Derived Categories, Arithmetic, and Reconstruction in Algebraic Geometry

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1 Overview of the Field

The workshop focused on various topics which all in some way concern the problem of reconstruction in algebraic geometry. The basic problem is to understand various properties of an algebraic variety, such as its birational class, whether it has a rational point, etc., from certain invariants which may be simpler to understand.

The main foci of the workshop were threefold:

(i) Derived Torelli theorems.

(ii) Derived categories, cohomology, and arithmetic applications.

(iii) Anabelian geometry and reconstruction results.

(i) It is well-known that the derived category of coherent sheaves is not a faithful invariant of algebraic varieties in the sense that one can have two non-isomorphic algebraic varieties with equivalent derived categories. This is, in fact, the origin of the subject of Fourier-Mukai transforms and derived equivalences. At the same time, the derived category appears to be a strong invariant. Recent conjectures of Lieblich and Olsson [9] predict that in many cases the derived category, together with some additional information related to numerical Chow groups, should suffice to recover the birational equivalence class of an algebraic variety. This is also closely related to an older conjecture of Orlov that varieties with equivalent derived categories should have isomorphic Chow motives.

(ii) Related to (i) is the basic question of what information is encoded in the derived category, or better the ∞-category, of coherent sheaves on a variety. A lot is known, especially in regards to cohomology. For example, one can show that certain twisted versions of any of the standard cohomology theories (Betti, de Rham, étale, crystalline) are preserved under derived equivalence. More recently, in the work of Bhatt, Morrow, and Scholze [3] it was shown how to relate topological Hochschild homology, obtained from the ∞-category of perfect complexes on a smooth projective variety over a perfect field of positive characteristic, to the de Rham-Witt complex, and therefore also crystalline cohomology. These results lead to numerous questions related to understanding the arithmetic information encoded in the derived category and/or the cohomology of algebraic varieties.

(iii) Inspired by work on derived categories, Kollár, Lieblich, Olsson, and Sawin [8] have obtained several new results about recovering the isomorphism class of a variety of dimension ≥ 2 purely from the Zariski topological space, and, in some cases, the Zariski topological space along with its divisor class group. Topaz has also obtained results which recover function fields of higher-dimensional algebraic varieties from data encoded in certain cohomological structures [12][13]. These results are closely related to work of Zilber on curves and their Jacobians [14], and to work of Bogomolov-Korotaev-Tschinkel [4], Bogomolov-Tschinkel [5][6], Pop [10], Pirutka-Cadoret [7], and others, from anabelian geometry. There are some further connections as well with the newly developed theory of exodromy due to Barwick, Glasman and Haine [2]. One of the main goals of the workshop was to bring together researchers working around such reconstruction results from these different points of view.

2 Discussion of open problems

An important component of the workshop was discussion around open problems in the field. To facilitate discussion around this the workshop included two scheduled events. Following the practice at many conferences a "problem session" was held on Tuesday (second day) of the workshop wherein participants shared problems, ideas, speculations, etc., about important open problems in the field. This was preceded by a session "organized discussion" at the end of Monday (first day) wherein participants were broken into small groups and discussed problems they might highlight for the entire group in Tuesday's problem session. This organized discussion served two purposes: (1) It focused Tuesday's problem session; (2) It helped foster interactions among participants and especially made it easier for junior participants to engage with the group. This approach appears to have worked well. Appended to this report is a writeup of the problems discussed during the problem session.

3 Hybrid format

This workshop was delivered in a hybrid format with 20 in-person participants and 14 hybrid participants. While the technological facilities could allow substantially more remote participants the organizers deemed it important to retain the workshop atmosphere for both in-person and remote participants and therefore extended invitations to participants who would take an active part (of course, with the lectures posted online this did not present an obstacle for others to view the research presentations). It should also be noted that for various pandemic-related reasons several participants (and one organizer) had to change their travel plans at the last minute and switch from in-person attendance to remote attendance.

4 Presentations

The meeting included presentations in various formats including three survey talks, one-hour research presentations, and 10-15 minutes "lightning talks" by more junior participants. While most presentations were in-person the workshop also included several talks delivered remotely. With this format, all in-person participants were given an opportunity to speak.

4.1 Survey Lectures

1. Lieblich, Max (U Washington): *Reconstruction*.

   Abstract: I will discuss various general types of reconstruction or characterization results, ranging from basic algebraic geometry to new results obtained in joint work with Kollr, Olsson, and Sawin, and with Alper and de Jong. In each case, one attempts to extract a complete algebraic invariant – for example, a ring, a group, a Hodge structure, an abelian category, a tensor triangulated category, a motive, a topos, an abstract projective structure, a poset – from a geometric object. Each type of invariant has successes and failures.

2. Stix, Jacob (Goethe-Universität Frankfurt), *An invitation to anabelian Geometry*.

   Abstract: This will be a survey talk about anabelian geometry.

3. Hassett, Brendan (Brown University): *Derived categories and rational points*.

   Abstract: This is a survey of the relationship between derived equivalence, the existence of rational points, and other arithmetic properties. Given smooth varieties X and Y over a field k, assume to be derived equivalent over k, how are the k-rational points of X and Y related? We summarize what is currently known for K3 surfaces as well as some important recent results of Addington-Antieau-Honigs-Frei in higher dimensions. (joint with Tschinkel)

4.2 One-hour research lectures

Talks indicated as "Zoom lecture" were delivered by remote participants via Zoom.

1. Ballard, Matthew (University of South Carolina), *Generation in prime characteristic/a GUT for flops*.

   Abstract: A double feature talk. During the first half, Ill discuss how, when, and where does the Frobenius pushforward generate the derived category. This is joint with Pat Lank. In the second half, Ill introduce a general construction which extracts integral kernels form flips and show how it gives an equivalence for stratified Mukai flops. This is joint with Nitin Chidambaram and David Favero.

2. Bragg, Daniel (University of California Berkeley), *A Stacky Murphys Law for the Stack of Curves*.

   Abstract: We show that every Deligne-Mumford gerbe over a field occurs as the residual gerbe of a point of the moduli stack of curves. Roughly speaking, this means that the moduli space of curves fails to be a fine moduli space in every possible way. This is joint work with Max Lieblich.

3. Cadoret, Anna (Sorbonne Universit), *Degeneracy locus of ℓ-adic local systems - an anabelian approach*.

   Abstract: Let X be a smooth variety over a number field k. I will review the general heuristic underlying our expectation that the set of k-rational points in the degeneracy locus of a p-adic local system whose geometric monodromy is semisimple (perfect?) are not Zariski-dense and recall some of the motivations for this question. This heuristic relies on a geometric conjecture and (a weak form of) the Bombieri-Lang conjecture. In the second part of the talk, I will give an hint of the proofs of the geometric conjecture when X is a curve (joint with A. Tamagawa) and a product of 2 curves.

4. Frei, Sarah (Rice University), *Symplectic involutions of hyperkahler fourfolds of Kummer type*.

   Abstract: The middle cohomology of hyperkahler fourfolds of Kummer type was studied by Hassett and Tschinkel, who showed that a large portion is generated by cycle classes of fixed-point loci of symplectic involutions. In recent joint work with Katrina Honigs, we study symplectic fourfolds over arbitrary fields which are constructed as fibers of the Albanese map on moduli spaces of stable sheaves on an abelian surface. We have extended the results of Hassett and Tschinkel and characterized the Galois action on the cohomology. We do this by giving an explicit description of the symplectic involutions on the fourfolds. This has natural consequences for derived equivalences between Kummer fourfolds.

5. Gaulhiac, Sylvain (University of Alberta), *Towards tempered anabelian recovery of lengths in Berkovich geometry*.

   Abstract: In the framework of non-archimedean Berkovich geometry, questions of anabelian type are best answered using the so-called tempered fundamental group, introduced by Yves Andr. It is now known that in many cases, the tempered group of a curve determines its skeleton as a graph. This graph also has a natural metric. Does the tempered fundamental group determine the length of each edge? If the answer is positive in some cases for algebraic curves due to some work of Lepage, it remains unknown otherwise, even for the most simple curves : annuli. I will present a partial result in this direction, using some interesting methods of splitting radius of torsors and resolution of non-singularities.

6. Haine, Peter (UC Berkeley), *Galois-theoretic reconstruction of schemes and Exodromy*.

Abstract: The classical theorem of Neukirch and Uchida says that number fields are completely determined by their absolute Galois groups. In this talk, well explain joint work with Clark Barwick and Saul Glasman generalizing this reconstruction result to schemes. Given a scheme S we construct a category $Gal(S)$ that records the Galois groups of all of the residue fields of S (with their profinite topologies) together with ramification data relating them. Well explain why the construction $S \mapsto Gal(S)$ is a complete invariant of normal schemes over a number field. The category $Gal(S)$ also plays some other roles. For example, just like how there is a monodromy equivalence between representations of tale fundamental group and local systems, there is an equivalence between representations of the category $Gal(S)$ and constructible sheaves. This invariant also gives rise to a new definition of the tale homotopy type.

7. Huang, Jesse (University of Alberta), *Homotopy Path Algebras*.

   Abstract: In this talk, I will define a basic class of algebras, "homotopy path algebras", and explain the relation between a homotopy path algebra and entrance/exit paths on an appropriately stratified classifying space that naturally gives a cellular resolution of the diagonal bimodule. An earlier result of mirror symmetry due to Bondal-Ruan and certain Berglund-Hübsch-Krawitz mirrors can be recovered as an application. I will also discuss some results on minimal cellular resolutions of diagonal bimodules. This is based on joint work with David Favero.

8. (Zoom lecture) Markman, Eyal (University of Massachusetts Amherst), *Rational Hodge isometries of hyper-Kahler varieties of $K3[n]$-type are algebraic*.

   Abstract: Let X and Y be compact hyper-Kahler manifolds deformation equivalence to the Hilbert scheme of length n subschemes of a K3 surface. A cohomology class in their product XxY is an analytic correspondence, if it belongs to the subalgebra generated by Chern classes of coherent analytic sheaves. Let f be a Hodge isometry of the second rational cohomologies of X and Y with respect to the Beauville-Bogomolov-Fujiki pairings. We prove that f is induced by an analytic correspondence. We furthermore lift f to an analytic correspondence F between their total rational cohomologies, which is a Hodge isometry with respect to the Mukai pairings, and which preserves the gradings up to sign. When X and Y are projective the correspondences f and F are algebraic.

9. Sankar, Soumya (Ohio State University), *Curve classes on conic bundle threefolds and applications to rationality*.

   Abstract: Conic bundles are a geometrically rich class of varieties. In the 70's, Beauville showed that over an algebraically closed field, the group of algebraically trivial curve classes on a conic bundle threefold is isomorphic to the Prym variety of a double cover naturally associated with it. In joint work with Sarah Frei, Lena Ji, Bianca Viray and Isabel Vogt, we study curve classes on (geometrically standard and geometrically ordinary) conic bundle threefolds over arbitrary fields of odd characteristic. We then use the description of these classes to study the rationality of such varieties. Indeed, Hassett-Tschinkel and Benoist-Wittenberg introduced an obstruction to rationality, namely the intermediate Jacobian torsor obstruction, closely related to the structure of the group of curve classes on threefolds. We show that this obstruction is insufficient to characterize rationality.

10. (Zoom lecture) Kaushal Srivastava, Tanya (IIT Gandhinagar), *Counting Twisted Fourier Mukai partners of an ordinary K3 surface*.

    Abstract: The talk is based on joint work with Sofia Tirabassi. I will be discussing tame twisted K3 surface over an algebraically closed field of positive characteristic and counting its untwisted FM partners. On the way to the counting results, we will also discuss that every tame twisted Fourier Mukai partner of a K3 surface of finite height is a moduli space of twisted sheaves over it.

11. (Zoom lecture) Zilber, Boris (Oxford University), *Arithmetic geometry through the eyes of model theory*.

    Abstract: I am going to discuss a progress in an ongoing project (since approx 2000) which aims to formalise the notion of an analytic covering space of a complex algebraic variety in such a way that the formal cover is unique up to abstract isomorphisms (categorically axiomatised). It turned out deeply dependent on and related to both arithmetic geometry and transcendental number theory. Model-theoretic geometry presents aspects of both in an explicit and predictive format.

4.3 Lightning talks

There were four lightning talks given by the most junior of the workshop participants. The talks were given by

1. Andrew Kwon (University of Pennsylvania)
2. Martin Lüdtke (Rijksuniversiteit Groningen)
3. Jack Petok (Dartmouth College)
4. Libby Taylor (Stanford University)

5 Outcome of the Meeting

The workshop was structured to be forward-looking with plentiful time for informal discussion of research directions. This was further facilitated in a conscious way with structured activities included the organized discussions, the problem session, and of course the shared meals and coffee breaks. Together with the wonderful facilities at BIRS the workshop created a creative environment reinvigorating existing and beginning new collaborations. This workshop also provided an important forum for researchers working in areas (derived categories of coherent sheaves, anabelian geometry, and algebraic topology) that don't normally interact a lot to learn more about the connections between these fields and to establish professional connections to aid in future research.

A Appendix. Problem List

The following is a list of problems discussed as part of the problem session, moderated by Max Lieblich on July 5, 2022, for the BIRS meeting: Derived Categories, Arithmetic, and Reconstruction in Algebraic Geometry.

The following summary is based on a notes of Brendan Hassett and Martin Olsson.

1. Given two K3 surfaces X and Y derived equivalent over a field K. Is it possible that $X(K) = \emptyset$ but $Y(K) \neq \emptyset$?

   Remark. Given X and (Y, α), where the latter is twisted by a Brauer class, there are counterexamples by Ascher-Dasaratha-Perry-Zhou.

   Remark. Given X and Y hyperkähler fourfolds of $K3^{[2]}$ type, there are counterexamples by Addington-Antieau-Frei-Honigs.

   General question: When is the existence of a K-point a derived invariant? How does this vary with K.

   The original question is known over a finite field K.

2. Let X be a smooth projective variety over a finite field k. What is meaning of the trace of Frobenius on the ℓ-adic Mukai lattice

   $$\oplus_{n\in\mathbf{Z}} H^*(X, \mathbf{Q}_\ell(n))[2n]?$$

   This is a module over $\mathbf{Q}_\ell[\beta^{\pm 1}]$ where $\deg(\beta) = 2$ with cyclotomic action.

   Remark. See work of Toen and Vezzossi [11].

3. Is $\mathrm{Br}(X) = \mathrm{Br}'(X)$ for smooth threefolds?

4. Is there a smooth projective X over k algebraically closed that is an "algebraic $K(\pi, 1)$" that is not obviously derived from curves or abelian varieties?

   Remark: Ball quotients don't seem to work as the cohomology of the group is different from its profinite completion.

5. Let C be a curve over an algebraically closed field. Is there a non-model-theoretic proof that the abstract Jacobian determines the $\mathbf{Z}$-scheme C.

   Related: Is there a version of the Rabinovich theorem without model theory.

6. What can one say about fields K for which $\mathrm{Gal}_K \simeq \mathrm{Gal}_{\mathbf{Q}}$.

   Is there a valuation w/divisible value group whose residue field is $\mathbf{Q}$?

7. It is known that in general one cannot reconstruct K from its Galois group Gal_K, where $K/\mathbf{Q}_p$ finite. One can have $\mathrm{Gal}_K \simeq \mathrm{Gal}_{K'}$ but $K \not\simeq K'$. Is there nonetheless some geometric object associated to K that is determined by the Galois group?

8. Given A and B abelian varieties over a field K. We know that A and B are derived equivalent if and only if
$$A \times \hat{A} \simeq B \times \hat{B}.$$
   Suppose now that T and T' are torsors under an abelian variety A. What ensures that $D(T) \simeq D(T')$?

   Remark: Article of Antieau-Krashen-Ward covers genus one [1].

   How does this interact with translation by $H^1(A)$? Can we reduce to the case $T = A$?

9. Many hyperkähler manifolds can be constructed as moduli spaces of objects of Bridgeland stable objects in Kuznetsov components of Fano varieties. Are all hyperkählers like this? What is the geometric relationship between the Fano and the hyperkähler manifold?

   Example: Cubic fourfolds and varieties of lines, hyperkähler of $K3^{[2]}$-type.

   Blow up $\mathbf{P}^N$ along embedded hyperkähler?

10. If X and Y admit a filtered derived equivalence, are they birational? This means $D(X) \simeq D(Y)$ such that the induced
$$\mathrm{CH}_{num}(X) \to \mathrm{CH}_{num}(Y)$$
    respects the codimension filtration.

    Evidence: assuming the standard conjectures then filtered derived equivalence implies an isomorphism of motives $M_X \simeq M_Y$.

11. Let X be a smooth projective geometrically rational variety over K. Suppose that X admits a full strong exceptional collection of line bundles over the base field. Does X have a K point?

    Orlov conjectures that X should be rational when there is a full exceptional collection, over an algebraically closed field. Elagin-Lunts say that there is a stratification by rational varieties.

    Ballard has obtained results for toric varieties.

12. Can the Kollár-Lieblich-Olsson-Sawin results be extended to weaker situations, e.g., X admitting an ample family of invertible sheaves?

    What about characteristic p? Can only hope to do this up to purely inseparable maps.

13. Recall a theorem of Voevodsky: normal varieties over finitely generated fields of characteristic 0 are determined by their étale topoi.

    What about characteristic p?

    Voevodsky says it goes through, without proof.

    - field should have transcendence degree at least one;

- only recover things up to purely inseparable ambiguity?

It was suggested this should be accessible via "Bogomolov program" after passing to field extensions; this approach can only work in higher dimensions.

B Appendix. List of participants

Olsson, Martin (University of California Berkeley)
Flapan, Laure (Michigan State University)
Honigs, Katrina (Simon Fraser University)
Topaz, Adam (University of Alberta)
Ballard, Matthew (University of South Carolina)
Bragg, Daniel (University of California Berkeley)
Brakkee, Emma (University of Amsterdam)
Cadoret, Anna (Sorbonne Universit)
Frei, Sarah (Rice University)
Gaulhiac, Sylvain (University of Alberta)
Haine, Peter (UC Berkeley)
Hassett, Brendan (Brown University)
Huang, Jesse (University of Alberta)
Kwon, Andrew (University of Pennsylvania)
Lieblich, Max (University of Washington)
Ldtke, Martin (Rijksuniversiteit Groningen)
Petok, Jack (Dartmouth College)
Sankar, Soumya (Ohio State University)
Stix, Jakob (Goethe-Universitt Frankfurt)
Taylor, Libby (Stanford University)
Barwick, Clark (Edinburgh)
Bogomolov, Fedor (New York University)
Favero, David (University of Alberta)
Kaushal Srivastava, Tanya (IIT Gandhinagar)
Markman, Eyal (University of Massachusetts Amherst)
O'Gorman, Ronan (University of California, Berkeley)
Pop, Florian (University of Pennsylvania)
Rapinchuk, Igor (Michigan State University)
Srinivasan, Padmavathi (ICERM)
Tevelev, Jenia (University of Massachusetts)
Tirabassi, Sofia (Stockholm University)
Torres, Sebastian (IMSA-Miami / ICMS-Sofia)
Voloch, Felipe (University of Canterbury)
Zilber, Boris (Oxford University)

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