BIRS REPORT: "NEW DIRECTIONS IN GEOMETRIC FLOWS"

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1. Overview

Geometric evolution equations describe physical phenomena ranging from a child's soap bubble to the evolution of the cosmos. The famous, Fields Medal winning work of Grigory Perelman used the "Ricci flow" equation to give a complete understanding of geometry in three dimensions. This was a huge leap forward in our understanding of these equations, and set off a firestorm of activity in the ensuing years.

In the ensuing years since Perelman's breakthrough resolution of the Poincaré and Geometrization conjectures, the subject of Ricci flow has continued to grow considerably in depth and breadth, in many distinct but interrelated directions. Most directly related to Perelman's work, the study of surgical Ricci flows in three dimensions has been considerably refined in recent years. In particular the works of Bamler, Kleiner-Lott, etc. have given a much more complete picture of the singularity formation and long time behavior of the Ricci flow in three dimensions. Recently Kleiner-Lott and Bamler-Kleiner have constructed a canonical surgical Ricci flow in three dimensions, which has been applied to study the topology of diffeomorphism groups and lead to a partial resolution of the Generalized Smale Conjecture. In a related direction, intriguing recent work of Haslhofer-Naber gives a characterization of the Ricci flow in terms of associated martingales on path spaces, leading to a new formulation of "weak Ricci flow" in arbitrary dimensions. Also there has been important progress on more general regularity theory for Ricci flow such as the work of Bamler on Ricci flow with bounded scalar curvature in the non-collapsing case.

Also there has been great progress in the use of Ricci flow to classify manifolds with natural curvature positivity conditions. The work of Bohm-Wilking

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used Ricci flow to classify manifolds with positive curvature operator, by developing new methods for analyzing the ODE governing the evolution of the curvature tensor. Later work of Brendle-Schoen used related methods to establish the 1/4-pinched differentiable sphere theorem. More recent works have appeared analyzing Ricci flows which preserve some curvature positivity conditions but still encounter local singularities, thus requiring the use of surgical solutions. Most notably Brendle has classified manifolds satisfying certain natural positivity conditions by this method.

Another branch of Ricci flow which has seen considerable progress is in the Kähler setting. Song-Tian introduced the 'analytic minimal model program,' a broad effort to use solutions of the Kähler-Ricci flow to give an analytic approach to the classical, algebraic, minimal model program. Since then many follow-up works in support of this program have appeared revealing the structure of singularities along the Kähler-Ricci flow, such as the work of Tian-Zhang, Chen-Wang and Bamler-Zhang, which made an alternative proof of the Yau-Tian-Donaldson conjecture possible. Despite these great advances many problems remain, and this remains an active area of research.

In another direction, many generalizations of Ricci flow have now appeared, aiming to address different questions in geometry and topology not naturally approachable via Ricci flow. Joint works of Streets-Tian introduced natural generalizations of the Ricci flow to the setting of Hermitian geometry, and since then other geometric flows in Hermitian geometry have also appeared, for instance the Chern-Ricci flow of Gil-Tosatti-Weinkove and the anomaly flow of Phong-Picard-Zhang, aiming to address different questions in the broad field of Hermitian geometry. Also, to aid in the search for special holonomy metrics, Bryant-Xu introduced a natural generalization of Ricci flow which aims to flow closed G_2 structures to genuine G_2 holonomy metrics. Recent work of Lotay has revealed more geometric structure of this flow, and obtained global existence results in some settings.

There has also been considerable progress in understanding mean curvature flow - the most important evolution equation in extrinsic geometry and a close cousin of the Ricci flow. Colding-Minicozzi have understood singularity formation in terms of "generic singularities", showing roughly speaking that singularities other than necks or round spheres can be perturbed away. This lead to a visionary picture of generic mean curvature flow, as well as very sharp structural results about the singular set. Building on the idea of generic flows, Bernstein-Wang proved a beautiful classification result for hypersurfaces of low entropy. In a series of impressive recent papers, Brendle-Huisken and Haslhofer-Kleiner constructed surgical solutions for mean convex surfaces and two-convex hypersurfaces in Euclidean space and other ambient manifolds. This in turn lead to new geometric and topological applications of the mean curvature flow, in particular the discovery of new minimal surfaces by Haslhofer-Ketover and Schulze's proof of a sharp isoperimetric inequality in Cartan-Hadamard manifolds. In a different direction, related to the work on canonical Ricci flow through singularities, Hershkovits-White proved that the mean curvature flow through singularities is unique provided all singularities have a mean convex neighborhood, a hypothesis verified by Hershkovits in the important special case of self-similar solutions. Related to this, in a recent breakthrough Brendle-Choi and Angenent-Daskalopoulos-Sesum established a complete classification of all blowup limits for the flow of mean convex surfaces and two-convex hypersurfaces.

2. Objectives:

As described in the overview, in the years following Perleman's work on the Poincaré and Geometrization conjectures, Ricci flow has experienced considerable growth in many directions. The subject has branched out into many directions, with different researchers working in the closely related but distinct areas of classical geometry, complex geometry, mathematical physics, PDE, special holonomy, etc. These works aim to exploit the powerful method of geometric flows to address natural questions in diverse areas of mathematics, making this topic relevant to a broad range of mathematicians. Classical Ricci flow is deeply intertwined with questions in topology and geometry, while Kähler-Ricci flow has many interesting interactions with algebraic geometry. Various generalizations of Kähler-Ricci flow to complex non-Kähler manifolds aim at applications in complex geometry and mathematical physics, while mean curvature flow and related hypersurface flows also have important applications in geometry, material science, image processing and general relativity.

Given this growth in disparate directions, we sought to bring these researchers together to process these works, reunify the field, and benefit from each other's insights. More directly, a primary objective of this workshop was to bring together experts working on different aspects of the subject of geometric flows in an effort to disseminate the most recent, deepest, advances and encourage collaboration in an effort to unify and exploit ideas from different subjects. To help achieve this objective we asked some participants deliver survey-style talks aimed at a broader audience, together with classical seminar-style talks.

Furthermore, the field of geometric analysis, and geometric flows in particular, continues to attract great attention from graduate students and postdocs in Canada, the US, and elsewhere. However, as the field continues to grow at such a rapid pace, it can be a daunting and bewildering task to keep up with all of the latest developments. Thus a second main objective was to expose younger mathematicians working generally in the subject of geometric flows to experts in the field and their most recent advances, in an effort to strengthen and renew the foundation of the field for the near future. We thus aimed to include large number of younger researchers to push towards this goal.

Women and minorities continue to be underrepresented in the highest levels of academia, and the subject of geometric evolution is no exception. With an eye towards this, a further goal was to increase the participation of women and minorities in the subject of geometric flows. The main effort towards this goal was in choosing the invited participants, where we made a direct point to be as inclusive as possible, while still maintaining the overall focus of the conference.

3. Conference highlights:

The conference included many informal discussion times, both in-person and online, which resulted in dissemination of new ideas across different subdisciplines in the field. The major activities of the conference were lectures delivered by leading experts on new progress in the field, which were followed up by frequently lively discussion and question sessions. Below we record the abstracts of all of these talks.

(1) **Pangiota Daskalopoulos: Type II smoothing in mean curvature flow:** In 1994 Velzquez constructed a smooth $O(4)O(4)$ invariant Mean Curvature Flow that forms a type-II singularity at the origin in space-time. Stolarski very recently showed that the mean curvature on this solution is uniformly bounded. Earlier, Velzquez also provided formal asymptotic expansions for a possible smooth continuation of the solution after the singularity. Jointly with S. Angenent and N. Sesum we establish the short time existence of Velzquez' formal continuation, and we verify that the mean curvature is also uniformly bounded on the continuation.

(2) **Paula Burkhardt-Guim: Pointwise lower scalar curvature bounds for C^0 metrics via regularizing Ricci flow:** We propose a class of

local definitions of weak lower scalar curvature bounds that is well defined for C^0 metrics. We show the following: that our definitions are stable under greater-than-second-order perturbation of the metric, that there exists a reasonable notion of a Ricci flow starting from C^0 initial data which is smooth for positive times, and that the weak lower scalar curvature bounds are preserved under evolution by the Ricci flow from C^0 initial data.

(3) **Yi Lai: Steady gradient Ricci solitons with positive curvature operators:** We find a family of 3d steady gradient Ricci solitons that are flying wings. This verifies a conjecture by Hamilton. For a 3d flying wing, we show that the scalar curvature does not vanish at infinity. The 3d flying wings are collapsed. For dimension $n \geq 4$, we find a family of $\mathbb{Z}_2 \times O(n-1)$-symmetric but non-rotationally symmetric n-dimensional steady gradient solitons with positive curvature operators. We show that these solitons are non-collapsed.

(4) **Alec Payne: Mass Drop and Multiplicity in Mean Curvature Flow:** Mean curvature flow can be continued through singularities via Brakke flow or level set flow. Brakke flow is defined with an inequality which makes it tantamount to a subsolution to smooth mean curvature flow. On the other hand, level set flow is like a supersolution, since it may attain positive measure. In this talk, we will discuss these weak solutions and will relate uniqueness problems for weak solutions to multiplicity problems in mean curvature flow. In particular, we discuss how Brakke flows with only generic singularities achieve equality in the inequality defining the Brakke flow. This uses an analysis of worldlines in the Brakke flow, analogous to the theory of singular Ricci flows.

(5) **Brian Harvie: The Inverse Mean Curvature Flow and Minimal Surfaces:** In this talk, I will discuss the relationship between Inverse Mean Curvature Flow (IMCF), an expanding extrinsic geometric flow, and minimal surfaces. A natural question about the IMCF of a closed hypersurface in Euclidean space is whether a finite-time singularity forms. When one does form, I will show how classical minimal surfaces may be used to characterize the flow behavior near the singular time: specifically, they allow one to establish a uniform bound on total curvature and hence a limit surface without rescaling the flow surfaces at the extinction. This singular profile contrasts sharply with the singular profiles of other extrinsic flows. When one does not form

and the evolution continues for all time, there is a connection to previous work by Meeks and Yau on the embedded Plateau problem. In particular, via a comparison principle arising from embedded global solutions of IMCF, I will show that global area-minimizers for Jordan curves confined to star-shaped or certain rotationally symmetric mean-convex surfaces in $\mathbb{R}^3$ are embedded. Furthermore, such curves admit only a finite number of stable minimal disks with areas smaller than any fixed number.

(6) **Jian Song: Long time solutions of the Kahler-Ricci flow:** The Kahler-Ricci flow admits a long-time solution if and only if the canonical bundle of the underlying Kahler manifold is nef. We prove that if the canonical bundle is semi-ample, the diameter is uniformly bounded for long-time solutions of the normalized Kahler-Ricci flow. Our diameter estimate combined with the scalar curvature estimate for long-time solutions of the Kahler-Ricci flow are natural extensions of Perelman's diameter and scalar curvature estimates for short-time solutions on Fano manifolds.

(7) **Otis Chodosh: Generic mean curvature flow of low entropy initial data:** I will describe recent work with Choi, Mantoulidis, Schulze concerning generic behavior of MCF. I will compare two potential approaches to this problem and describe one of them (based on entropy drop near non-generic singularities) in detail.

(8) **Or Hershkovits: Noncollapsed translators in R^4:** Translating solution to the mean curvature flow form, together with self-shrinking solutions, the most important class of singularity models of the flow. When a translator arises as a blow-up of a mean convex mean curvature flow, it also naturally satisfies a noncollapsing condition. In this talk, I will report on a recent work with Kyeongsu Choi and Robert Haslhofer, in which we show that every mean convex, noncollapsed, translator in R^4 is a member of a one parameter family of translators, which was earlier constructed by Hoffman, Ilmanen, Martin and White.

(9) **Jonathan Zhu: Explicit Lojasiewicz inequalities for shrinking solitons:** ojasiewicz inequalities are a popular tool for studying the stability of geometric structures. For mean curvature flow, Schulze used Simons reduction to the classical ojasiewicz inequality to study compact tangent flows. For round cylinders, Colding and Minicozzi instead used a direct method to prove ojasiewicz inequalities. Well

discuss similarly explicit ojasiewicz inequalities and applications for other shrinking cylinders and products of spheres.

(10) **Maxwell Stolarski: Mean Curvature Flow Singularities with Bounded Mean Curvature:** In 1984, Huisken showed that the second fundamental form always blows up at a finite-time singularity for the mean curvature flow. Naturally, one might then ask if the mean curvature must also blow up at a finite-time singularity. We'll discuss work that shows the answer is "no" in general, that is, there exist mean curvature flow solutions that become singular with uniformly bounded mean curvature.

(11) **Bruce Kleiner: Ricci flow through singularities, and applications:** The talk will survey Ricci flow through singularities in dimension three, and some applications to topology; the lecture is intended for nonexperts. This is joint work with Richard Bamler and John Lott.

(12) **Mario Garcia-Fernandez: Non-Khler Calabi-Yau geometry and pluriclosed flow:** In this talk I will overview joint work with J. Jordan and J. Streets, in arXiv:2106.13716, about Hermitian, pluriclosed metrics with vanishing Bismut-Ricci form. These metrics give a natural extension of Calabi-Yau metrics to the setting of complex, non-Khler manifolds, and arise independently in mathematical physics. We reinterpret this condition in terms of the Hermitian-Einstein equation on an associated holomorphic Courant algebroid, and thus refer to solutions as Bismut Hermitian-Einstein. This implies Mumford-Takemoto slope stability obstructions, and using these we exhibit infinitely many topologically distinct complex manifolds in every dimension with vanishing first Chern class which do not admit Bismut Hermitian-Einstein metrics. This reformulation also leads to a new description of pluriclosed flow, as introduced by Streets and Tian, implying new global existence results. In particular, on all complex non-Khler surfaces of nonnegative Kodaira dimension. On complex manifolds which admit Bismut-flat metrics we show global existence and convergence of pluriclosed flow to a Bismut-flat metric.

(13) **Felix Schulze: A relative entropy and a unique continuation result for Ricci expanders:** We prove an optimal relative integral convergence rate for two expanding gradient Ricci solitons coming out of the same cone. As a consequence, we obtain a unique continuation

result at infinity and we prove that a relative entropy for two such self-similar solutions to the Ricci flow is well-defined. This is joint work with Alix Deruelle.

(14) **Keaton Naff: A neck improvement theorem in higher codimension MCF:**In both Ricci flow and mean curvature flow, there have recently been significant advances in our understanding of ancient solutions which model singularity formation. One of the crucial tools to this advance has been the development of local symmetry improvement results, as first introduced in mean curvature flow by Brendle and Choi, and later to the Ricci flow by Brendle. In this talk, we would like to discuss how the technique can be adapted to higher codimension mean curvature flow, exhibiting how both rotational symmetry and flatness improve along the flow.

(15) **Zhichao Wang: Uryson width of three dimensional mean convex domains with non-negative Ricci curvature:** In this joint work with B. Zhu, we prove that for every three dimensional manifold with non-negative Ricci curvature and strictly mean convex boundary, there exists a Morse function so that each connected component of its level sets has a uniform diameter bound, which depends only on the lower bound of mean curvature. This gives an upper bound of Uryson 1-width for those three manifolds with boundary. Our proof uses mean curvature flow with free boundary proved by Edelen-Haslhofer-Ivaki-Zhu.

(16) **Lu Wang: Closed hypersurfaces of low entropy are isotopically trivial:** We show that any closed connected hypersurface with sufficient low entropy is smoothly isotopic to the standard round sphere.

(17) **Natasa Sesum: Survey of recent classification results of ancient solutions:** We will discuss recent results and progress made on classifying ancient solutions in geometric flows. We will also mention very nice applications of these results that play an important role in singularity analysis of mean curvature flow and Ricci flow.

(18) **Ronan Conlon: Steady gradient Kahler-Ricci solitons:** Steady gradient Khler-Ricci solitons are fixed points of the Khler-Ricci flow evolving only by the action of biholomorphisms generated by a real holomorphic vector field. We show that there is a unique steady gradient Khler-Ricci soliton in each Khler class of a crepant resolution of a

Calabi-Yau cone. To do this, we solve a complex Monge-Ampere equation via a continuity method. Our construction is based on an ansatz due to Cao in the 90s which was utilized by Biquard-MacBeth in 2017. This is joint work with Alix Deruelle.

4. Outcomes:

Due to the onset of the COVID pandemic, we were forced to move the conference from the intended in-person event to a hybrid in-person and online format. This unfortunately resulted in reduced attendance and further challenges for setting up mathematical interactions. Nonetheless we were successful in having talks delivered by top experts in the field. Many of the talks inspired lengthy questions and discussions which aided in our objective of unifying disparate parts of the field. The final attendee and speakers list indicates that we were successful in our objectives of increasing the participation of recent Ph.D's and underrepresented groups.

Scientifically we were successful in having lectures delivered on topics representing a large cross-section of the field of geometric evolution equations. The lectures by Bruce Kleiner and Natasa Sesum gave a broad overview of the great recent progress in Ricci flow in recent years. These were naturally complemented by lectures on intriguing recent results of Paula Burkhardt-Guim on regularization, Felix Schulze on a relative entropy functional for Ricci expanders, and Yi Lai on the construction of new solitons.

In the direction of complex geometry Jian Song discussed recent results on the Kähler-Ricci flow and their application in Kähler and algebraic geometry. Also Ronan Conlon discussed recent progress on the classification of steady gradient Kähler-Ricci solitons. This was further complemented by a lecture by Mario Garcia-Fernandez on new results for the pluriclosed flow in complex geometry with new applications to the understanding of non-Kähler Calabi-Yau manifolds.

Another major theme of the lectures was minimal surfaces and mean curvature flow. The talks by Otis Chodosh and Lu Wang concerned mean curvature flows with low initial entropy, with Chodosh describing the flow for generic initial data and Wang showing that hypersurfaces with sufficiently small entropy are isotopic to the round sphere. In the direction of understanding the singularity formation, Maxwell Stolarski described a proof of a fundamental result that the mean curvature need not blow up at a finite time singularity. Furthermore Keaton Naff exhibited results describing the singular region of certain flows in terms of neck-like regions. Also Alec Payne described the refined structure of the weak Brakke flow solutions to mean curvature flow. Also Pangiota Daskalopoulos showed the existence of a smooth solution emanating from certain symmetric initial data. The lectures by Or Hershkovits and Jonathan Zhu described the structure and classification of shrinking and steady solitons for mean curvature flow. In a slightly different direction Brian Harvie spoke on the inverse Mean curvature flow, which was famously used to resolve the Penrose conjecture in the case of a connected horizon, in particular showing a new embeddedness result for certain minimal surfaces. Also Zhichao Wang described how to use the mean curvature flow with free-boundary to characterize the Uryson width of manifolds with nonnegative Ricci curvature.