Kontsevich's hairy Lie graph complex $\mathrm{GC}^m(n,s)$.

If X is a smooth manifold, we can use these to define characteristic classes of oriented block bundles with fiber X by setting

$$\widetilde{\kappa}^\alpha_{c_1,\ldots,c_s} = \int_\pi^\alpha c_1(T_\pi) \otimes \cdots \otimes c_s(T_\pi) \in H^*(B\widetilde{\mathrm{Diff}}(X);\mathbb{Q}) \tag{2}$$

for $c_1,\ldots,c_s \in H^*(BO;\mathbb{Q})$, where π is the universal oriented block bundle over $B\widetilde{\mathrm{Diff}}(X)$ and T_π is the stable fiberwise tangent bundle. We can also consider the pullbacks of these classes along the canonical map $I\colon B\mathrm{Diff}(X) \to B\widetilde{\mathrm{Diff}}(X)$,

$$\kappa^\alpha_{c_1,\ldots,c_s} = I^*(\widetilde{\kappa}^\alpha_{c_1,\ldots,c_s}) \in H^*(B\mathrm{Diff}o(X);\mathbb{Q}). \tag{3}$$

These generalize the well-known Miller–Morita–Mumford classes in the sense that

$$\kappa^{\epsilon_{n,s}}_{c_1,\ldots,c_s} = \kappa_{e^n c_1 \ldots c_s} \tag{4}$$

for certain classes $\epsilon_{n,s} \in H_0(\mathrm{GC}^m(n,s))$.

Now consider the manifold $W_g = \#^g S^d \times S^d$ and let $D^{2d} \subset W_g$ be an embedded disc. Building on [BM20], we can show that classes of the form (2) freely generate the stable rational cohomology algebra of $B\widetilde{\mathrm{Diff}}(W_g, D^{2d})$ (and all non-trivial α are needed). In contrast, Galatius–Randal-Williams [GRW14] show that the stable rational cohomology algebra of $B\mathrm{Diff}(W_g, D^{2d})$ can be freely generated by classes of the form (4) only.

This suggests that the classes (3) should satisfy a number of algebraic relations that are not satisfied by the classes (2). Whatever these relations are, they give obstructions for "unblocking" block bundles.

Question 4.31. *What algebraic relations do the classes $\kappa^\alpha_{c_1,\ldots,c_s}$ satisfy and how much of the cohomology of $B\mathrm{Diff}o(X)$ do they account for?*

A naive guess would be that the classes (3) are zero whenever $|\alpha| > 0$, but we do not even know whether this is the case for $X = W_g$.

5 Further program highlights

In addition to the talks, the workshop featured several other components.

1. *Online learning pre-seminar.* In the run-up to the workshop, we organized an online seminar which covered the background knowledge required to get the most out of the talks. The talks were given by pairs of junior participants, mentored by a more senior expert attending the workshop. It was highly successful in improving the interaction between participants, allowing the speakers to assume more prerequisites during the workshop, and hence allowing the talks during the workshop to focus on cutting-edge research. The pre-seminar was open to any interested party and well-attended by a wider mathematical community than the group of workshop-attendees. We expect that the connections fostered through this mentoring program will be helpful for the participating early career participants, especially those from under-represented groups.

2. *Problem session.* To set the stage for the next decade of innovations in the study of manifolds through higher-algebraic invariants, we organized a problem session on the Thursday afternoon of the workshop. Having asked the participants in general and the speakers in particular to think about open problems in advance, we collected them and included them in this report.

3. *Career panel.* During the Thursday evening of the workshop we organized an informal career panel. This presented an opportunity for graduate students and postdocs to ask questions about the job search process and to learn from the first-hand experiences of those who have recently gone through this process, as well as experiences of more senior participants who serve on committees and can provide an alternative perspective. It was attended by nearly all junior participants and was very well received.

6 Outcome of the Meeting

The objective of this meeting was to bring together experts working on moduli spaces of manifolds, including those more geometrically driven and those working on higher algebraic structures which can be applied to manifolds. Our intention was for them to exchange not only results but also speculations and problems, so as to spur innovation and collaboration. We believe the workshop was successful at this, as many of the talks led to lively discussions about the "big picture". This workshop was timely; as indicated in this report, while recent years have seen a number of important breakthroughs, the complete picture remains mysterious. Our hope is that this workshop set the stage for the next decade of innovations in this direction. A further objective was to decrease the gap between textbooks and the forefront of research—inevitably given the amount of activity in the field of this workshop—which can be hard to bridge for graduate students and early-career researchers. The workshop and in particular its adjacent learning seminar were a valuable resource in doing so, as during the latter we brought younger participants—and any other interested parties—to the level where they could understand the talks during the workshop and see how they fit into the big picture.

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