LINEARIZATION OF HOLOMORPHIC LIPSCHITZ FUNCTIONS

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Abstract. Let X and Y be complex Banach spaces with B_X denoting the open unit ball of X. This paper studies various aspects of the *holomorphic Lipschitz space* $\mathcal{H}L_0(B_X, Y)$, endowed with the Lipschitz norm. This space is the intersection of the spaces, $\mathrm{Lip}_0(B_X, Y)$ of Lipschitz mappings and $\mathcal{H}^\infty(B_X, Y)$ of bounded holomorphic mappings, from B_X to Y. Thanks to the Dixmier-Ng theorem, $\mathcal{H}L_0(B_X, \mathbb{C})$ is indeed a dual space, whose predual $\mathcal{G}_0(B_X)$ shares linearization properties with both the Lipschitz-free space and Dineen-Mujica predual of $\mathcal{H}^\infty(B_X)$. We explore the similarities and differences between these spaces, and combine techniques to study the properties of the space of holomorphic Lipschitz functions. In particular, we get that $\mathcal{G}_0(B_X)$ contains a 1-complemented subspace isometric to X, we analyze when $\mathcal{G}_0(B_X)$ is a subspace of $\mathcal{G}_0(B_Y)$, and we obtain an analogous to Godefroy's characterization of functionals with a unique norm preserving extension to the holomorphic Lipschitz context.

1. Introduction

Linearizing non-linear functions is a typical procedure in infinite dimensional analysis. Originating nearly 70 years ago with Grothendieck [30] (and his research about linearization of bilinear mappings through the projective tensor product), the practice of identifying spaces of continuous *non-linear* functions with spaces of continuous *linear* mappings defined on Banach spaces has proved to be a useful technique. Accordingly, the study of geometric and topological properties of these *linearizing Banach spaces* has increasingly attracted interest.

Lipschitz functions (defined on pointed metric spaces) and holomorphic bounded functions (defined on the open unit ball of a Banach space) are really different both as sets and as function spaces. However, when looking at their linearization processes several similarities emerge. The purpose of this article is to study, in light of these resemblances, the new set of functions consisting of the intersection of the previous sets. Lipschitz holomorphic functions defined on the open unit ball of a Banach space taking the value 0 at 0 will be our focus of attention. In the exploration of this set we take advantage of a result of Ng [38] concerning the existence of preduals and all the background about related linearization processes.

We begin with a brief review of important terms and concepts. General references for Lipschitz functions include [29] and [42] and a standard reference for holomorphic functions on finite or infinite dimensional domains is [37]. The linearization process for bounded holomorphic functions is originally developed in [35]. A review about linearization procedures both for Lipschitz functions and for bounded holomorphic functions appeared in the recent survey [26] while a general approach to linearizing non-linear sets of functions was settled in [17].

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For a metric space (M,d) and a Banach space Y, let $\mathrm{Lip}(M,Y)$ be the vector space of all $f\colon M \to Y$ such that $\|f(x_1)-f(x_2)\| \leqslant Cd(x_1,x_2)$ for some $C>0$ and for all $x_1 \neq x_2 \in M$. The smallest C in the above definition is the *Lipschitz constant* of $f, L(f)$. Let $0 \in M$ denote an arbitrary fixed point. In order to get a normed space, we will be particularly interested in the subspace $\mathrm{Lip}_0(M,Y)$ consisting of those $f \in \mathrm{Lip}(M,Y)$ such that $f(0)=0$. In this way, $L(f)=0$ if and only if $f=0$, and so $\|\cdot\| = L(\cdot)$ defines a norm on $\mathrm{Lip}_0(M,Y)$.

For complex Banach spaces X and Y and open set $U \subset X$, denote by $\mathcal{H}^\infty(U,Y)$ the vector space of all $f\colon U \to Y$ such that f is holomorphic (i.e. complex Fréchet differentiable) and bounded on U, endowed with the supremum norm. In both the Lipschitz and $\mathcal{H}^\infty$ situations, if the range $Y=\mathbb{K}$, then the notation is shortened to $\mathrm{Lip}_0(M)$ and $\mathcal{H}^\infty(U)$.

It is known that $\mathrm{Lip}_0(M)$ and $\mathcal{H}^\infty(U)$ are dual spaces and that in some special situations, the predual is unique. The construction of a (or, in some cases, the) predual follows the same lines for both the Lipschitz and $\mathcal{H}^\infty$ situations: Calling X one of Lip_0 or $\mathcal{H}^\infty$, we consider those functionals $\varphi \in X^*$ such that $\varphi|_{\overline{B}_X}$ is continuous when $\overline{B}_X$ is endowed with the compact-open topology. Among such functionals are the evaluations $f \rightsquigarrow \delta(x)(f) \equiv f(x)$ where x ranges over the domain of $f \in X$. In the case of $\mathrm{Lip}_0(M)$, the closed span of the set of such φ will be denoted $\mathcal{F}(M)$ while the analogous closed span for $\mathcal{H}^\infty(U)$ is $\mathcal{G}^\infty(U)$. Each of these is a Banach space, being a closed subspace of $\mathrm{Lip}_0(M)^*$, and $\mathcal{H}^\infty(U)^*$, respectively. Using a standard technique developed by Ng [38], it follows that $\mathcal{F}(M)^* \equiv \mathrm{Lip}_0(M)$ and $\mathcal{G}^\infty(U)^* \equiv \mathcal{H}^\infty(U)$.

Among the most important common features of Lip_0 and $\mathcal{H}^\infty$ is *linearization.* In each of the two cases below, δ is the evaluation inclusion taking $x \rightsquigarrow \delta(x)$. Also, for f in either $\mathrm{Lip}_0(M,Y)$ or $\mathcal{H}^\infty(U,Y)$, T_f is the unique linear mapping making the diagram commute. Moreover, $\|f\| = \|T_f\|$.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\delta"'] & Y \\ \mathcal{F}(M) \arrow[ur, "T_f"'] & \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\delta"'] & Y \\ \mathcal{G}^\infty(U) \arrow[ur, "T_f"'] & \end{tikzcd}$$

Notation. X, Y will stand for complex Banach spaces. We denote by B_X (respectively S_X) its open unit ball (respectively unit sphere). $\mathcal{L}(X,Y)$ denotes the space of continuous linear maps from X to Y, and $X^* = \mathcal{L}(X,\mathbb{C})$. $\mathcal{P}(^mX,Y)$ stands for the space of continuous m-homogeneous polynomials, that is, those $P\colon X \to Y$ so that there exists a continuous m-linear symmetric map $\check{P}\colon X \times \cdots \times X \to Y$ with $P(x) = \check{P}(x,\ldots,x)$. We also write $\mathcal{P}(^mX) = \mathcal{P}(^mX,\mathbb{C})$. We say that $P \in \mathcal{P}(^mX,Y)$ is of *finite type* if $P(x) = \sum_{j=1}^n [x_j^*(x)]^m y_j$ for certain $x_j^* \in X^*$ and $y_j \in Y$. $\mathcal{P}_f(^mX,Y)$ stands for the space of finite type m-homogeneous polynomials. Moreover, we set $\mathcal{P}(X,Y)$ (resp. $\mathcal{P}_f(X,Y)$) to be the space of finite sums of continuous homogeneous polynomials (resp. homogeneous polynomials of finite type) from X to Y. Also, $\mathbb{D}(z,r)$ (resp. $C(z,r)$) denotes the open disc (resp. the circumference) in $\mathbb{C}$ centered at z with radius r, in particular $\mathbb{D} = \mathbb{D}(0,1)$.

Recall that X is said to have the Bounded Approximation Property (BAP) if there is $\lambda > 0$ such that the identity $I\colon X \to X$ can be approximated by finite-rank operators in $\lambda B_{\mathcal{L}(X,X)}$ uniformly on compact sets. If $\lambda = 1$, then X is said to have the Metric Approximation Property

(MAP). If X has λ-BAP and Y is λ'-complemented in X, then Y has $\lambda\lambda'$-BAP. We refer the reader to [19] for examples and applications.

Organization of the paper. Section 2 introduces the main space of interest, $\mathcal{H}L_0(B_X, Y)$, consisting of those functions that are in both $\mathrm{Lip}_0(B_X, Y)$ and $\mathcal{H}^\infty(B_X, Y)$. A number of properties of $\mathcal{H}L_0(B_X, Y)$ are discussed and it is proved that this space really differs from $\mathrm{Lip}_0(B_X, Y)$ and $\mathcal{H}^\infty(B_X, Y)$ (in the sense that a nonseparable space can be injected in between). Section 3 contains a study of the predual $\mathcal{G}_0(B_X)$ of $\mathcal{H}L_0(B_X)$ (where $Y = \mathbb{C}$). Specifically, we will see that $\mathcal{H}L_0(B_X)$ has a canonical predual whose properties echo those of $\mathcal{H}L_0(B_X)$ and $\mathrm{Lip}_0(B_X)$. When $X = \mathbb{C}$ with open unit disc $\mathbb{D}$, one consequence of our work is a characterization of the extreme points of the closed ball of $\mathcal{H}L_0(\mathbb{D})$ and of the norm attaining elements of $\mathcal{H}L_0(\mathbb{D})$ considered as the dual of $\mathcal{G}_0(\mathbb{D})$. The final two sections involve a closer inspection of $\mathcal{G}_0(B_X)$. Section 4 considers the relation between $\mathcal{G}_0(B_X)$ and $\mathcal{G}_0(B_Y)$ when $X \subset Y$. The final section specializes to the case of $X \subset X^{**}$. Among other things, under natural conditions on X and X^{**}, we characterize when a function $f \in \mathcal{H}L_0(B_X)$ has a unique norm preserving extension to $\mathcal{H}L_0(B_{X^{**}})$. Both sections make use of the concept of *(Arens) symmetric regularity,* which is reviewed in Section 4.

2. Holomorphic and Lipschitz functions

In the case that the metric space M is B_X, the open unit ball of a complex Banach space X, and Y is another complex Banach space, $\mathrm{Lip}_0(B_X, Y)$ is the space of Lipschitz functions $f\colon B_X \to Y$ with $f(0) = 0$ and:

$$L(f) = \sup\left\{\frac{\|f(x) - f(y)\|}{\|x - y\|} \, : \, x \neq y \in B_X\right\}.$$

It is well known that $L(\cdot)$ defines a norm on $\mathrm{Lip}_0(B_X, Y)$ and $(\mathrm{Lip}_0(B_X, Y), L(\cdot))$ is a Banach space. Indeed, $\mathrm{Lip}_0(B_X, Y)$ is isometrically isomorphic to the space of operators $\mathcal{L}(\mathcal{F}(B_X), Y)$, where $\mathcal{F}(B_X)$ denotes the *Lipschitz-free space* over B_X (see e.g. [27, 42]).

Next, $\mathcal{H}^\infty(B_X, Y)$ stands for the space of bounded holomorphic functions from B_X to Y, which is a Banach space when endowed with the supremum norm. Analogous to the Lipschitz case above, we have that $\mathcal{H}^\infty(B_X, Y)$ is isometrically isomorphic to $\mathcal{L}(\mathcal{G}^\infty(B_X), Y)$, where $\mathcal{G}^\infty(B_X)$ is Mujica's canonical predual of $\mathcal{H}^\infty(B_X)$ [35] (we will review the space $\mathcal{G}^\infty(B_X)$ in Section 3).

The parallel behavior of these Lipschitz and $\mathcal{H}^\infty$ spaces was the authors' motivation to introduce and study the following space and its canonical predual:

$$\mathcal{H}L_0(B_X, Y) = \mathrm{Lip}_0(B_X, Y) \cap \mathcal{H}^\infty(B_X, Y).$$

We will also denote $\mathcal{H}L_0(B_X) = \mathcal{H}L_0(B_X, \mathbb{C})$. Sometimes we will deal with holomorphic Lipschitz functions without assuming $f(0) = 0$, and then we use the notation $\mathcal{H}L(B_X, Y)$ and $\mathcal{H}L(B_X)$.

Since both normed spaces $\mathcal{H}^\infty(B_X, Y)$ and $\mathrm{Lip}_0(B_X, Y)$ are complete (with their respective norms) and each $f \in \mathcal{H}L_0(B_X, Y)$ satisfies $\|f\|_\infty \leqslant L(f)$ we easily derive that $\mathcal{H}L_0(B_X, Y)$ is a Banach space with norm $L(\cdot)$.

Given $f \in \mathcal{H}^\infty(B_X, Y)$ such that $df \in \mathcal{H}^\infty(B_X, \mathcal{L}(X, Y))$ and $f(0) = 0$, by the Mean Value Theorem, we have that $\|f(x) - f(y)\| \leqslant \|df\|\|x - y\|$ for any $x, y \in B_X$. Then, $f \in \mathrm{Lip}_0(B_X, Y)$

and $L(f) \leqslant \|df\|$. Conversely, if $f \in \mathcal{H}L_0(B_X, Y)$ we know that $df \in \mathcal{H}(B_X, \mathcal{L}(X,Y))$. Also, for $x, y \in B_X$,

$$\|df(x)(y)\| = \lim_{h\to 0} \left\| \frac{f(x+hy) - f(x)}{h} \right\| \leqslant L(f).$$

This means that df belongs to $\mathcal{H}^\infty(B_X, \mathcal{L}(X,Y))$ and $\|df\| \leqslant L(f)$.

This shows that there is another useful representation of our primary space of interest.

Proposition 2.1. $\mathcal{H}L_0(B_X, Y) = \{f \in \mathcal{H}^\infty(B_X, Y) : \ df \in \mathcal{H}^\infty(B_X, \mathcal{L}(X,Y));\ f(0) = 0\}$. *Moreover, for every* $f \in \mathcal{H}L_0(B_X, Y)$, $L(f) = \|df\|$; *that is,* $L(f) = \sup_{x\in B_X} \|df(x)\|$.

Note that $P|_{B_X} \in \mathcal{H}L_0(B_X, Y)$ for every $P \in \mathcal{P}(X, Y)$ such that $P(0) = 0$, a fact that will be useful later.

When $Y = \mathbb{C}$, we can define a mapping

$$\begin{aligned} \Phi\colon \mathcal{H}L_0(B_X) &\to \mathcal{H}^\infty(B_X, X^*) \\ f &\mapsto df \end{aligned}$$

In general, Φ is an isometry *into* $\mathcal{H}^\infty(B_X, X^*)$, although if X also equals $\mathbb{C}$, then Φ is *onto*. Indeed, in the one-dimensional case, Φ is surjective since every holomorphic function f on $\mathbb{D}$ has a primitive that is Lipschitz whenever f is bounded. However, Φ is not surjective for $X \neq \mathbb{C}$. Indeed, given $P \in \mathcal{P}(^2X)$, we have that $P|_{B_X} \in \mathcal{H}L_0(B_X)$ and $dP \in \mathcal{L}(X, X^*)$ is symmetric (i.e. $dP(x)(y) = dP(y)(x)$ for every $x, y \in X$). Note that df is linear only when f is a 2-homogeneous polynomial. Hence, a non-symmetric element of $\mathcal{L}(X, X^*)$ (which always exists whenever the dimension of X is strictly bigger than one) cannot be in the range of Φ.

In particular, we see that

$$\mathcal{H}L_0(\mathbb{D}) = \{f \in \mathcal{H}^\infty(\mathbb{D}) :\ f(0) = 0 \text{ and } f' \in \mathcal{H}^\infty(\mathbb{D})\}.$$

A lot of research has been done on $\mathcal{H}L_0(\mathbb{D})$ and on $\mathcal{H}L_0(U)$ for certain domains $U \subset \mathbb{C}^n$ such as the Euclidean ball. See, e.g., [1, 10, 11, 12, 14, 18, 39, 40] where this topic is approached from different viewpoints than what is done here.

For the case of $\mathcal{H}L(B_X, Y)$ we consider the norm $\|f\|_{\mathcal{H}L} = \max\{\|f(0)\|, L(f)\}$. The fact that this is a norm and that $(\mathcal{H}L(B_X, Y), \|\cdot\|_{\mathcal{H}L})$ is a Banach space follows easily. Note that $\|f\|_\infty \leqslant 2\|f\|_{\mathcal{H}L}$ for any $f \in \mathcal{H}L(B_X, Y)$. Also, it is plain to see that $\mathcal{H}L_0(B_X, Y)$ is a 1-complemented subspace of $\mathcal{H}L(B_X, Y)$. Moreover, motivated by a similar result for Lip_0-spaces (see [42, Th. 1.7.2]) we get:

Proposition 2.2. *Let X, Y be complex Banach spaces. Then $\mathcal{H}L(B_X, Y)$ is isometric to a 1-complemented subspace of $\mathcal{H}L_0(B_{X\oplus_1\mathbb{C}}, Y)$.*

Proof. Consider $\Phi\colon \mathcal{H}L(B_X, Y) \to \mathcal{H}L_0(B_{X\oplus_1\mathbb{C}}, Y)$ given by $\Phi f(x, \lambda) = f(x) + (\lambda - 1)f(0)$. It is easy to check that Φf is Lipschitz with $L(\Phi f) \leqslant \|f\|_{\mathcal{H}L}$ for every $f \in \mathcal{H}L(B_X, Y)$. Note that

$$L(\Phi f) \geqslant \sup\left\{ \frac{\|\Phi f(x,0) - \Phi f(y,0)\|}{\|x - y\|} : x \neq y \in B_X \right\} = L(f)$$

and also

$$L(\Phi f) \geqslant \frac{\|\Phi f(0,1) - \Phi f(0,0)\|}{\|(0,1) - (0,0)\|_1} = \|f(0)\|,$$

so we actually have $L(\Phi f) = \|f\|_{\mathcal{H}L}$. Thus Φ is an into isometry.

Now consider $T\colon \mathcal{H}L_0(B_{X\oplus_1\mathbb{C}}, Y) \to \mathcal{H}L(B_X, Y)$ given by $Tg(x) = g(x,0) + g(0,1)$. One can easily check that $\|T\| \leqslant 1$ and $T \circ \Phi = I_{\mathcal{H}L(B_X,Y)}$. Therefore $P = \Phi \circ T$ is a norm-one projection from $\mathcal{H}L_0(B_{X\oplus_1\mathbb{C}}, Y)$ onto $\Phi(\mathcal{H}L(B_X, Y))$.

$\square$

Note that there are plenty of examples of non-Lipschitz functions in $\mathcal{H}^\infty(\mathbb{D})$. Indeed, given a sequence $(b_n) \subset \mathbb{C}\backslash\{1\}$ with $|b_n| = 1$ and $b_n \to 1$, define $f\colon \{b_n\} \cup \{1\} \to \mathbb{C}$ by $f(1) = 0$ and $f(b_n) = \sqrt{|b_n - 1|}$. Then the Rudin-Carleson theorem provides an extension of f which lies in the disc algebra $\mathcal{A}(\mathbb{D})$ and has the same supremum norm, but it is not Lipschitz.

We will devote the rest of the section to show that $\mathcal{H}L_0(B_X)$ is much smaller than both $\mathcal{H}^\infty(B_X)$ and $\mathrm{Lip}_0(B_X)$. More precisely, we will show that ℓ_∞ is isomorphic to a subspace of $\mathcal{H}^\infty(B_X)$ made up of non-Lipschitz functions (expect, of course, the function 0), and that ℓ_∞ is isomorphic to a subspace of $\mathrm{Lip}_0(B_X)$ made up of non-holomorphic functions (except 0).

In the following results, we will use the function $\varphi_\lambda\colon \mathbb{C} \to \mathbb{C}$ given by

$$\varphi_\lambda(z) = \frac{\overline{\lambda}z + 1}{2}.$$

It is a standard fact that

$$\varphi_\lambda(\lambda) = 1, \quad |\varphi_\lambda(z)| < 1 \text{ for all } z \in \overline{\mathbb{D}}\backslash\{\lambda\}. \tag{1}$$

We also need the following technical lemma, which in particular provides another example of a non-Lipschitz function in the disc algebra $\mathcal{A}(\mathbb{D})$ (that is, the space of uniformly continuous functions in $\mathcal{H}^\infty(\mathbb{D})$).

Lemma 2.3. *Fix $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and define $f_\lambda\colon \mathbb{C} \to \mathbb{C}$ by*

$$f_\lambda(z) = \begin{cases} 1 + (\overline{\lambda}z - 1)e^{1/(\overline{\lambda}z-1)} & \text{if } z \neq \lambda \\ 1 & \text{if } z = \lambda. \end{cases}$$

Then

(a) f_λ is holomorphic in $\mathbb{C}\backslash\{\lambda\}$.
(b) The restriction of f_λ to $\overline{\mathbb{D}}$ belongs to $\mathcal{A}(\mathbb{D})\backslash\mathcal{H}L(\mathbb{D})$.
(c) $|f_\lambda(z)| \leqslant 3$ for all $z \in \overline{\mathbb{D}}$.
(d) If $0 < s < 1$, then $|f_\lambda'(z)| \leqslant \frac{s+1}{s}$ for all $z \in \mathbb{D}$ such that $|z - \lambda| \geqslant s$.
(e) Given $k \in \mathbb{N}$ and $0 < \delta < 1$, we have that

$$\sup_{z\in\mathbb{D}(\lambda,\delta)\cap\mathbb{D}} |(f_\lambda \cdot \varphi_\lambda^k)'(z)| = +\infty.$$

Proof. A standard computation shows that (a) holds. Now, to prove the rest of the claims it is enough to consider the case $\lambda = 1$. Denote $f = f_1$ and take $z = a + ib \in \overline{\mathbb{D}}\backslash\{1\}$, with $a, b \in \mathbb{R}$. We have that

$$\left|e^{\frac{1}{z-1}}\right| = e^{\mathrm{Re}\,\frac{1}{z-1}} = e^{\frac{a-1}{(a-1)^2+b^2}} \leqslant e^0 = 1.$$

Hence f, defined as $f(z) = 1 + (z-1)e^{\frac{1}{z-1}}$ is holomorphic on $\mathbb{C}\backslash\{1\}$ and continuously extends to $\overline{\mathbb{D}}$. Further $|f(z)| \leqslant 3$ for every $z \in \overline{\mathbb{D}}$. Let us show that f is not a Lipschitz function, i.e. f

belongs to $\mathcal{A}(\mathbb{D})\backslash\mathcal{H}L(\mathbb{D})$. For that, it is enough to check that f' is not a bounded function on $\mathbb{D}$. Observe that

$$f'(z) = \frac{z-2}{z-1} e^{\frac{1}{z-1}},$$

for every $z \in \mathbb{C}\backslash\{1\}$. Therefore, taking a null sequence $0 < \theta_n < 1$ and setting $z_n := \cos\theta_n(\cos\theta_n + i\sin\theta_n)$, we obtain that the sequence $(z_n) \subset \mathbb{D}$ converges to 1 and

$$\operatorname{Re}\left(\frac{1}{z_n - 1}\right) = \frac{\cos^2\theta_n - 1}{(\cos^2\theta_n - 1)^2 + \sin^2\theta_n \cos^2\theta_n} = -1$$

for every n. Thus,

$$|f'(z_n)| = \left|\frac{z_n - 2}{z_n - 1}\right| e^{\operatorname{Re}\left(\frac{1}{z_n-1}\right)} = \left|\frac{z_n-2}{z_n-1}\right| e^{-1}.$$

Consequently, $\lim_{n\to+\infty} |f'(z_n)| = +\infty$. Thus far we have proved *(a)*, *(b)* and *(c)*. Let's check *(d)*. We have

$$|f'(z)| = \left|\frac{z-2}{z-1}\right| \cdot \left|e^{\frac{1}{z-1}}\right| \leqslant 1 + \frac{1}{|z-1|},$$

for all $z \in \mathbb{D}$. Hence, if $0 < s < 1$ and $z \in \mathbb{D}$ with $|z-1| \geqslant s$ we have that $|f'(z)| \leqslant \frac{s+1}{s}$.

Finally we prove *(e)*. Again it is enough to consider the case $\lambda = 1$ and we denote $\varphi = \varphi_1$.

Since $(f\varphi^k)'(z) = f'(z)\varphi^k(z) + f(z)(\varphi^k)'(z)$ for all $z \in \mathbb{C}\backslash\{1\}$ and $f \cdot (\varphi^k)'$ is continuous on $\overline{\mathbb{D}}$ and hence bounded on $\overline{\mathbb{D}}$, it is enough to prove that $f' \cdot \varphi^k$ is unbounded on $\mathbb{D}(1,\delta) \cap \mathbb{D}$ for any $\delta > 0$. But using the same sequence (z_n),

$$\lim_{n\to+\infty} |f'(z_n)\varphi^k(z_n)| = \lim_{n\to+\infty} e^{-1}\left|\frac{z_n-2}{z_n-1}\right| \left|\left(\frac{z_n+1}{2}\right)^k\right| = +\infty.$$

Since there exists $n_0 \in \mathbb{N}$ such that $z_n \in \mathbb{D}(1,\delta) \cap \mathbb{D}$ for any $n \geqslant n_0$, it follows that *(e)* holds.

□

We denote, for convenience $\mathcal{H}_0^\infty(\mathbb{D}) = \{f \in \mathcal{H}^\infty(\mathbb{D}) : \; f(0) = 0\}$.

Theorem 2.4. *The space ℓ_∞ is isomorphic to a subspace of $\mathcal{H}_0^\infty(\mathbb{D})\backslash\mathcal{H}L(\mathbb{D}) \cup \{0\}$.*

Proof. To begin with, we choose a sequence $(\lambda_n) \subset \mathbb{C}\backslash\{1\}$ convergent to 1 with $|\lambda_n| = 1$ and $\lambda_n \neq \lambda_m$ for every $n \neq m$. Consider the function $\Phi\colon \mathbb{C}^2 \to \mathbb{C}$ defined as

$$\Phi(z,\lambda) = \frac{\overline{\lambda}z + 1}{2}$$

and, for each $p \in \mathbb{N}$, the compact subset of $\mathbb{C}^2$

$$K_p = \{(\lambda_p, \lambda_n) : n \in \mathbb{N}, n \neq p\} \cup \{(\lambda_p, 1)\}.$$

We have $\left|\Phi(z,\lambda)\right| < 1$ for every $(z,\lambda) \in K_p$ by (1), and Φ is continuous on $\mathbb{C}^2$. Hence, there exists $0 < s_p < 1$ such that $\left|\Phi(z,\lambda)\right| < 1$ for every $(z,\lambda) \in K_p + \overline{\mathbb{D}}((0,0), s_p)$. In particular, if we denote $\varphi_n(\cdot) := \Phi(\cdot, \lambda_n)$,

$$|\varphi_n(z)| = \left|\Phi(z,\lambda_n)\right| < 1, \tag{2}$$

for all $z \in \overline{\mathbb{D}}(\lambda_p, s_p)$ and all $n \neq p$.

Now, since the sequence (λ_n) is convergent to 1 we can find a sequence of positive numbers (r_n) that tends to 0 such that $0 < 2r_n < s_n$ for all $n \in \mathbb{N}$ and such that

$$\overline{\mathbb{D}}(\lambda_n, 2r_n) \cap \overline{\mathbb{D}}(\lambda_p, 2r_p) = \varnothing,$$

for all $n \neq p$. Moreover, as (r_n) converges to 0, for each $n \in \mathbb{N}$ the set

$$L_n := \bigcup_{p\neq n} \overline{\mathbb{D}}(\lambda_p, 2r_p) \cup \{1\},$$

is also a compact subset of $\mathbb{C}$, (although it is not a subset of $\overline{\mathbb{D}}$) and $|\varphi_n(z)| < 1$ for all $z \in L_n$. Since $|\varphi_n|$ is continuous on $\mathbb{C}$ we obtain that

$$\max\{|\varphi_n(z)| \,:\, z \in C_n \cup L_n\} < 1,$$

for all n, where $C_n = \overline{\mathbb{D}}\backslash\mathbb{D}(\lambda_n, r_n)$. As a consequence, for each n the sequence $\left(\varphi_n^k\right)_{k=1}^{\infty}$ converges uniformly to 0 on $C_n \cup L_n$ and we can find a $k_n \in \mathbb{N}$ such that

$$|\varphi_n^{k_n}(z)| < \frac{r_n}{3^{n+1}}, \tag{3}$$

for every $z \in C_n \cup L_n$.

We denote $f_n := f_{\lambda_n}$, for $n \in \mathbb{N}$ and we define $F\colon \ell_\infty \longrightarrow \mathcal{H}^\infty(\mathbb{D})$ by

$$F(a_n) := \sum_{n=1}^{\infty} a_n f_n \varphi_n^{k_n},$$

We claim that F is a topological isomorphism onto its image. Let us check that for each $(a_n) \in \ell_\infty$ the series $F(a_n)$ is pointwise convergent in the closed disc.

Consider $z \in \overline{\mathbb{D}}$. We have two possibilities:

a) $z \in \overline{\mathbb{D}}\backslash \left(\bigcup_{n=1}^{\infty} \overline{\mathbb{D}}(\lambda_n, r_n)\right)$. In that case, by (3) and Lemma 2.3. (c),

$$\sum_{n=1}^{\infty} |a_n f_n(z)\varphi_n^{k_n}(z)| \leqslant \sum_{n=1}^{\infty} 3|a_n|\frac{r_n}{3^{n+1}} \leqslant \frac{1}{2}\|(a_n)\|_\infty. \tag{4}$$

Hence $F(a_n)(z)$ exists for each z. Moreover, we have proved that the series $F(a_n)$ converges absolutely and uniformly on the open set $\mathbb{D}\backslash\left(\bigcup_{n=1}^{\infty} \overline{\mathbb{D}}(\lambda_n, r_n)\right)$. Thus $F(a_n)$ is holomorphic in that open set.

b) there exists a unique $n_0 \in \mathbb{N}$ such that $z \in \mathbb{D}(\lambda_{n_0}, 2r_{n_0})$. By (3), for every $u \in \mathbb{D}(\lambda_{n_0}, 2r_{n_0})$ we have that

$$|a_n f_n(u)\varphi_n^{k_n}(u)| \leqslant 3|a_n|\frac{r_n}{3^{n+1}} < \frac{|a_n|}{3^n},$$

for all $n \neq n_0$ and

$$|a_{n_0} f_{n_0}(u)\varphi_{n_0}^{k_{n_0}}(u)| \leqslant 3|a_{n_0}|.$$

Hence,

$$\sum_{n=1}^{\infty} |a_n f_n(z)\varphi_n^{k_n}(z)| \leqslant 4\|(a_n)\|_\infty, \tag{5}$$

and we have obtained that for every $z \in \mathbb{D}(\lambda_{n_0}, 2r_{n_0})$, $F(a_n)(z)$ exists and in fact $|F(a_n)(z)| \leqslant 4\|(a_n)\|_\infty$. But our argument actually shows that the series $F(a_n)$ is absolutely and uniformly convergent in the open disc $\mathbb{D}(\lambda_{n_0}, 2r_{n_0})$. Hence, $F(a_n)$ exists and it is holomorphic on that set.

Thus $F(a_n)$ is holomorphic on $\mathbb{D}\cup\bigcup_{n=1}^{\infty}\mathbb{D}(\lambda_n,2r_n)$ and $F\colon \ell_\infty\to\mathcal{H}^\infty(\mathbb{D})$ is a continuous mapping since it is linear and

$$\|F(a_n)\|\leqslant 4\|(a_n)\|_\infty,$$

for all $(a_n)\in\ell_\infty$.

Now we check that F is bounded below. We already know that for each $(a_n)\in\ell_\infty$, the function $F(a_n)$ is holomorphic on $\mathbb{D}\cup\bigcup_{n=1}^{\infty}\mathbb{D}(\lambda_n,2r_n)$ and bounded on $\mathbb{D}$. Thus, using (3) and the fact that $\lambda_p\in\overline{\mathbb{D}}$, we get

$$\begin{aligned}\|F(a_n)\| &= \sup_{z\in\mathbb{D}}|F(a_n)(z)|\geqslant \sup_{p\in\mathbb{N}}|F(a_n)(\lambda_p)|\geqslant \sup_{p\in\mathbb{N}}\left\{|a_p|-\sum_{n\neq p}3|a_n|\frac{r_n}{3^{n+1}}\right\}\\ &\geqslant \sup_{p\in\mathbb{N}}\left\{|a_p|-\frac{\|(a_n)\|_\infty}{2}\right\}=\frac{\|(a_n)\|_\infty}{2}\end{aligned}$$

for every $(a_n)\in\ell_\infty$.

Now we check that if $(a_n)\in\ell_\infty\backslash\{0\}$, then $F(a_n)$ is not Lipschitz.

Consider $(a_n)\in\ell_\infty\backslash\{0\}$. There exists n_0 such that $a_{n_0}\neq 0$. We are going to show that $F(a_n)'(z)$ is not bounded on $\mathbb{D}(\lambda_{n_0},\frac{r_{n_0}}{3})\cap\mathbb{D}$.

By the Weierstrass theorem,

$$F(a_n)'(z)=\sum_{n=1}^{+\infty}a_n\big(f_n\varphi_n^{k_n}\big)'(z),$$

for every $z\in\mathbb{D}\cup\bigcup_{n=1}^{\infty}\mathbb{D}(\lambda_n,2r_n)$. If $n\neq n_0$, then by the Cauchy integral formula

$$\big(\varphi_n^{k_n}\big)'(z)=\frac{1}{2\pi i}\int_{C(\lambda_{n_0},r_{n_0})}\frac{\varphi_n^{k_n}(u)}{(u-z)^2}du,$$

for every $z\in\mathbb{D}(\lambda_{n_0},\frac{r_{n_0}}{3})$. Thus, by (2) and (3), we obtain

$$\sup_{z\in\mathbb{D}(\lambda_{n_0},\frac{r_{n_0}}{3})}|\big(\varphi_n^{k_n}\big)'(z)|\leqslant\frac{r_{n_0}}{(\frac{2}{3}r_{n_0})^2}\sup_{|u-\lambda_{n_0}|=r_{n_0}}|\varphi_n^{k_n}(u)|<\frac{9}{4r_{n_0}}\frac{r_n}{3^{n+1}}<\frac{1}{r_{n_0}}\frac{1}{3^n},$$

and we get

$$\begin{aligned}|\big(f_n\varphi_n^{k_n}\big)'(z)| &\leqslant |f_n'(z)||\varphi_n^{k_n}(z)|+|f_n(z)||(\varphi_n^{k_n})'(z)|\\ &\leqslant\frac{1+r_n}{r_n}\frac{r_n}{3^{n+1}}+3\frac{1}{r_{n_0}}\frac{1}{3^n}<\frac{1}{3^n}+\frac{1}{r_{n_0}}\frac{1}{3^{n-1}}\end{aligned}$$

where in the second inequality we have applied, (2), (3) and the properties of f_n and f_n' given in Lemma 2.3. Hence,

$$\begin{aligned}|F(a_n)'(z)| &\geqslant |a_{n_0}||\big(f_{n_0}\varphi_{n_0}^{k_{n_0}}\big)'(z)|-\sum_{n\neq n_0}|a_n||\big(f_n\varphi_n^{k_n}\big)'(z)|\\ &\geqslant |a_{n_0}||\big(f_{n_0}\varphi_{n_0}^{k_{n_0}}\big)'(z)|-\|(a_n)\|_\infty\big(\frac{1}{2}+\frac{3}{2r_{n_0}}\big),\end{aligned}$$

for every $z\in\mathbb{D}(\lambda_{n_0},\frac{r_{n_0}}{3})$. But, by Lemma 2.3.(e), we know that

$$\sup_{z\in\mathbb{D}(\lambda_{n_0},\frac{r_{n_0}}{3})\cap\mathbb{D}}|\big(f_{n_0}\varphi_{n_0}^{k_{n_0}}\big)'(z)|=+\infty.$$

Thus, $F(a_n)'$ is unbounded on $\mathbb{D}(\lambda_{n_0}, \frac{r_{n_0}}{3}) \cap \mathbb{D}$ and we have obtained that $F(a_n)$ does not belong to $\mathcal{H}L(\mathbb{D})$.

Finally, if we define $F_1 \colon \ell_\infty \to \mathcal{H}_0^\infty(\mathbb{D})$ by $F_1(a_n)(z) := zF(a_n)(z)$ for $(a_n) \in \ell_\infty$ and $z \in \mathbb{D}$, it is clear that F_1 is an isomorphism onto its image and that $F_1(\ell_\infty \backslash \{0\}) \subset \mathcal{H}_0^\infty(\mathbb{D}) \backslash \mathcal{H}L(\mathbb{D})$.

□

Straightforward modifications of the above arguments show that the space c_0 is isomorphic to a subspace of $\mathcal{A}(\mathbb{D}) \backslash \mathcal{H}L(\mathbb{D}) \cup \{0\}$, where $\mathcal{A}(\mathbb{D})$ denotes the disc algebra. We note that there are known results far stronger than this. Indeed, in three relevant papers [10, 11, 12], L. Bernal et al. have obtained many results on the existence of large subspaces of functions that belong to $\mathcal{A}(\mathbb{D}) \backslash \mathcal{H}L(\mathbb{D}) \cup \{0\}$. In particular, in [10, Th. 4.1.c] the authors show that there exists an infinite dimensional Banach space X contained in $\mathcal{A}(\mathbb{D})$ such that any non-null function in X is not differentiable on any point of a fixed dense subset of $\mathbb{T}$. Also, in [12, Th. 3.4], they prove that there exists an infinite dimensional Banach space X, contained in $\mathcal{A}(\mathbb{D})$, (which, however, is endowed with a stronger norm than the one inherited from $\mathcal{A}(\mathbb{D})$) such that if $f \in X$, then the restriction of f to $\mathbb{T}$ is nowhere Hölder on $\mathbb{T}$.

Theorem 2.4 can be extended to any complex Banach space. We denote

$$\mathcal{H}_0^\infty(B_X) = \{f \in \mathcal{H}^\infty(B_X) : \ f(0) = 0\}.$$

Corollary 2.5. *Let X be a complex Banach space. Then ℓ_∞ is isomorphic to a subspace of $\mathcal{H}_0^\infty(B_X) \backslash \mathcal{H}L(B_X) \cup \{0\}$.*

Proof. We fix $x_0 \in X$ such that $\|x_0\| = 1$ and consider $x^* \in X^*$ such that $x^*(x_0) = 1 = \|x^*\|$. We define

$$\Psi \colon \mathcal{H}^\infty(\mathbb{D}) \longrightarrow \mathcal{H}^\infty(B_X)$$

by $\Psi(f) = f \circ x^*$, for $f \in \mathcal{H}^\infty(\mathbb{D})$. Clearly Ψ is a well-defined linear mapping and since $x^*(B_X) = \mathbb{D}$ we have that Ψ is an isometry onto its image. Moreover, considering the restriction of Ψ to $\mathcal{H}L(\mathbb{D})$ we are going to have

$$\Psi \colon \mathcal{H}L(\mathbb{D}) \longrightarrow \mathcal{H}L(B_X),$$

that again is an isometry, now with the Lipschitz norms. Indeed, if $f \in \mathcal{H}L(\mathbb{D})$ and $x, y \in B_X$ then

$$|\Psi(f)(x) - \Psi(f)(y)| \leqslant L(f)|x^*(x) - x^*(y)| \leqslant L(f)\|x^*\|\|x - y\| = L(f)\|x - y\|.$$

Thus $L(\Psi(f)) \leqslant L(f)$. But if $\lambda, \mu \in \mathbb{D}$, then

$$\begin{aligned} |f(\lambda) - f(\mu)| =& |f \circ x^*(\lambda x_0) - f \circ x^*(\mu x_0)| = |\Psi(f)(\lambda x_0) - \Psi(f)(\mu x_0)| \\ \leqslant& L(\Psi(f))\|\lambda x_0 - \mu x_0\| = L(\Psi(f))|\lambda - \mu|, \end{aligned}$$

and we get $L(f) \leqslant L(\Psi(f))$. Finally, due to the injectivity of Ψ (a direct proof is also elementary) we have that

$$\Psi(\mathcal{H}_0^\infty(\mathbb{D}) \backslash \mathcal{H}L(\mathbb{D})) \subset \mathcal{H}_0^\infty(B_X) \backslash \mathcal{H}L(B_X).$$

Now the claim is a straightforward consequence of Theorem 2.4.

□

We finish this section with a counterpart of Corollary 2.5; that is, there is a copy of ℓ_∞ made up of Lipschitz non-holomorphic functions.

Proposition 2.6. *Let X be a complex Banach space. Then ℓ_∞ is isomorphic to a subspace of* $\mathrm{Lip}_0(B_X)\backslash\mathcal{H}L_0(B_X)\cup\{0\}$.

Proof. First we consider the 1-dimensional case $X=\mathbb{C}$. Let $l\colon\mathbb{R}\to[0,1]$ be a C^∞ function such that l is strictly increasing on $(1/2,1)$, $0<l(x)<1$ if $1/2<x<1$, $l(x)=0$ for $x\leqslant 1/2$, $l(x)=1$ for $x\geqslant 1$, $l^{(k)}(1/2)=0$ if $k\geqslant 0$ and $l^{(k)}(1)=0$ if $k\geqslant 1$. Define $f\colon\mathbb{C}\to[0,1]$ as $f(z)=l(|z|)$. Considered it as being defined on $\mathbb{R}^2$, f is C^∞ and $df\colon\mathbb{R}^2\to\mathbb{R}$ is a continuous function. Hence, by the Mean Value Theorem, $f\in\mathrm{Lip}_0(\mathbb{D})$. Now we define $T\colon\mathcal{H}L_0(\mathbb{D})\to\mathrm{Lip}_0(\mathbb{D})$ as $T(g)=f\cdot g$. We claim that T is an isomorphism onto its image. Indeed, given $g\in\mathcal{H}L_0(\mathbb{D})$ and $z,u\in\mathbb{D}$,

$$|f(z)g(z)-f(u)g(u)|\leqslant|f(z)-f(u)||g(z)|+|f(u)||g(z)-g(u)|\leqslant 2L(f)L(g)|z-u|.$$

Thus T is a continuous linear mapping with $\|T\|\leqslant 2L(f)$. Now we check that T is bounded below. As $f(x,y)=l(\sqrt{x^2+y^2})$ we have that $df(x,y)=0$ if $z=x+iy$ satisfies $|z|\geqslant 1$. By continuity on a compact set, given $\varepsilon>0$ there exists $0<r<1$ such that if $|z|\geqslant r$, then both $\|df(x,y)\|<\varepsilon$ and $f(z)>1-\varepsilon$. Thus, for $g\in\mathcal{H}L_0(\mathbb{D})$,

$$L(fg)=\|d(fg)\|_{\mathbb{D}}=\|gdf+fg'\|_{\mathbb{D}}\geqslant\|fg'\|_{\mathbb{D}\backslash r\mathbb{D}}-\|g\|_{\mathbb{D}}\|df\|_{\mathbb{D}\backslash r\mathbb{D}}\geqslant\|fg'\|_{\mathbb{D}\backslash r\mathbb{D}}-L(g)\varepsilon.$$

But, by the maximum modulus theorem

$$\|fg'\|_{\mathbb{D}\backslash r\mathbb{D}}\geqslant(1-\varepsilon)\|g'\|_{\mathbb{D}\backslash r\mathbb{D}}=(1-\varepsilon)\|g'\|_{\mathbb{D}}=(1-\varepsilon)L(g).$$

and we get $L(fg)\geqslant(1-2\varepsilon)L(g)$, for every $\varepsilon>0$. As a consequence

$$L(Tg)=L(fg)\geqslant L(g),$$

and T is bounded below. Moreover, $T(g)=f\cdot g$ is never holomorphic on $\mathbb{D}$ for any $g\in\mathcal{H}L_0(\mathbb{D})\backslash\{0\}$, and $T(\mathcal{H}L_0)(\mathbb{D})$ is isomorphic to $\mathcal{H}L_0(\mathbb{D})$ which in turn is isometric to $\mathcal{H}^\infty(\mathbb{D})$ that has a subspace isomorphic to ℓ_∞.

The general case is a straightforward consequence of the above argument in the following natural way. Let X a non-null complex Banach space and take $x^*\in S_{X^*}$. Defining $R\colon\mathrm{Lip}_0(\mathbb{D})\to\mathrm{Lip}_0(B_X)$ by $R(h)=h\circ x^*$, we are going to have that R is an isometry into. Hence, $R\circ T\colon\mathcal{H}L_0(\mathbb{D})\to\mathrm{Lip}_0(B_X)$ is an isomorphism into its image and we get that ℓ_∞ is isomorphic to a subspace of $\mathcal{H}L_0(B_X)$. But if $g\in\mathcal{H}L_0(\mathbb{D})\backslash\{0\}$, then $R\circ T(g)=(f\cdot g)x^*$ is not a Gateaux holomorphic function since its restriction to $\{zx : z\in\mathbb{D}\}$ is not holomorphic. We conclude that $\ell_\infty\backslash\{0\}\subset\mathrm{Lip}_0(B_X)\backslash\mathcal{H}L_0(B_X)$. □

3. The predual of the space of holomorphic Lipschitz functions

In this section, we will show that the space $\mathcal{H}L_0(B_X)$ has a canonical predual with very similar properties to the canonical preduals of $\mathcal{H}^\infty(B_X)$ and $\mathrm{Lip}_0(B_X)$.

Let us denote by τ_0 the compact-open topology on $\mathcal{H}L_0(B_X)$. An easy argument using Montel's theorem [22, Th. 15.50] shows that $\overline{B}_{\mathcal{H}L_0(B_X)}$ is τ_0-compact. In fact, on this ball, convergence in the topology τ_0 coincides with pointwise convergence. Thus, the Dixmier-Ng theorem [38] says that $\mathcal{H}L_0(B_X)$ is a dual space with predual given by

$$\mathcal{G}_0(B_X):=\{\varphi\in\mathcal{H}L_0(B_X)^*:\ \varphi|_{\overline{B}_{\mathcal{H}L_0(B_X)}}\text{ is }\tau_0-\text{continuous}\}.$$

For $x\in B_X$ and $f\in\mathcal{H}L_0(B_X)$, denote $\delta(x)(f)=f(x)$. Clearly $\delta(x)\colon\mathcal{H}L_0(B_X)\to\mathbb{C}$ is linear and continuous meaning that $\delta(x)\in\mathcal{H}L_0(B_X)^*$. Also, $\delta(x)|_{\overline{B}_{\mathcal{H}L_0(B)}}$ is τ_0-continuous so $\delta(x)\in\mathcal{G}_0(B_X)$.

Proposition 3.1. *Let X be a complex Banach space.*

(a) The mapping

$$\begin{aligned}\delta\colon B_X &\to \mathcal{G}_0(B_X)\\ x &\mapsto \delta(x)\end{aligned}$$

is holomorphic and $\|\delta(x)-\delta(y)\| = \|x-y\|$ for every $x,y \in B_X$. In particular, $\delta \in \mathcal{H}L_0(B_X, \mathcal{G}_0(B_X))$ with $L(\delta)=1$.

(b) $\mathcal{G}_0(B_X) = \overline{\operatorname{span}}\{\delta(x) : x \in B_X\}$.

(c) For any complex Banach space Y and any $f \in \mathcal{H}L_0(B_X, Y)$, there is a unique operator $T_f \in \mathcal{L}(\mathcal{G}_0(B_X), Y)$ such that the following diagram commutes:

$$\begin{tikzcd} B_X \arrow[r, "f"] \arrow[d, "\delta"'] & Y \\ \mathcal{G}_0(B_X) \arrow[ur, "T_f"'] & \end{tikzcd}$$

The map $f \mapsto T_f$ defines an isometric isomorphism from $\mathcal{H}L_0(B_X, Y)$ onto $\mathcal{L}(\mathcal{G}_0(B_X), Y)$. These properties characterize $\mathcal{G}_0(B_X)$ uniquely up to an isometric isomorphism.

(d) A bounded net $(f_\alpha) \subset \mathcal{H}L_0(B_X)$ is weak-star convergent to a function $f \in \mathcal{H}L_0(B_X)$ if and only if $f_\alpha(x) \to f(x)$ for every $x \in B_X$.

Proof. (a) The map δ is weakly holomorphic since for any $f \in \mathcal{G}_0(B_X)^* = \mathcal{H}L_0(B_X)$ we have that $f \circ \delta = f$ is holomorphic. Thus, δ is holomorphic (see [37, Th. 8.12]). Also, given $x, y \in B_X$, we have

$$\|\delta(x)-\delta(y)\| = \sup_{f \in B_{\mathcal{H}L_0(B_X)}} |\langle f, \delta(x)-\delta(y)\rangle| = \sup_{f \in B_{\mathcal{H}L_0(B_X)}} |f(x)-f(y)| \leqslant \|x-y\|,$$

and equality holds since we may take $f = x^*|_{B_X}$ where $\|x^*\| = 1$ and $x^*(x-y) = \|x-y\|$.

(b) Just observe that for every $f \in \mathcal{H}L_0(B_X) = \mathcal{G}_0(B_X)^*$ we have that $f = 0$ whenever $f|_{\{\delta(x):x \in B_X\}} = 0$.

(c) First, note that an interpolation argument shows that the set $\{\delta(x) : x \in B_X \backslash \{0\}\}$ is linearly independent in $\mathcal{G}_0(B_X)$. Indeed, assume that $\sum_{j=1}^n \lambda_j \delta(x_j) = 0$ for different points $x_j \in B_X \backslash \{0\}$ and $\lambda_j \in \mathbb{C}$. Let $x_0 = 0$ and $\lambda_0 = 0$. Take $x_{ij}^* \in S_{X^*}$ with $x_{ij}^*(x_i - x_j) = \|x_i - x_j\|$ and define $f(x) = \sum_{j=0}^n \overline{\lambda_j} \prod_{i \neq j} \frac{x_{ij}^*(x_i - x)}{\|x_i - x_j\|}$. Then $f \in \mathcal{H}L_0(B_X)$ and $0 = \langle f, \sum_{j=1}^n \lambda_j \delta(x_j)\rangle = \sum_{j=1}^n |\lambda_j|^2$.

Now, given $f \in \mathcal{H}L_0(B_X, Y)$, we define $T_f(\delta(x)) := f(x)$ for every $x \in B_X$ (this is the only possibility to get a commutative diagram) and extend it linearly to $\operatorname{span}\{\delta(x) : x \in B_X\}$. Note that, given $u = \sum_{j=1}^n \lambda_j \delta(x_j)$,

$$\begin{aligned}\|T_f u\| = \left\| \sum_{j=1}^n \lambda_j f(x_j) \right\| &= \sup_{y^* \in B_{Y^*}} \left| \sum_{j=1}^n \lambda_j (y^* \circ f)(x_j) \right| = \sup_{y^* \in B_{Y^*}} |\langle u, y^* \circ f\rangle| \\ &\leqslant \sup\{L(y^* \circ f) : y^* \in B_Y\} \|u\| = L(f)\|u\|.\end{aligned}$$

Thus, T_f extends uniquely to an operator $T_f \in \mathcal{L}(\mathcal{G}_0(B), Y)$ with $\|T_f\| \leqslant L(f)$. Since $L(\delta) = 1$ and $f = T_f \circ \delta$, indeed we get that $\|T_f\| = L(f)$.

Moreover, the map $f \mapsto T_f$ is onto since, given any $T \in L(\mathcal{G}_0(B_X), Y)$, we have that $f := T \circ \delta$ is a holomorphic Lipschitz map with $f(0) = 0$ and $T = T_f$.

The uniqueness of $\mathcal{G}_0(B_X)$ follows from the diagram property and the fact that $\|T_f\| = L(f)$.

(d) The ball $\overline{B}_{\mathcal{H}L_0(B_X)}$ is τ_0-compact and the weak-star topology is coarser than τ_0, so they coincide on $\overline{B}_{\mathcal{H}L_0(B_X)}$. □

Proposition 3.2. *For every complex Banach space X we have that X is isometric to a 1-complemented subspace of $\mathcal{G}_0(B_X)$.*

Proof. In the particular case of $f = Id\colon B_X \to X$, differentiating the diagram in Proposition 3.1 and using that $d(Id)(x) = Id$ for all $x \in B_X$, we obtain another commutative diagram where all the arrows are linear:

$$\begin{tikzcd} X \arrow[r, "Id"] \arrow[d, "d\delta(0)"'] & X \\ \mathcal{G}_0(B_X) \arrow[ur, "T_{Id}"'] & \end{tikzcd}$$

Moreover, $d\delta(0)$ is an isometry. Indeed, given $x \in X$ and $f \in \mathcal{H}L_0(B_X)$ we have

$$\langle f, d\delta(0)(x)\rangle = \lim_{t\to 0}\langle f, \frac{\delta(tx) - \delta(0)}{t}\rangle = \lim_{t\to 0} \frac{f(tx) - f(0)}{t} = df(0)(x)$$

and so

$$\|d\delta(0)(x)\| = \sup\{|df(0)(x)| : f \in B_{\mathcal{H}L_0(B_X)}\} \leqslant \|x\|.$$

In addition, if we take $f = x^*|_{B_X}$ then $\langle f, d\delta(0)(x)\rangle = x^*(x)$, so $\|d\delta(0)(x)\| = \|x\|$ for every $x \in X$.

Finally, let $P = d\delta(0) \circ T_{Id}$. Then, using that $T_{Id} \circ d\delta(0) = Id$, we have

$$P^2 = d\delta(0) \circ T_{Id} \circ d\delta(0) \circ T_{Id} = d\delta(0) \circ T_{Id} = P,$$

so P is a norm-one projection from $\mathcal{G}_0(B_X)$ onto $d\delta(0)(X)$. □

Note that this result also holds for $\mathcal{G}^\infty(B_X)$ [35] but not in general for $\mathcal{F}(B_X)$. In [29] it is proved that this is true for X separable although for nonseparable X it could even occur that $\mathcal{F}(B_X)$ does not contain a subspace isomorphic to X. Another useful property of Lipschitz-free spaces is the fact that they contain a complemented copy of ℓ_1 [20], the same holds for $\mathcal{G}_0(B_X)$.

Proposition 3.3. *Let X be a complex Banach space. Then there is a complemented subspace of $\mathcal{G}_0(B_X)$ isomorphic to ℓ_1.*

Proof. ℓ_∞ is isomorphic to a subspace of $\mathcal{H}^\infty(\mathbb{D})$. Since $\mathcal{H}^\infty(\mathbb{D})$ is isometric to $\mathcal{H}L_0(\mathbb{D})$, which is a complemented subspace in $\mathcal{H}L_0(B_X)$, the same holds for $\mathcal{H}L_0(B_X)$. It is a classical result (see [13, Th. 4]) that this implies its predual $\mathcal{G}_0(B_X)$ contains a complemented copy of ℓ_1. □

Next, we want to describe the closed unit ball of $\mathcal{G}_0(B_X)$. For that, we introduce some more notation. We denote by conv the convex hull of a set and by Γ the absolute convex hull of a set. As usual in the Lipschitz world, for every $x, y \in B_X$ with $x \neq y$, $m_{x,y}$ stands for the *elementary molecule* $\frac{\delta(x)-\delta(y)}{\|x-y\|}$. Also, for every $x \in B_X$, $y \in X$ and $f \in \mathcal{H}L_0(B_X)$, we denote $e_{x,y}(f) := df(x)(y)$. Then $e_{x,y} \in \mathcal{G}_0(B_X)$ with $\|e_{x,y}\| = \|y\|$. Indeed, it is clear that

$$\|e_{x,y}\| = \sup\{|df(x)(y)| : f \in B_{\mathcal{H}L_0(B_X)}\} \leqslant \sup\{\|df(x)\| : f \in B_{\mathcal{H}L_0(B_X)}\} \|y\| \leqslant \|y\|.$$

Conversely, take $x^* \in X^*$ with $x^*(y) = \|y\|$ and $\|x^*\| = 1$. Then $x^*|_{B_X} \in \mathcal{H}L_0(B_X)$ and $e_{x,y}(x^*|_{B_X}) = x^*(y) = \|y\|$. This shows that $e_{x,y}$ belongs to $\mathcal{H}L_0(B_X)^*$ and the equality of norms. Finally, by a simple application of a Cauchy's integral formula we derive that the restriction of $e_{x,y}$ to $\overline{B}_{\mathcal{H}L_0(B_X)}$ is τ_0-continuous and so it belongs to $\mathcal{G}_0(B_X)$.

Proposition 3.4. *Let X be a complex Banach space. Then,*

$$\overline{B}_{\mathcal{G}_0(B_X)} = \overline{\Gamma}\{m_{x,y} : \ x, y \in B_X, \ x \neq y\} = \overline{\mathrm{conv}}\{e_{x,y} : \ x \in B_X, y \in S_X\}$$

Proof. By Proposition 3.1, we have that $\|m_{x,y}\| = 1$ for every $x, y \in B_X$ with $x \neq y$. Also,

$$L(f) = \sup\{|\langle f, m_{x,y}\rangle| : x, y \in B_X, x \neq y\} \text{ for all } f \in \mathcal{H}L_0(B_X).$$

Thus, $\{m_{x,y} : x, y \in B_X, x \neq y\}$ is 1-norming for $\mathcal{H}L_0(B_X)$. Equivalently, $\overline{B}_{\mathcal{G}_0(B_X)} = \overline{\Gamma}\{m_{x,y} : x, y \in B_X, \ x \neq y\}$. Analogously, we have that

$$L(f) = \|df\| = \sup\{\|df(x)\| : x \in B_X\} = \sup\{|\langle f, e_{x,y}\rangle| : x \in B_X, y \in S_X\}$$

and so $\overline{B}_{\mathcal{G}_0(B_X)} = \overline{\Gamma\{e_{x,y} : x \in B_X, y \in S_X\}}$. But $e_{x,\lambda y_1 + \eta y_2} = \lambda e_{x,y_1} + \eta e_{x,y_2}$ for every $\lambda, \eta \in \mathbb{C}$ so actually $\overline{B}_{\mathcal{G}_0(B_X)} = \overline{\mathrm{conv}}\{e_{x,y} : x \in B_X, y \in S_X\}$. □

As a consequence, the density characters of X and $\mathcal{G}_0(B_X)$ coincide. In particular X is separable if and only if $\mathcal{G}_0(B_X)$ is separable.

We will now relate $\mathcal{G}_0(B_X)$ with the Lipschitz-free space $\mathcal{F}(B_X)$ and Mujica's predual $\mathcal{G}^\infty(B_X)$ of $\mathcal{H}^\infty(B_X)$. Note that each element of $\mathcal{F}(B_X)$ can be seen also as an element of $\mathcal{G}_0(B_X)$, but maybe with a different behavior. For instance, consider μ given by $\langle \mu, f\rangle = \int_{C(0,1/2)} f(z)dz$ for $f \in \mathrm{Lip}_0(B_X)$. Then $\mu \neq 0$ in $\mathcal{F}(B_X)$ but $\langle \mu, f\rangle = 0$ for all $f \in \mathcal{H}L_0(B_X)$, so $\mu = 0$ when considered as an element of $\mathcal{G}_0(B_X)$. The next proposition formalizes this situation. We say that an operator $T\colon X \to Y$ is a *quotient operator* if T is surjective and $\|y\| = \inf\{\|x\| : Tx = y\}$ for every $y \in Y$; this implies that $X/\ker T$ is isometrically isomorphic to Y.

Proposition 3.5. *Let X be a complex Banach space.*

(a) The operator

$$\begin{aligned} \pi\colon \mathcal{F}(B_X) &\to \mathcal{G}_0(B_X) \\ \delta(x) &\mapsto \delta(x) \end{aligned}$$

is a quotient operator with kernel $\mathcal{H}L_0(B_X)_\perp = \{\mu \in \mathcal{F}(B_X) : \langle f, \mu\rangle = 0 \, \forall f \in \mathcal{H}L_0(B_X)\}$. Thus $\mathcal{G}_0(B_X) \equiv \mathcal{F}(B_X)/\mathcal{H}L_0(B_X)_\perp$ isometrically.

(b) The operator

$$\begin{aligned} \Psi\colon \mathcal{G}^\infty(B_X)\widehat{\otimes}_\pi X &\to \mathcal{G}_0(B_X) \\ \delta(x) \otimes y &\mapsto e_{x,y} \end{aligned}$$

is a quotient map with $\|\Psi\| = 1$. In addition, the operator Ψ is injective if and only if $X = \mathbb{C}$.

Proof. (a) First note that the existence of such an operator π follows from the linearization property of Lipschitz-free spaces applied to the 1-Lipschitz map $B_X \to \mathcal{G}_0(B_X)$ given by $x \mapsto \delta(x)$. Also, $\pi^*\colon \mathcal{H}L_0(B_X) \to \mathrm{Lip}_0(B_X)$ is just the inclusion map since

$$\pi^* f(x) = \langle \pi^* f, \delta(x)\rangle = \langle f, \pi(\delta(x))\rangle = \langle f, \delta(x)\rangle = f(x) \quad \forall f \in \mathcal{H}L_0(B_X), \forall x \in B_X.$$

Thus, π^* is an isometry into. It is a standard fact that this implies that π is a quotient operator. Moreover, $\ker \pi = \pi^*(\mathcal{H}L_0(B_X))_\perp = \mathcal{H}L_0(B_X)_\perp$.

(b) Consider the into isometry

$$\begin{aligned} \Phi\colon \mathcal{H}L_0(B_X) &\to \mathcal{H}^\infty(B_X, X^*) \\ f &\mapsto df \end{aligned}$$

defined after Proposition 2.1. Recall that $\mathcal{G}^\infty(B_X)\widehat{\otimes}_\pi X$ is a predual of $\mathcal{L}(\mathcal{G}^\infty(B_X), X^*) \simeq \mathcal{H}^\infty(B_X, X^*)$ (see e.g. [41]). Thus, if we restrict Φ^* to this predual we obtain $\Psi = \Phi^*|_{\mathcal{G}^\infty(B_X)\widehat{\otimes}_\pi X}$, note that $\Psi(\delta(x) \otimes y) = e_{x,y} \in \mathcal{G}_0(B_X)$ for all x, y and so $\Psi(\mathcal{G}^\infty(B_X)\widehat{\otimes}_\pi X) \subset \mathcal{G}_0(B_X)$. Then $\|\Psi\| = 1$ and Ψ is a quotient operator since $\Psi^* = \Phi$ is an into isometry. In the case $X = \mathbb{C}$, we have indeed that $\Phi\colon \mathcal{H}L_0(\mathbb{D}) \to \mathcal{H}^\infty(\mathbb{D})$ is an onto isometry, and thus Ψ is also an isometry from $\mathcal{G}^\infty(\mathbb{D})$ onto $\mathcal{G}_0(\mathbb{D})$. However, Ψ is not injective for $X \neq \mathbb{C}$ since Φ is not surjective.

□

Thus, $\mathcal{G}_0(\mathbb{D})$ is isometric to $\mathcal{G}^\infty(\mathbb{D})$ (which is the unique predual of $\mathcal{H}^\infty(\mathbb{D})$ [3]). We have some immediate consequences.

Corollary 3.6. *A function f is an extreme point of $\overline{B}_{\mathcal{H}L_0(\mathbb{D})}$ if and only if f' is an extreme point of $\overline{B}_{\mathcal{H}^\infty(\mathbb{D})}$.*

Corollary 3.7. *A function $f \in \mathcal{H}L_0(\mathbb{D})$ attains its norm as a functional on $\mathcal{G}_0(\mathbb{D})$ if and only if $f' \in \mathcal{H}^\infty(\mathbb{D})$ attains its norm as a functional on $\mathcal{G}^\infty(\mathbb{D})$.*

Let us state one more consequence of Proposition 3.5.

Corollary 3.8. *Let X be a complex Banach space and $\varphi \in \mathcal{G}_0(B_X)$.*

(a) There are sequences $(x_n), (y_n) \subset B_X$ with $x_n \neq y_n$ and $(a_n) \subset \ell_1$ such that

$$\varphi = \sum_{n=1}^{\infty} a_n m_{x_n, y_n}.$$

Moreover, $\|\varphi\| = \inf \sum_{n=1}^\infty |a_n|$ where the infimum is taken over all such representations of φ.

(b) There are sequences $(x_n) \subset B_X$, $(y_n) \subset S_X$ and $(a_n) \subset \ell_1$ such that

$$\varphi = \sum_{n=1}^{\infty} a_n e_{x_n, y_n}.$$

Moreover, $\|\varphi\| = \inf \sum_{n=1}^\infty |a_n|$ where the infimum is taken over all such representations of φ.

Proof. Given $\varepsilon > 0$, Proposition 3.5 (a) provides an element $\mu \in \mathcal{F}(B_X)$ with $\pi(\mu) = \varphi$ and $\|\pi(\mu)\| \leqslant \|\varphi\| + \varepsilon$. It is known (see e.g. [2, Lem. 3.3]) that there are points $x_n, y_n \in B_X$ and $(a_n) \subset \ell_1$ with $\mu = \sum_{n=1}^\infty a_n \frac{\delta(x_n) - \delta(y_n)}{\|x_n - y_n\|}$ and $\sum_{n=1}^\infty |a_n| \leqslant \|\mu\| + \varepsilon \leqslant \|\varphi\| + 2\varepsilon$ (here δ denotes the canonical embedding $\delta\colon B_X \to \mathcal{F}(B_X)$). Then $\varphi = \sum_{n=1}^\infty a_n \pi(\frac{\delta(x_n) - \delta(y_n)}{\|x_n - y_n\|}) = \sum_{n=1}^\infty a_n m_{x_n, y_n}$.

Item (b) follows similarly using the corresponding property for projective tensor products (see e.g. [41, Prop. 2.8]) and $\mathcal{G}^\infty(B_X)$ [36, Th. 5.1]. □

Following Mujica's ideas we study the metric approximation property (MAP) for $\mathcal{G}_0(B_X)$. For that we first prove the following result about approximation of elements in the closed unit ball of the dual space. We introduce the notation:

- $\mathcal{P}_0(X, Y)$: The vector space of polynomials from $P\colon X \to Y$ such that $P(0) = 0$ endowed with the norm $\|dP\| = L(P|_{B_X})$.
- $\mathcal{P}_{f,0}(X, Y)$: The subspace of $\mathcal{P}_0(X, Y)$ consisting of finite type polynomials.

Proposition 3.9. *Let X and Y be complex Banach spaces. Then*

(a) $\overline{B}_{\mathcal{H}L_0(B_X,Y)} = \overline{B_{\mathcal{P}_0(X,Y)}}^{\tau_0}$.
(b) If X has the MAP then $\overline{B}_{\mathcal{H}L_0(B_X,Y)} = \overline{B_{\mathcal{P}_{f,0}(X,Y)}}^{\tau_0}$.

Proof. (a) If $f \in \overline{B}_{\mathcal{H}L_0(B_X,Y)}$ then $f \in \mathcal{H}^\infty(B_X, Y)$ and $f(0) = 0$. Consider the Taylor series expansion of f at 0: $f(x) = \sum_{k=0}^{\infty} P^k f(0)(x)$. As in [35], for each $m \in \mathbb{N} \cup \{0\}$, we denote

$$S_m f(x) = \sum_{k=0}^{m} P^k f(0)(x) \qquad \text{and} \qquad \sigma_m f(x) = \frac{1}{m+1} \sum_{k=0}^{m} S_k f(x).$$

Since $df = \sum_{k=0}^{\infty} dP^k f(0) \in \mathcal{H}^\infty(B_X, \mathcal{L}(X, Y))$ it follows from [35, Prop. 5.2] that $\sigma_m f(x) \to f(x)$ for all $x \in B_X$ and

$$\|d\sigma_m f\| = \|\sigma_m(df)\| \leqslant \|df\| \leqslant 1.$$

This implies that $f \in \overline{B_{\mathcal{P}_0(X,Y)}}^{\tau_0}$.

For the reverse inclusion, let $f \in \mathcal{H}L_0(B_X, Y)$ and $(P_\alpha) \subset B_{\mathcal{P}_0(X,Y)}$ such that $P_\alpha(x) \to f(x)$ for all $x \in B_X$. Then $L(f) \leqslant 1$ and so $f \in \overline{B}_{\mathcal{H}L_0(B_X,Y)}$.

(b) If X has the MAP there is a net of finite rank operators $(T_\alpha) \subset \mathcal{L}(X, X)$ such that $T_\alpha(x) \to x$ for all $x \in X$ and $\|T_\alpha\| \leqslant 1$ for every α. Given $P \in B_{\mathcal{P}_0(X,Y)}$ we have that $P \circ T_\alpha$ belongs to $B_{\mathcal{P}_{f,0}(X,Y)}$ (since $L(P \circ T_\alpha|_{B_X}) < 1$) and $P(T_\alpha x) \to P(x)$ for every x. This means that $P \in \overline{B_{\mathcal{P}_{f,0}(X,Y)}}^{\tau_0}$. Finally, an appeal to (a) yields the result. $\square$

Theorem 3.10. *X has the MAP if and only if $\mathcal{G}_0(B_X)$ has the MAP.*

Proof. X being isometric to a 1-complemented subspace of $\mathcal{G}_0(B_X)$ it is clear that X has the MAP when $\mathcal{G}_0(B_X)$ has it.

Now, suppose that X has the MAP and consider the mapping $\delta \in \overline{B}_{\mathcal{H}L_0(B_X,\mathcal{G}_0(B_X))}$. By Proposition 3.9 there exist a net $(P_\alpha) \subset B_{\mathcal{P}_{f,0}(X,\mathcal{G}_0(B_X))}$ such that $P_\alpha(x) \to \delta(x)$ for all $x \in B_X$. Applying a linearization as in Proposition 3.1 we obtain finite rank linear mappings T_{P_α} with norm bounded by 1, such that the following diagram commutes:

$$\begin{tikzcd}
B_X \arrow[r, "P_\alpha"] \arrow[d, "\delta"'] & \mathcal{G}_0(B_X) \\
\mathcal{G}_0(B_X) \arrow[ur, "T_{P_\alpha}"'] &
\end{tikzcd}$$

Note that $T_{P_\alpha}(\delta(x)) = P_\alpha(x) \to \delta(x) = Id(\delta(x))$. Then, we have that $T_{P_\alpha} \to Id$ on $\operatorname{span}\{\delta(x) : x \in B_X\}$. Since the net (T_{P_α}) is bounded the same holds for the closure. Hence, $\mathcal{G}_0(B_X)$ has the MAP. $\square$

Note that our arguments cannot be adapted to the case in which X has the BAP since the approximations of the identity could send the unit ball B_X to a bigger ball.

Question 1. Does $\mathcal{G}_0(B_X)$ have the BAP whenever X has the BAP?

The same question for $\mathcal{G}^\infty(B_X)$ was posed by Mujica in [35]. As far we know, this question is still open.

Another consequence of the linearization process shows that functions in $\mathcal{H}L_0$ behave similarly to functions in $\mathrm{Lip}_0(B_X, B_Y)$ (that can be isometrically factorized through the free-Lipschitz spaces $\mathcal{F}(B_X)$ and $\mathcal{F}(B_Y)$). Given $f \in \mathcal{H}L_0(B_X, Y)$ with $f(B_X) \subset B_Y$ we can easily obtain a commutative diagram:

$$\begin{tikzcd} B_X \arrow[r, "f"] \arrow[d, "\delta_X"'] & B_Y \arrow[d, "\delta_Y"] \\ \mathcal{G}_0(B_X) \arrow[r, "T_{\delta_Y \circ f}"'] & \mathcal{G}_0(B_Y), \end{tikzcd} \tag{6}$$

where $T_{\delta_Y \circ f}$ is linear and $\|T_{\delta_Y \circ f}\| = L(f)$.

Remark 3.11. With the same procedure as at the beginning of the section we can produce a canonical predual $\mathcal{G}(B_X)$ of $\mathcal{H}L(B_X)$ made up of elements of $\mathcal{H}L(B_X)^*$ which are τ_0-continuous when restricted to the closed unit ball. The fact that $\mathcal{H}L_0(B_X)$ is a 1-complemented subspace of $\mathcal{H}L(B_X)$ and that the projection from $\mathcal{H}L(B_X)$ onto $\mathcal{H}L_0(B_X)$ is $\tau_0 - \tau_0$ continuous allow us to derive that $\mathcal{G}_0(B_X)$ is isometric to a 1-complemented subspace of $\mathcal{G}(B_X)$.

With standard adaptations most of the results of this section can be stated for $\mathcal{G}(B_X)$ instead of $\mathcal{G}_0(B_X)$. That is the case of Propositions 3.1, 3.2, 3.9 and Theorem 3.10. The version of Proposition 3.4 for $\mathcal{G}(B_X)$ requires the addition of $\delta(0)$ to both considered sets. Also note that the square diagram (6) can be made for $\mathcal{G}(B_X)$ but there is no equality between the norms of $T_{\delta_Y \circ f}$ and f.

4. Relation between $\mathcal{G}_0(B_X)$ and $\mathcal{G}_0(B_Y)$ when $X \subset Y$

Recall that, given metric spaces M, N with $0 \in M \subset N$, the (real) Lipschitz-free space $\mathcal{F}(M)$ canonically identifies with a subspace of $\mathcal{F}(N)$. This follows from the McShane extension theorem asserting that for every $f \in \mathrm{Lip}_0(M, \mathbb{R})$ there is $\tilde{f} \in \mathrm{Lip}_0(N, \mathbb{R})$ with $\tilde{f}|_M = f$ and $L(f) = L(\tilde{f})$, see e.g. [42, Th. 1.33]. Note in passing that in the complex-valued case all extensions can have a larger Lipschitz constant. This is why our next goal is to analyze the corresponding relation between $\mathcal{G}_0(B_X)$ and $\mathcal{G}_0(B_Y)$ when $X \subset Y$. Then $B_X \subset B_Y$ and the restriction mapping has norm one:

$$\begin{aligned} \mathcal{H}L_0(B_Y) &\to \mathcal{H}L_0(B_X) \\ f &\mapsto f|_{B_X}. \end{aligned}$$

Then, the following mapping also has norm one:

$$\begin{aligned} \rho : \mathcal{G}_0(B_X) &\to \mathcal{G}_0(B_Y) \\ \varphi &\mapsto \widehat{\varphi}, \end{aligned}$$

where $\widehat{\varphi}(f) = \varphi(f|_{B_X})$.

Whenever ρ is an isometry, we write $\mathcal{G}_0(B_X) \subset \mathcal{G}_0(B_Y)$. Then, by the Hahn-Banach theorem, every element of $\mathcal{H}L_0(B_X)$ would have a norm preserving extension to $\mathcal{H}L_0(B_Y)$. Since there exist polynomials which cannot be extended to a larger space it is not always true that $\mathcal{G}_0(B_X) \subset \mathcal{G}_0(B_Y)$. Moreover, the previous argument can be clearly reversed, so: $\mathcal{G}_0(B_X) \subset \mathcal{G}_0(B_Y)$ if and only if every $f \in \mathcal{H}L_0(B_X)$ has a norm preserving extension to $\mathcal{H}L_0(B_Y)$.

We study some cases where this norm preserving extension occurs. All are cases where we have an extension morphism. The simplest occurs when X is 1-complemented in Y. Here, the complementation also spreads to $\mathcal{G}_0(B_X)$.

Proposition 4.1. *If X is 1-complemented in Y then ρ is an isometry and $\mathcal{G}_0(B_X)$ is a 1-complemented subspace of $\mathcal{G}_0(B_Y)$.*

Proof. Let $\pi\colon Y \to X$ be a norm-one projection. Given $f \in \mathcal{H}L_0(B_X)$ the mapping $f \circ \pi$ belongs to $\mathcal{H}L_0(B_Y)$ with $L(f \circ \pi) \leqslant L(f)$ and $(f \circ \pi)|_{B_X} = f$. Now, for each $\varphi \in \mathcal{G}_0(B_X)$,

$$\|\varphi\| = \sup_{f \in B_{\mathcal{H}L_0(B_X)}} |\varphi(f)| = \sup_{f \in B_{\mathcal{H}L_0(B_X)}} |\widehat{\varphi}(f \circ \pi)| \leqslant \|\widehat{\varphi}\|.$$

Thus, $\|\varphi\| = \|\widehat{\varphi}\|$, meaning that ρ is an isometry. Finally, we derive that $\mathcal{G}_0(B_X)$ is 1-complemented in $\mathcal{G}_0(B_Y)$ through the following projection:

$$\begin{aligned} \mathcal{G}_0(B_Y) &\to \mathcal{G}_0(B_X) \\ \psi &\mapsto [f \mapsto \psi(f \circ \pi)]. \end{aligned}$$

□

M. Jung has proved recently that $\mathcal{G}^\infty(B_X)$ does not have the Radon-Nikodym property (RNP) for any X [32]. Here we obtain the same result for $\mathcal{G}_0(B_X)$.

Corollary 4.2. *The space $\mathcal{G}_0(B_X)$ fails to have the Radon-Nikodym Property for every complex Banach space X.*

Proof. The space $\mathcal{G}^\infty(\mathbb{D})$ fails to have the RNP since its the unit ball does not have extreme points [3]. Thus, by the isometry presented in Proposition 3.5, the same holds for $\mathcal{G}_0(\mathbb{D})$. Since $\mathbb{C}$ is 1-complemented in X, Proposition 4.1 yields that $\mathcal{G}_0(\mathbb{D})$ is a subspace of $\mathcal{G}_0(B_X)$ and we are done. □

Another situation when we have an extension morphism is when $Y = X^{**}$. Recall that, given $f \in \mathcal{H}^\infty(B_X)$, we can consider its AB extension $\tilde{f} \in \mathcal{H}^\infty(B_{X^{**}})$ [6]. The AB extension, which defines an isometry from $\mathcal{H}^\infty(B_X)$ to $\mathcal{H}^\infty(B_{X^{**}})$ [21], is a topic widely developed in the literature. For instance, it is essential in the description of the spectrum (or maximal ideal space) of the Banach algebra $\mathcal{H}^\infty(B_X)$. Another ingredient that usually appears associated with the AB extension and its properties is the notion of *symmetrically regular space.* Both these concepts have their origin in the study initiated by Arens [4, 5] about extending the product of a Banach algebra to its bidual.

For an n-linear mapping $A : X \times \cdots \times X \to Y$ the canonical extension $\widetilde{A} : X^{**} \times \cdots \times X^{**} \to Y^{**}$ is given by consecutive weak-star convergence in the following way:

$$\widetilde{A}(x_1^{**}, \ldots, x_n^{**})(y^*) = \lim_{\alpha_1} \ldots \lim_{\alpha_n} y^*(A(x_{\alpha_1}, \ldots, x_{\alpha_n}))$$

where each $(x_{\alpha_i}) \subset X$ is a net which is weak-star convergent to x_i^{**} and $y^* \in Y^*$. Now, the AB extension of a homogeneous polynomial $P \in \mathcal{P}(^nX, Y)$ is given by $\widetilde{P} \in \mathcal{P}(^nX^{**}, Y^{**})$ which is defined, for $x^{**} \in X^{**}$, in the expected way:

$$\widetilde{P}(x^{**}) = \widetilde{\widetilde{\overset{\vee}{P}}}(x^{**}, \dots, x^{**}).$$

This provides a way to extend bounded holomorphic functions $f \in \mathcal{H}^\infty(B_X, Y) \rightsquigarrow \widetilde{f} \in \mathcal{H}^\infty(B_{X^{**}}, Y^{**})$ and we know from [21] that this extension is an isometry: $\|f\| = \|\widetilde{f}\|$.

Recall that X is said to be *regular* if every continuous bilinear mapping $A : X \times X \to \mathbb{C}$ is Arens regular. That is, the following two extensions of A to $X^{**} \times X^{**} \to \mathbb{C}$ coincide:

$$\lim_\alpha \lim_\beta A(x_\alpha, y_\beta) \quad \text{and} \quad \lim_\beta \lim_\alpha A(x_\alpha, y_\beta),$$

where (x_α) and (y_β) are nets in X converging weak-star to points x_0^{**} and y_0^{**} in X^{**}. The space X is *symmetrically regular* if the above holds for every continuous symmetric bilinear form. Equivalently, X is (symmetrically) regular if any continuous (symmetric) linear mapping $T : X \to X^*$ is weakly compact. Several equivalent characterizations of this notion can be seen in [8, Th. 8.3] and some interesting properties appeared in [9, Section 1]. As examples of non reflexive regular (and hence, symmetrically regular) Banach spaces we have, for instance, those that satisfy property (V) of Pełczyński, like c_0, $C(K)$ or $\mathcal{H}^\infty(\mathbb{D})$ while typical non symmetrically regular spaces are ℓ_1 and $X \oplus X^*$, for any non reflexive space X. Also, Leung [34, Th. 12] provided an example of a symmetrically regular space that is not regular and in [9] it is showed that $c_0(\ell_1^n)$ is regular but its bidual $\ell_\infty(\ell_1^n)$ is not symmetrically regular.

We now want to work with the AB extension for elements in $\mathcal{H}L_0(B_X)$. For $f \in \mathcal{H}L_0(B_X)$, in order to compute the Lipschitz constant of $\widetilde{f}$ we need to deal with the differential of the AB extension, $d\widetilde{f}$ which belongs to $\mathcal{H}(B_{X^{**}}, X^{***})$. Instead, we do know the norm of the AB extension of the differential $\widetilde{df} \in \mathcal{H}^\infty(B_{X^{**}}, X^{***})$. Fortunately, on symmetrically regular spaces they coincide:

Proposition 4.3. *If X is symmetrically regular and $f \in \mathcal{H}L_0(B_X)$ then $d\widetilde{f} = \widetilde{df}$.*

Proof. If $f = \sum_{k=0}^\infty P^k f(0)$ then the series expansion of df at 0 is given by $df = \sum_{k=0}^\infty dP^k f(0)$. Thus, $\widetilde{df} = \sum_{k=0}^\infty \widetilde{(dP^k f(0))}$. On the other hand, $\widetilde{f} = \sum_{k=0}^\infty \widetilde{P^k f(0)}$ and so $d\widetilde{f} = \sum_{k=0}^\infty d(\widetilde{P^k f(0)})$.

Therefore, the result is proved once we show that for any given $m \in \mathbb{N}$ and any $P \in \mathcal{P}(^mX)$, $\widetilde{dP} = d\widetilde{P}$. Note that in this case $\widetilde{P} \in \mathcal{P}(^mX^{**})$, $dP \in \mathcal{P}(^{m-1}X, X^*)$ while both $\widetilde{dP}$ and $d\widetilde{P}$ belong to $\mathcal{P}(^{m-1}X^{**}, X^{***})$.

When X is symmetrically regular, it follows from [8, Th. 8.3] that $\widetilde{\widetilde{\overset{\vee}{P}}} = \overset{\vee}{\widetilde{P}}$. The argument is now complete because, for each $x^{**}, y^{**} \in X^{**}$ we have $\widetilde{dP}(x^{**})(y^{**}) = m\widetilde{\widetilde{\overset{\vee}{P}}}(x^{**}, \dots, x^{**}, y^{**})$ and $d\widetilde{P}(x^{**})(y^{**}) = m\overset{\vee}{\widetilde{P}}(x^{**}, \dots, x^{**}, y^{**})$. $\square$

Proposition 4.4. *If X is symmetrically regular then the AB extension mapping*

$$E : \mathcal{H}L_0(B_X) \to \mathcal{H}L_0(B_{X^{**}})$$
$$f \mapsto \widetilde{f}$$

is an isometry.

Proof. If $f \in \mathcal{H}L_0(B_X)$ then its norm is given by $\|df\|$. By [21], $\|df\| = \|\widetilde{df}\|$. Also, by the previous proposition we know that $d\widetilde{f} = \widetilde{df}$. So, we obtain that $\|df\| = \|d\widetilde{f}\|$, meaning that $\widetilde{f}$ does indeed belong to $\mathcal{H}L_0(B_{X^{**}})$ and that the mapping $f \mapsto \widetilde{f}$ is an isometry. □

In the previous result symmetric regularity is used to obtain that $d\widetilde{f} = \widetilde{df}$. Actually we only need the identity of their norms: $\|d\widetilde{f}\| = \|\widetilde{df}\|$. We do not know if this equality holds in general.

Corollary 4.5. *If X is symmetrically regular then $\mathcal{G}_0(B_X) \subset \mathcal{G}_0(B_{X^{**}})$.*

Note that in the above corollary the hypothesis of symmetric regularity is not a necessary condition since, for example, for $X = \ell_1$ the result holds due to Proposition 4.1.

A generalization of this procedure (which, however, uses the AB extension in its definition) is when there exists an isometric extension morphism $s\colon X^* \to Y^*$. This happens, for instance, when X is an M-ideal in Y. More generally, if $X \subset Y$ then the existence of an isometric extension morphism $s\colon X^* \to Y^*$ is equivalent to X^{**} being 1-complemented in Y^{**}.

Note that $s(x^*)(x) = x^*(x)$ for all $x \in X$, $x^* \in X^*$ and that $\|s(x^*)\| = \|x^*\|$. This extension transfers to $\mathcal{H}^\infty(B_X)$ in the following way:

$$\begin{aligned} \overline{s} : \mathcal{H}^\infty(B_X) &\to \mathcal{H}^\infty(B_Y) \\ f &\mapsto \widetilde{f} \circ s^* \circ i_Y, \end{aligned}$$

where $i_Y : Y \to Y^{**}$ is the canonical inclusion.

The mapping $\overline{s}$ is an isometric extension from $\mathcal{H}^\infty(B_X)$ to $\mathcal{H}^\infty(B_{X^{**}})$. Again, to work in $\mathcal{H}L_0(B_X)$ we require a symmetrically regular hypothesis.

Proposition 4.6. *If X is symmetrically regular, $X \subset Y$ and there is an isometric extension morphism $s : X^* \to Y^*$ then*

$$\begin{aligned} \overline{s} : \mathcal{H}L_0(B_X) &\to \mathcal{H}L_0(B_Y) \\ f &\mapsto \widetilde{f} \circ s^* \circ i_Y \end{aligned}$$

is an isometric extension.

Proof. For any $P \in \mathcal{P}(^mX)$ we have that $\overline{s}(P) \in \mathcal{P}(^mY)$ and $d(\overline{s}(P)) \in \mathcal{P}(^{m-1}Y, Y^*)$. Now, for $y, z \in B_Y$,

$$\begin{aligned} d(\overline{s}(P))(y)(z) &= m\widecheck{(\overline{s}(P))}(y, \ldots, y, z) = m\check{\widetilde{P}}\left(s^*(i_Y(y)), \ldots, s^*(i_Y(y)), s^*(i_Y(z))\right) \\ &= d\widetilde{P}(s^*(i_Y(y)))(s^*(i_Y(z))) = (i_Y^* \circ s^{**} \circ d\widetilde{P} \circ s^* \circ i_Y)(y)(z). \end{aligned}$$

This says that $d(\overline{s}(P)) = i_Y^* \circ s^{**} \circ d\widetilde{P} \circ s^* \circ i_Y$ for every polynomial $P \in \mathcal{P}(^mX)$. Then, the same equality holds for every $f \in \mathcal{H}L_0(B_X)$:

$$d(\overline{s}(f)) = i_Y^* \circ s^{**} \circ d\widetilde{f} \circ s^* \circ i_Y.$$

Since X is symmetrically regular, by Proposition 4.4 we obtain that $\|d(\overline{s}(f))\| \leqslant \|d\widetilde{f}\| = \|df\|$. Also, note that for $x \in B_X$, we have $s^* \circ i_Y(x) = i_X(x)$. This implies that $d\widetilde{f}(s^*(i_Y(x)) = i_{X^*}(df(x))$. Therefore,

$$d(\overline{s}(f))(x) = i_Y^* \circ s^{**}\left(i_{X^*}(df(x))\right) = s(df(x)).$$

This equality and the fact that s is an isometry allow us to derive the other inequality:

$$\begin{aligned}\|d(\overline{s}(f))\| &\geqslant \sup_{x\in B_X}\|d(\overline{s}(f))(x)\| = \sup_{x\in B_X}\|s(df(x))\|\\ &= \sup_{x\in B_X}\|df(x)\| = \|df\|,\end{aligned}$$

which concludes the proof. ☐

Corollary 4.7. *If X is symmetrically regular, $X\subset Y$ and there is an isometric extension morphism $s: X^*\to Y^*$ then $\mathcal{G}_0(B_X)\subset\mathcal{G}_0(B_Y)$.*

4.1. **Dual isometric spaces.** It is known that there exist non isomorphic Banach spaces with isomorphic duals. Attending to that, Díaz and Dineen [23] posed the following question: if X and Y are Banach spaces such that X^* and Y^* are isomorphic, under which conditions is it true that $\mathcal{P}(^nX)$ and $\mathcal{P}(^nY)$ are isomorphic for every $n\geqslant 1$? That is, if X^* and Y^* are isomorphic (i. e. the spaces of 1-homogeneous polynomials are isomorphic) does it imply that the spaces of n-homogeneous polynomials are isomorphic for every n? They also gave a partial answer to this question. Later, a relaxation of the conditions was obtained by Cabello-Sánchez, Castillo and García [15, Th. 1] and Lassalle and Zalduendo [33, Th. 4] independently, proving that the answer is affirmative whenever X and Y are symmetrically regular. We present here a version of this result for holomorphic Lipschitz functions on the ball. Since we need to remain inside the ball when changing the space we have to restrict ourselves to the case of isometric isomorphisms.

Proposition 4.8. *If X and Y are symmetrically regular Banach spaces such that X^* and Y^* are isometrically isomorphic then $\mathcal{H}L_0(B_X)$ and $\mathcal{H}L_0(B_Y)$ are isometrically isomorphic as well.*

Proof. Let us denote by $s: X^*\to Y^*$ the isometric isomorphism and consider the mapping $\overline{s}:\mathcal{H}L_0(B_X)\to\mathcal{H}L_0(B_Y)$ as in Proposition 4.6. By the proof of that proposition we derive that $\overline{s}$ is continuous and $\|\overline{s}\|\leqslant 1$. Since Y is symmetrically regular, we can use the same procedure for the mapping $\overline{s^{-1}}:\mathcal{H}L_0(B_Y)\to\mathcal{H}L_0(B_X)$ leading to $\|\overline{s^{-1}}\|\leqslant 1$. Finally, appealing to [33, Cor. 3] we obtain that $\overline{s^{-1}}\circ\overline{s}(P)=P$ for every homogeneous polynomial P on X and, hence, $\overline{s^{-1}}\circ\overline{s}(f)=f$ for every $f\in\mathcal{H}L_0(B_X)$. Indeed, if $\sum_{k=0}^\infty P^k$ is the Taylor series expansion of a given $f\in\mathcal{H}L_0(B_X)$, then $\tilde{f}(z)=\sum_{k=0}^\infty\tilde{P}^k(z)$ for every $z\in B_{X^{**}}$. Thus

$$\overline{s}(f)(y)=\tilde{f}(s^*(i_Y(y)))=\sum_{k=0}^\infty\tilde{P}^k(s^*(y))=\sum_{k=0}^\infty\overline{s}(P^k)(y),$$

for every $y\in Y$. From here

$$\begin{aligned}\overline{s^{-1}}(\overline{s}(f))(x) &= \widetilde{\overline{s}(f)}\big((s^{-1})^*(i_X(x))\big)=\sum_{k=0}^\infty\widetilde{\overline{s}(P^k)}\big((s^{-1})^*(i_X(x)\big)\\ &= \sum_{k=0}^\infty\overline{s^{-1}}(\overline{s}(P^k))(x)=\sum_{k=0}^\infty P^k(x)=f(x),\end{aligned}$$

for every $x\in X$. Analogously one can check that $\overline{s}\circ\overline{s^{-1}}(f)=f$ for every $f\in\mathcal{H}L(B_Y)$. ☐

In the previous proposition we can change the hypothesis of X and Y being symmetrically regular by X or Y being regular. Indeed, it is proved in [33, Rmk. 2] (see also [15, Prop. 1]) that if X^* and Y^* are isomorphic and X is regular then so is Y.

4.2. **Mapping between $\mathcal{G}_0(B_X)$ and $\mathcal{G}_0(B_Y)$.** Any linear mapping between X and Y produces a mapping between $\mathcal{G}_0(B_X)$ and $\mathcal{G}_0(B_Y)$ by a canonical procedure (actually, two canonical procedures depending on the norm of the mapping).

(i) Let $\psi\colon X \to Y$ a linear mapping with $\|\psi\| \leqslant 1$. Note that $L(\psi) = \|\psi\|$ in this case. Since $\psi(B_X) \subset B_Y$ we can define the canonical mapping with norm $\leqslant 1$:

$$\begin{aligned} \mathcal{H}L_0(B_Y) &\to \mathcal{H}L_0(B_X) \\ f &\mapsto f \circ \psi. \end{aligned}$$

Thus, the following also has norm $\leqslant 1$:

$$\begin{aligned} T_{\delta_Y \circ \psi} \colon \mathcal{G}_0(B_X) &\to \mathcal{G}_0(B_Y) \\ \varphi &\mapsto \widehat{\varphi}, \end{aligned}$$

where $\widehat{\varphi}(f) = \varphi(f \circ \psi)$.

(ii) When $\|\psi\| > 1$ the previous construction does not work but we can appeal to a linearization plus differentiation process (as we used to show that X is a 1-complemented subspace of $\mathcal{G}_0(B_X)$).

Let $\psi \in \mathcal{L}(X, Y)$ so that $\psi|_{B_X} \in \mathcal{H}L_0(B_X, Y)$. We have the usual commutative diagram:

$$\begin{tikzcd} B_X \arrow[r, "\psi|_{B_X}"] \arrow[d, "\delta_X"'] & Y \arrow[d, "\delta_Y"] \\ \mathcal{G}_0(B_X) \arrow[ur, "T_\psi"'] & \mathcal{G}_0(B_Y) \end{tikzcd}$$

where $T_\psi \in \mathcal{L}(\mathcal{G}_0(B_X), Y)$.

Applying the differential at 0 to the equality $\psi|_{B_X} = T_\psi \circ \delta_X$ we get the commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "d\delta_X(0)"'] & Y \arrow[d, "d\delta_Y(0)"] \\ \mathcal{G}_0(B_X) \arrow[ur, "T_\psi"'] \arrow[r, "d\delta_Y(0) \circ T_\psi"'] & \mathcal{G}_0(B_Y). \end{tikzcd}$$

Note that the linear mapping $d\delta_Y(0) \circ T_\psi : \mathcal{G}_0(B_X) \to \mathcal{G}_0(B_Y)$ has norm less than or equal to $\|\psi\|$.

5. Local complementation in the bidual

In this section, we are interested in the relationship between $\mathcal{G}_0(B_{X^{**}})$ and $\mathcal{G}_0(B_X)^{**}$ under the hypothesis of X^{**} having the MAP, in the spirit of what is done in [16].

We begin with a result about a special approximation behavior in the case that the bidual space has the MAP.

Proposition 5.1. *Let X, Y be Banach spaces such that X^{**} has the MAP. For each $f \in \mathcal{H}L_0(B_{X^{**}}, Y)$ with $L(f) = 1$ there exists a net $(Q_\alpha) \subset \mathcal{P}_{f,0}(X, Y)$ with $L(Q_\alpha|_{B_X}) \leqslant 1$ satisfying $\widetilde{Q}_\alpha(x^{**}) \to f(x^{**})$ for all $x^{**} \in B_{X^{**}}$.*

Proof. By Proposition 3.9 it is enough to consider $f = P \in \mathcal{P}_0(X^{**}, Y)$ with $L(P|_{B_{X^{**}}}) = 1$. If X^{**} has the MAP we can appeal to [16, Cor. 1] to obtain a net of finite rank mappings $(t_\alpha) \subset \mathcal{L}(X, X^{**})$ with $\|t_\alpha\| \leqslant 1$ and $t_\alpha^{**}(x^{**}) \to x^{**}$ for all $x^{**} \in X^{**}$. Now we define $Q_\alpha = P \circ t_\alpha$, which clearly belongs to $\mathcal{P}_{f,0}(X, Y)$. Note that, for any $x, y \in B_X$,

$$\|Q_\alpha(x) - Q_\alpha(y)\| = \|P(t_\alpha(x)) - P(t_\alpha(y))\| \leqslant L(P|_{B_{X^{**}}})\|t_\alpha\|\|x - y\| \leqslant \|x - y\|.$$

Then, $L(Q_\alpha|_{B_X}) \leqslant 1$. Since t_α is a finite rank mapping, we have that $t_\alpha^{**} \in \mathcal{L}(X^{**}, X^{**})$. Hence, $\widetilde{Q}_\alpha = \widetilde{P} \circ t_\alpha^{**} = P \circ t_\alpha^{**}$. As a consequence, $\widetilde{Q}_\alpha(x^{**}) = P(t_\alpha^{**}(x^{**})) \to P(x^{**})$ for all $x^{**} \in B_{X^{**}}$. □

For a symmetrically regular space X, we consider the following mapping

$$\begin{aligned}\Theta : B_{X^{**}} &\to \mathcal{G}_0(B_X)^{**} = \mathcal{H}L_0(B_X)^* \\ x^{**} &\mapsto [f \in \mathcal{H}L_0(B_X) \mapsto \widetilde{f}(x^{**})].\end{aligned}$$

Proposition 5.2. *If X is symmetrically regular then Θ belongs to $\mathcal{H}L_0(B_{X^{**}}, \mathcal{G}_0(B_X)^{**})$ with $L(\Theta) = 1$.*

Proof. If X is symmetrically regular, by Proposition 4.4, the AB extension is an isometry from $\mathcal{H}L_0(B_X)$ into $\mathcal{H}L_0(B_{X^{**}})$, so Θ is well defined. For any $f \in \mathcal{H}L_0(B_X)$, we have $\Theta(\cdot)(f) = \widetilde{f}$, meaning that Θ is weak-star holomorphic and thus, it is holomorphic. Also, $\Theta(0) = 0$ and for any $x^{**}, y^{**} \in B_{X^{**}}$, once again by the symmetric regularity of X we have

$$\|\Theta(x^{**}) - \Theta(y^{**})\| = \sup_{f \in B_{\mathcal{H}L_0(B_X)}} \|\widetilde{f}(x^{**}) - \widetilde{f}(y^{**})\| \leqslant \|x^{**} - y^{**}\|.$$

This means that $\Theta \in \mathcal{H}L_0(B_{X^{**}}, \mathcal{G}_0(B_X)^{**})$ with $L(\Theta) \leqslant 1$. On the other hand,

$$\|\Theta(x^{**}) - \Theta(y^{**})\| \geqslant \sup_{x^* \in B_{X^*}} |x^{**}(x^*) - y^{**}(x^*)| = \|x^{**} - y^{**}\|.$$

Therefore, $L(\Theta) = 1$. □

As a consequence of the previous proposition, if X is symmetrically regular we can linearize the mapping Θ:

$$\begin{tikzcd} B_{X^{**}} \arrow[r, "\Theta"] \arrow[d, "\delta_{X^{**}}"'] & \mathcal{G}_0(B_X)^{**} \\ \mathcal{G}_0(B_{X^{**}}) \arrow[ur, "T_\Theta"'] & \end{tikzcd}$$

This produces a linear mapping $T_\Theta \in \mathcal{L}(\mathcal{G}_0(B_{X^{**}}), \mathcal{G}_0(B_X)^{**})$ with $\|T_\Theta\| = L(\Theta) = 1$.

Motivated by the Principle of Local Reflexivity, Kalton [31] introduced the following definition:

Definition 5.3. Given Banach spaces $X \subset Y$ we say that X is 1-locally complemented in Y if for every $\varepsilon > 0$ and every finite dimensional subspace F of Y there exist a linear mapping $T : F \to X$ such that $\|T\| \leqslant 1 + \varepsilon$ and $T(x) = x$ for all $x \in F \cap X$.

Note that the Principle of Local Reflexivity says that X is 1-locally complemented in X^{**}, for any Banach space X.

Theorem 5.4. *If X is symmetrically regular and X^{**} has the MAP then T_Θ embeds $\mathcal{G}_0(B_{X^{**}})$ as a 1-locally complemented subspace of $\mathcal{G}_0(B_X)^{**}$. In particular, T_Θ is an isometry.*

Proof. We know that the mapping $\delta_{X^{**}}$ belongs to $\mathcal{H}L_0(B_{X^{**}}, \mathcal{G}_0(B_{X^{**}}))$ with $L(\delta_{X^{**}}) = 1$. Thus, we can apply Proposition 5.1 to get a net $(Q_\alpha) \subset \mathcal{P}_{f,0}(X, \mathcal{G}_0(B_{X^{**}}))$ with $L(Q_\alpha|_{B_X}) \leqslant 1$ such that $\widetilde{Q}_\alpha(x^{**}) \to \delta_{X^{**}}(x^{**})$ for all $x^{**} \in B_{X^{**}}$.

Consider the following two commutative diagrams:

$$\begin{tikzcd} B_X \arrow[r, "Q_\alpha|_{B_X}"] \arrow[d, "\delta_X"'] & \mathcal{G}_0(B_{X^{**}}) \\ \mathcal{G}_0(B_X) \arrow[ur, "T_{Q_\alpha}"'] & \end{tikzcd} \qquad \begin{tikzcd} B_{X^{**}} \arrow[r, "\widetilde{Q}_\alpha|_{B_{X^{**}}}"] \arrow[d, "\delta_{X^{**}}"'] & \mathcal{G}_0(B_{X^{**}}) \\ \mathcal{G}_0(B_{X^{**}}) \arrow[ur, "T_{\widetilde{Q}_\alpha}"'] & \end{tikzcd}$$

Note that, since X is symmetrically regular we have

$$\|T_{Q_\alpha}\| = L(Q_\alpha|_{B_X}) = L(\widetilde{Q}_\alpha|_{B_X}) = \|T_{\widetilde{Q}_\alpha}\| \leqslant 1.$$

For each α, since T_{Q_α} is a finite rank operator we have that $T_{Q_\alpha}^{**}$ belongs to $\mathcal{L}(\mathcal{G}_0(B_X)^{**}, \mathcal{G}_0(B_{X^{**}}))$. Thus, we have the following diagram

$$\begin{tikzcd} \mathcal{G}_0(B_{X^{**}}) \arrow[rr, "T_\Theta"] \arrow[dr, "T_{\widetilde{Q}_\alpha}"'] & & \mathcal{G}_0(B_X)^{**} \arrow[dl, "T_{Q_\alpha}^{**}"] \\ & \mathcal{G}_0(B_{X^{**}}) & \end{tikzcd}$$

The space $\mathcal{G}_0(B_{X^{**}})$ has the MAP witnessed by the net $(T_{\widetilde{Q}_\alpha})$ thanks to (the proof of) Theorem 3.10. Appealing to [16, Lem. 4], the proof will be completed once we check that the previous diagram is commutative. For this, it is enough to prove that $T_{\widetilde{Q}_\alpha}(\delta_{X^{**}}(x^{**})) = T_{Q_\alpha}^{**} \circ T_\Theta(\delta_{X^{**}}(x^{**}))$ for every $x^{**} \in B_{X^{**}}$.

On the one hand we know that $T_{\widetilde{Q}_\alpha}(\delta_{X^{**}}(x^{**})) = \widetilde{Q}_\alpha(x^{**})$. On the other hand, $T_{Q_\alpha}^{**} \circ T_\Theta(\delta_{X^{**}}(x^{**})) = T_{Q_\alpha}^{**}(\Theta(x^{**}))$. To understand this element of $\mathcal{G}_0(B_{X^{**}})$ let us see how it acts on any $f \in \mathcal{H}L_0(B_{X^{**}})$:

$$\langle T_{Q_\alpha}^{**}(\Theta(x^{**})), f\rangle = \langle \Theta(x^{**}), T_{Q_\alpha}^{*}(f)\rangle. \tag{7}$$

Now, $T_{Q_\alpha}^{*}(f)$ belongs to $\mathcal{H}L_0(B_X)$ and for any $x \in B_X$ satisfies

$$T_{Q_\alpha}^{*}(f)(x) = \langle T_{Q_\alpha}^{*}(f), \delta_X(x)\rangle = \langle f, T_{Q_\alpha}(\delta_X(x))\rangle = \langle f, Q_\alpha(x)\rangle = (T_f \circ Q_\alpha)(x).$$

Then, $T_{Q_\alpha}^{*}(f) = T_f \circ Q_\alpha$. Replacing this equality in (7) and using the definition of Θ and the fact that the range of $\widetilde{Q}_\alpha$ is contained in $\mathcal{G}_0(B_{X^{**}})$ we derive

$$\begin{aligned} \langle T_{Q_\alpha}^{**}(\Theta(x^{**})), f\rangle &= \langle \Theta(x^{**}), T_f \circ Q_\alpha\rangle = \widetilde{T_f \circ Q_\alpha}(x^{**}) = T_f^{**} \circ \widetilde{Q}_\alpha(x^{**}) \\ &= T_f(\widetilde{Q}_\alpha(x^{**})) = \langle \widetilde{Q}_\alpha(x^{**}), f\rangle, \quad \text{for all } f \in \mathcal{H}L_0(B_{X^{**}}). \end{aligned}$$

Therefore, $T_{Q_\alpha}^{**}(\Theta(x^{**})) = \widetilde{Q}_\alpha(x^{**})$ and thus $T_{Q_\alpha}^{**} \circ T_\Theta(\delta_{X^{**}}(x^{**})) = T_{\widetilde{Q}_\alpha}(\delta_{X^{**}}(x^{**}))$ for every $x^{**} \in B_{X^{**}}$, which finishes the proof.

$\square$

It is known (see, for instance, [16, Lem. 3] or [31, Th. 3.5]) that X is 1-locally complemented in Y if and only if X^* is 1-complemented in Y^*.

Corollary 5.5. *If X is symmetrically regular and X^{**} has the MAP then $\mathcal{H}L_0(B_{X^{**}})$ is isometric to a 1-complemented subspace of $\mathcal{H}L_0(B_X)^{**}$.*

Under the same conditions of the previous results we can also obtained a version for holomorphic Lipschitz functions of the following characterization of unique norm preserving extensions to the bidual proved by Godefroy in [28].

Lemma 5.6. *Let X be a Banach space and $x^* \in X^*$ with $\|x^*\| = 1$. The following are equivalent:*

(i) x^* has a unique norm preserving extension to a functional on X^{**}.
(ii) The function $Id_{\overline{B}_{X^*}} : (\overline{B}_{X^*}, w^*) \longrightarrow (\overline{B}_{X^*}, w)$ is continuous at x^*.

Aron, Boyd and Choi [7] gave a version of this result for homogeneous polynomials. Later, other extensions appeared (for instance, in [25] for ideals of homogeneous polynomials and in [24] for bilinear mappings in operator spaces).

Now, the statement of the theorem in our setting is the following:

Theorem 5.7. *Suppose X is symmetrically regular and X^{**} has the MAP. Consider a function $f \in \mathcal{H}L_0(B_X)$ with $L(f) = 1$. Then, the following are equivalent:*

(i) f has a unique norm preserving extension to $\mathcal{H}L_0(B_{X^{**}})$.
(ii) The AB extension from $(\overline{B}_{\mathcal{H}L_0(B_X)}, w^*)$ to $(\overline{B}_{\mathcal{H}L_0(B_{X^{**}})}, w^*)$ is continuous at f.
(iii) If the net $(f_\alpha) \subset \overline{B}_{\mathcal{H}L_0(B_X)}$ converges pointwise to f, then $(\widetilde{f}_\alpha) \subset \overline{B}_{\mathcal{H}L_0(B_{X^{**}})}$ converges pointwise to $\widetilde{f}$.

Proof. $(i) \Rightarrow (ii)$ Let $(f_\alpha) \subset \overline{B}_{\mathcal{H}L_0(B_X)}$ be a net weak-star convergent to a function $f \in \overline{B}_{\mathcal{H}L_0(B_X)}$. By the weak-star compactness of the ball $\overline{B}_{\mathcal{H}L_0(B_{X^{**}})}$ there is a subnet $(\widetilde{f}_\beta)$ weak-star convergent to a function $g \in \overline{B}_{\mathcal{H}L_0(B_{X^{**}})}$. Since for each $x \in B_X$, $\widetilde{f}_\alpha(x) = f_\alpha(x) \to f(x)$ we derive that $g|_{B_X} = f$. Also, since $L(g) \leqslant 1 = L(f)$, it follows that $L(g) = L(f)$, which means that g is a norm preserving extension of f. By (i) and Proposition 4.4 we obtain that $g = \widetilde{f}$. Now, a standard subnet argument shows that the whole net $(\widetilde{f}_\alpha)$ must converge weak-star to $\widetilde{f}$.

$(ii) \Rightarrow (iii)$ It is clear due to Proposition 3.1 (d).

$(iii) \Rightarrow (i)$ Let $g \in \overline{B}_{\mathcal{H}L_0(B_{X^{**}})}$ be a norm preserving extension of f. By Proposition 5.1 there is a net $(Q_\alpha) \subset \mathcal{P}_{f,0}(X, Y)$ with $L(Q_\alpha|_{B_X}) \leqslant 1$ satisfying $\widetilde{Q}_\alpha(x^{**}) \to g(x^{**})$ for all $x^{**} \in B_{X^{**}}$. But for any $x \in B_X$ we have $\widetilde{Q}_\alpha(x) = Q_\alpha(x) \to g(x) = f(x)$. Now, (iii) clearly implies that $g = \widetilde{f}$. □

All the numbered results of Sections 4 and 5 have easily adapted analogous versions for $\mathcal{G}$ and $\mathcal{H}L$ instead of $\mathcal{G}_0$ and $\mathcal{H}L_0$.

5.1. **The case of $\mathcal{H}^\infty(B_X)$ and $\mathcal{G}^\infty(B_X)$.** The arguments of this section can be canonically translated to prove analogous results for the case of $\mathcal{G}^\infty$ instead of $\mathcal{G}_0$ (and $\mathcal{H}^\infty$ instead of $\mathcal{H}L_0$). Moreover, for this case the hypothesis of symmetrical regularity is unnecessary. Let us state the results without proofs, since they are similar to the previous arguments.

Theorem 5.8. *If X^{**} has the MAP then $\mathcal{G}^\infty(B_{X^{**}})$ is isometric to a 1-locally complemented subspace of $\mathcal{G}^\infty(B_X)^{**}$ and $\mathcal{H}^\infty(B_{X^{**}})$ is isometric to a 1-complemented subspace of $\mathcal{H}^\infty(B_X)^{**}$.*

The following question is posed in [16]: when X^{**} has the BAP, is it true that $\mathcal{H}^\infty(B_{X^{**}})$ is isomorphic to a complemented subspace of $\mathcal{H}^\infty(B_X)^{**}$? Note that the previous theorem answers affirmatively this open question for the case X^{**} having MAP.

Theorem 5.9. *Suppose X^{**} has the MAP. Consider a function $f \in \mathcal{H}^\infty(B_X)$ with $\|f\| = 1$. Then, the following are equivalent:*

(i) f has a unique norm preserving extension to $\mathcal{H}^\infty(B_{X^{**}})$.
(ii) The AB extension from $(\overline{B}_{\mathcal{H}^\infty(B_X)}, w^*)$ to $(\overline{B}_{\mathcal{H}^\infty(B_{X^{**}})}, w^*)$ is continuous at f.
(iii) If the net $(f_\alpha) \subset \overline{B}_{\mathcal{H}^\infty(B_X)}$ converges pointwise to f, then $(\widetilde{f}_\alpha) \subset \overline{B}_{\mathcal{H}^\infty(B_{X^{**}})}$ converges pointwise to $\widetilde{f}$.

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References

[1] P. Ahern, R. Schneider, Holomorphic Lipschitz functions in pseudoconvex domains. *Amer. J. Math.* **101** (1979), no. 3, 543–565.
[2] R. J. Aliaga, E. Pernecká, C. Petitjean, and A. Procházka, Supports in Lipschitz-free spaces and applications to extremal structure, *J. Math. Anal. Appl.* **489** (2020), no. 1, 124128, 14.
[3] T. Ando, On the predual of $\mathcal{H}^\infty$, *Comment. Math.* **1** (1978) 33–40 (special issue).
[4] R. Arens, The adjoint of a bilinear operation, *Proc. Amer. Math. Soc.*, **2** (1951), 839-848.
[5] R. Arens, Operations induced in function classes, *Monatsh. Math.* , **55** (1951), 1-19.
[6] R. Aron, P. Berner, A Hahn–Banach extensions theorem for analytic mappings. *Bull. Soc. Math. France*, **106** (1978) 3–24.
[7] R. Aron, C. Boyd, Y. S. Choi. Unique Hahn–Banach theorems for spaces homogeneous polynomials. *J. Aust. Math. Soc.*, **70** (2001) 387–400.
[8] R. Aron, B. J. Cole and T. W. Gamelin, Spectra of algebras of analytic functions on a Banach space, *J. Reine Angew. Math.*, **415** (1991) 51-93.
[9] R. Aron, P. Galindo, D. García, M. Maestre, Regularity and algebras of analytic functions in infinite dimensions. *Trans. Amer. Math. Soc.*, **348** (1996), no. 2, 543-559.
[10] L. Bernal-González, J López-Salazar, J. B. Seoane-Sepúlveda, On Weierstrass' monsters in the disc algebra. *Bull. Belg. Math. Soc. Simon Stevin* **25** (2018), no. 2, 241–262.
[11] L. Bernal-González, A. Bonilla, J. López-Salazar, J. B. Seoane-Sepúlveda, Boundary-nonregular functions in the disc algebra and in holomorphic Lipschitz spaces. *Mediterr. J. Math.* **15** (2018), no. 3, Paper No. 114, 20 pp.
[12] L. Bernal-González, A. Bonilla, J. López-Salazar, J. B. Seoane-Sepúlveda, Nowhere hölderian functions and Pringsheim singular functions in the disc algebra. *Monatsh. Math.* **188** (2019), no. 4, 591–609.
[13] C. Bessaga and A. Pełczyński, On bases and unconditional convergence of series in Banach spaces, *Studia Math.* , **17** (1958), 151–164.
[14] J. Bruna and F. Tugores. Free interpolation for holomorphic functions regular to the boundary. *Pac. J. Math.*, **108** (1983), no. 1, 31–49.

[15] F. Cabello Sánchez, J. Castillo, R. García. Polynomials on dual-isomorphic spaces. *Ark. Mat.*, **38** (2000), no. 1, 37-44.
[16] F. Cabello Sánchez, R. García. The bidual of tensor product of Banach spaces, *Rev. Mat. Iberoam.*, **21** (2005), 843-861.
[17] D. Carando, I. Zalduendo. Linearization of functions. *Math. Ann.* **328** (2004), 683-700.
[18] P.L. Duren, Theory of Hp spaces. Dover, Mineola, New York (2000)
[19] Peter G. Casazza. Approximation properties. Handbook of the geometry of Banach spaces, Vol. I, 271–316, North-Holland, Amsterdam, 2001.
[20] M. Cúth, M. Doucha, and P. Wojtaszczyk, On the structure of Lipschitz-free spaces, *Proc. Amer. Math. Soc.*, **144** (2016), pp. 3833–3846.
[21] A. M. Davie, T. W. Gamelin. A theorem on polynomial-star approximation. *Proc. Amer. Math. Soc.*, **106** (1989), no. 2, 351–356.
[22] A. Defant, D. García, M. Maestre, P. Sevilla-Peris. Dirichlet Series and Holomorphic Functions in High Dimensions, New Mathematical Monographs, Cambridge University Press, 37, 2019.
[23] J. C. Díaz, S. Dineen, Polynomials on stable spaces, *Ark. Mat.*, **36** (1998), 87–96.
[24] V. Dimant, M. Fernández-Unzueta. Biduals of tensor products in operator spaces. *Studia Math.*, **230** (2015), no. 2, 165-185.
[25] V. Dimant, D. Galicer, R. García. Geometry of integral polynomials, M-ideals and unique norm preserving extensions, *J. Funct. Anal.*, **262** (2012), 1987-2012.
[26] E. García-Sánchez, D. de Hevia, P. Tradacete. Free objects in Banach space theory. https://arxiv.org/abs/2302.10807.
[27] G. Godefroy, A survey on Lipschitz-free Banach spaces, *Comment. Math.*, **55** (2015), 89–118.
[28] G. Godefroy. Points de Namioka, espaces normante, applications á la Théorie isométrique de la dualité. *Israel J. Math.*, **38** (1981) 209–220.
[29] G. Godefroy, N. Kalton. Lipschitz-free Banach spaces. *Studia Math.*, **159** (2003), 121-141.
[30] A. Grothendieck, Résumé de la théorie métrique des produits tensoriels topologiques, *Soc. de Matemática de São Paulo*, (1956).
[31] N. Kalton. Locally complemented subspaces and $\mathcal{L}_p$-spaces for $0 < p < 1$. *Math. Nachr.*, **115** (1984), 71–97.
[32] M. Jung. Daugavet property of Banach algebras of holomorphic functions and norm-attaining holomorphic functions, *Adv. Math.*, **421** (2023).
[33] S. Lassalle, I. Zalduendo, To what extent does the dual Banach space E' determine the polynomials over E?, *Ark. Mat.*, **38** (2000), no. 2, 343-354.
[34] D. Leung, Some remarks on regular Banach spaces, *Glasgow Math. J.* **38** (1996), no. 2, 243-248.
[35] J. Mujica. Linearization of bounded holomorphic mappings on Banach spaces. *Trans. Amer. Math. Soc.*, **324** (1991), 867-887.
[36] J. Mujica, Linearization of holomorphic mappings of bounded type, in: Progress in Functional Analysis, in: North-Holl. Math. Stud., vol. 170, North-Holland, Amsterdam, 1992, pp. 149-162.
[37] J. Mujica. Complex analysis in Banach spaces, Dover Publications, Inc. Mineola, New. York, 2010.
[38] K. F. Ng, On a theorem of Dixmier, *Math. Scand.*, **29** (1971), 279-280.
[39] M. Pavlovic, Function Classes on the Unit Disc. An Introduction.Walter de Gruyter, Berlin (2014)
[40] W. Rudin, Holomorphic Lipschitz functions in balls, *Comment. Math. Helv.* **53** (1978), no. 1, 143–147.
[41] R. A. Ryan, Introduction to Tensor Products of Banach Spaces, Springer Monogr. Math., Springer, London, 2002.
[42] N. Weaver, Lipschitz Algebras, 2nd ed., World Sci., Hackensack, NJ, 2018.

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