20w5064: Fractons and Beyond

Xie Chen (Caltech),
Rahul Nandkishore (University of Colorado at Boulder),
Yong Baek Kim (University of Toronto),
Zhenghan Wang (Microsoft Research)

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1 Overview of the Field

The field of condensed matter physics studies the complex and often surprising collective behavior of systems containing many particles. One of the most striking examples of new physics which arises in such many-body systems is the concept of an emergent quasiparticle. Strong interactions between the microscopic particles can often drive the formation of emergent quasiparticle excitations with vastly different properties from any known fundamental particle. The concept of a quasiparticle dates back to Landau's theory of Fermi liquids, in which interactions between electrons lead to the formation of quasiparticle excitations with the same charge as an electron, but with a different mass. A more dramatic example of an emergent quasiparticle was later found in the context of fractional quantum Hall systems, where Laughlin quasiparticles carry only a fraction of the elementary electric charge. Since then, a wide array of quasiparticles has been discovered, often possessing fractionalized quantum numbers or anyonic quantum statistics.

Recently, however, a new type of emergent quasiparticle has been encountered which differs from all previously known particles in an unusual way. Fractons are quasiparticles which lack an ability previously assumed to be inherent to all particles: namely the ability to move. A fracton is a quasiparticle which, in isolation, is unable to move in response to an

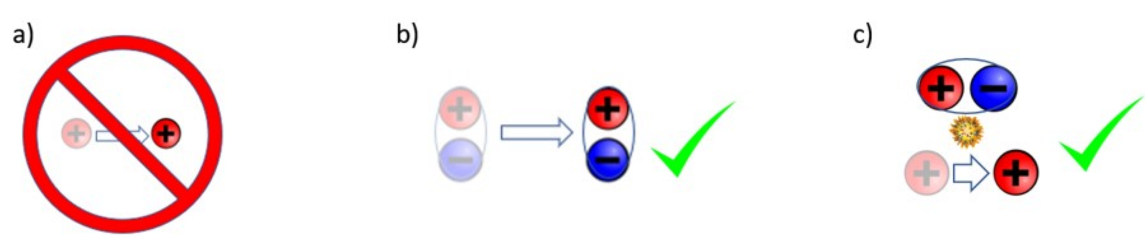

Figure 1: a) A single fracton cannot move freely in any direction. b) Fractons can sometimes move by forming certain bound states, such as dipoles. c) It is also possible for a fracton to move at the expense of creating new particles out of the vacuum.

applied force [6, 10, 33, 34, 20]. However, depending on the details of the model, fractons can sometimes move by combining to form certain bound states, as depicted in Figure 1. Fracton models are often classified as "type-I" if they possess stable mobile bound states, and as "type-II" if all mobile bound states can decay directly into the vacuum [34]. It is also possible for an individual fracton to move at the cost of creating new fractons out of the vacuum at each step of its motion. However, in the absence of a constant energy input to sustain this particle creation, an individual fracton will remain immobile. These unusual new particles were first encountered in certain exactly-solvable three-dimensional spin and Majorana models [6, 5, 10, 33, 34, 2, 38], but have since been shown to arise in contexts ranging from topological crystalline defects [21] to plaquette-ordered paramagnets [39] (see also precursor work in Ref. [15]). Furthermore, the restricted mobility of fractons causes them to exhibit a variety of unusual properties, such as nonergodic behavior [16, 14] and even gravitational physics [18, 37]. At a practical level, there is hope that the immobility of fractons may even be harnessed for the purposes of quantum information storage [10, 1, 31, 3].

It is generally agreed upon that the first manifestation of fracton behavior was encountered in a spin model exhibiting glassy dynamics constructed by Chamon [6], though there is also important conceptual overlap between fractons and earlier work on kinetically constrained models [23, 13, 7]. Later, Haah designed the paradigmatic type-II fracton model, featuring a characteristic fractal structure, with the goal of creating a self-correcting quantum memory [10]. However, the significance of these two models, often known as the Chamon model and Haah's code respectively, was not immediately appreciated. It was not until the seminal work of Vijay, Haah, and Fu that it became clear that these models were only two examples of a much larger class of fracton systems, representing a fundamentally new type of phase of matter [33, 34]. Vijay, Haah, and Fu constructed several now-prototypical fracton models in three dimensions, such as the X-cube model. Additionally, they recognized the existence of several close cousins of fractons: particles which can only move along a one- or two-dimensional subspace of a three-dimensional system. These particles have since come to be known as lineons and planons respectively, or sometimes more generally as subdimensional particles.

The next major advance in the understanding of fractons came with the realization by Pretko that the restricted mobility of fractons can be naturally understood in terms of a set of higher moment conservation laws, which often arise as a consequence of an emergent symmetric tensor gauge theory [20, 19]. For example, the simplest such gauge theories feature conservation of both charge and dipole moment, which immobilizes individual charges but allows for motion of stable dipolar bound states. Building on earlier work on symmetric tensor gauge theories [35, 22, 9, 8, 36], Pretko showed that these gauge theories provide an effective description of a broad class of fracton phases featuring emergent gapless gauge modes. It was later shown by Ma, Hermele, and Chen [11], and independently by Bulmash and Barkeshli [4], that certain symmetric tensor gauge theories give rise to the previously studied gapped fracton models via the Higgs mechanism. From this viewpoint, various spin-1/2 fracton models can be understood as types of Z_2 symmetric tensor gauge theories. In addition to shedding internal light on the field of fractons, the symmetric tensor gauge theory formalism has also drawn unexpected connections between fractons and other areas of physics, such as elasticity theory [21] and gravity [18].

Recently, there has been further significant progress on the understanding of fracton phases in gapped spin models. Useful tools have now been developed for relating such fracton phases to more familiar topological phases of matter. For example, it has been shown how certain three-dimensional fracton phases can arise via strongly coupling together layers of two-dimensional topological phases [12, 32]. Various schemes have also been proposed for generalizing the string-net condensate picture for ordinary topological phases to fracton phases [17, 30]. Moreover, an idea based on the notion of "foliation" was used to define "fracton phases" by Shirley, Slagle, and Chen [29, 28, 26, 25, 24, 27]. Based on this definition, various universal quantities were identified which lead to the classification of many of the known spin models into different fracton phases.

While much of the work on fractons takes place in the context of abstract spin models and gauge theories, it is important to note that fracton physics has a very concrete realization as the topological lattice defects of ordinary crystals. Specifically, the disclinations and dislocations of two-dimensional crystals exhibit the restricted mobility of fractons and lineons, respectively. This connection is made precise via a duality transformation, often referred to as "fracton-elasticity duality," which maps the elasticity theory of crystals onto a symmetric tensor gauge theory [21].

2 Open Problems

Beyond established results, there are also numerous open questions in the field of fractons, which range from the practical to the highly abstract.

First of all, the types of models that are known to exhibit fracton order are still very limited. New toy models and new approaches to construct toy models are being proposed constantly. This would not only expand our knowledge about what kind of fracton phenomena is possible, but also bring new perspective to study them. For example, there has also been only limited exploration of fractons in fermion systems, and the known models all have natural analogues in boson systems. Are there examples of intrinsically fermionic fracton models? Can fermion systems give rise to tensor gauge theories with half-integer higher-spin gauge modes? On the other hand, one important line of research is a push towards a complete classification of fracton systems, with a full characterization of all statistical processes. This line of effort is trying to extract universal properties of the fracton models and organize what we know about individual models into a more systematic framework.

One major issue in the field of fracton is its connection to field theory. Field theory has been very successful in describing almost all other phenomena in condensed matter systems – symmetry breaking, topological order, etc. Fracton models, on the other hand, exhibit features that seem hard to fit into a continuous framework. For example, the growth of ground state degeneracy with system size, fractal pattern of coordinated fracton motion, etc. Whether we can still capture fracton physics using continuous field theory or whether we need to extend the field theory framework in some fundamental way to accommodate them is an issue that is being actively investigated.

On the more practical side, one important line of research is the search for more experimentally-relevant spin models which may be realized in actual materials exhibiting frustrated magnetism. It will also be important to develop more experimental signatures of fractons in spin systems, particularly for gapped models. However, recent developments have made it clear that fracton physics is a much broader paradigm than its humble beginnings in exactly solvable spin models. Fractons are already known to be realized in a diverse set of systems, such as elasticity theory, plaquette paramagnets, hole-doped antiferromagnets, and more. As such, it is natural to ask what other platforms may host fractons, and how fracton physics is concretely manifested in experimental signatures.

Given that fractons are on the cusp of physical realization, it is also important to ask what we will do with fractons once we have them. How can we practically manipulate fractons in some useful way? It has been widely suggested that the properties of fractons will be useful for the purpose of quantum information storage [10, 1, 31, 3], but we lack any concrete roadmap for the precise implementation of this proposal. Much more work will be required to figure out how to usefully store and manipulate quantum information using a fracton system. It is also unclear whether or not the mobility restrictions of fractons can be harnessed for constructing any other sort of useful quantum devices. These questions are fundamentally related to the dynamics of the fracton models, which are now being explored in various settings.

Another interesting question is what we can learn about real gravitational systems from the connection between fractons and gravity. Can fracton physics provide new insights into more traditional gravitational theories? Can fracton models be used to simulate more complicated gravitational phenomena, such as black holes?

3 Presentation Highlights

At this workshop, a variety of fracton related topics were addressed, including the construction of toy models, classification of fracton phases, field theory description, physical realization schemes, dynamics in fracton models, etc. Due to the closeness of the topic of fracton to several other actively-developing topics in condensed matter nowadays, such as topological order, non-ergodic dynamics, etc., there were also talks which focused on 'beyond fracton' topics but with ideas highly pertinent to the study of fracton. Here are the highlights of the talks given on each topic.

3.1 Model construction

Sheng-Jie Huang introduced a class of gapped non-Abelian fracton models, dubbed "cage-net fracton models," which host immobile fracton excitations in addition to non-Abelian particles with restricted mobility. Starting from layers of two-dimensional string-net models, whose spectrum includes non-Abelian anyons, the extended one-dimensional "flux strings" built out of pointlike excitations are condensed. This flux-string condensation generalizes the concept of anyon condensation familiar from conventional topological order and allows us to establish properties of the fracton phase, such as its ground-state wave function and spectrum of excitations. To illustrate the main idea, Huang focused on a simple example: doubled-Ising cage-net model. He showed that there are non-Abelian excitations with restricted-mobility in this model and these are a fundamentally three-dimensional phenomenon, as they cannot be understood as bound states among two-dimensional non-Abelian anyons and Abelian particles.

Hao Song talked about his study of novel three-dimensional gapped quantum phases of matter which support quasiparticles with restricted mobility, including immobile ”fracton” excitations. So far, most existing fracton models may be instructively viewed as generalized Abelian lattice gauge theories. Here, by analogy with Dijkgraaf-Witten topological gauge theories, Song and collaborators discovered a natural generalization of fracton models, obtained by twisting the gauge symmetries. Introducing generalized gauge transformation operators carrying an extra phase factor depending on local configurations, they constructed a plethora of exactly solvable three-dimensional models, which they dub ”twisted fracton models.” A key result of their approach is to demonstrate the existence of rich non-Abelian fracton phases of distinct varieties in a three-dimensional system with finite-range interactions. For an accurate characterization of these novel phases, the notion of being inextricably non-Abelian is introduced for fractons and quasiparticles with one-dimensional mobility, referring to their new behavior of displaying braiding statistics that is, and remains, non-Abelian regardless of which quasiparticles with higher mobility are added to or removed from them. Song also analyzed these models by embedding them on a three-torus and computing their ground state degeneracies, which exhibit a surprising and novel dependence on the system size in the non-Abelian fracton phases.

Jeongwan Haah presented an exactly solvable model for a 4+1D beyond-cohomology symmetry protected topological phase. It has been proposed that there are four symmetry protected topological phases in 4+1D under internal Z_2 symmetry. One generator of these phases is a well understood Dijkgraaf-Witten cohomology theory, but the other is speculatively identified with the so-called generalized double semion theory. Haah discussed a new construction of an exactly solved model that was believed to represent the beyond cohomology phase. It is a decorated domain wall construction by the Walker-Wang state based on three-fermion anyon theory. An important property of this state is that the induced action of the symmetry on a 3+1D boundary is a nontrivial quantum cellular automaton — a locality preserving unitary that does not admit any constant depth quantum circuit decomposition.

3.2 Universal properties and Classification

Wilbur Shirley talked about entanglement renormalization of gapped fracton models. Gapped fractonic gauge theories are fracton models that arise from gauging the discrete subsystem symmetries of a quantum paramagnet. The entanglement renormalization group transformation for such theories is known to exhibit novel bifurcation phenomena. In this talk, Shirley discussed how these bifurcation phenomena reflect the structure of the subsystem symmetry of the ungauged model. He illustrated this principle in the case of both foliated and fractal fractonic orders.

Sagar Vijay presented a set of constraints on the ground-state wavefunctions of fracton phases, which provide a possible generalization of the string-net equations used to characterize topological orders in two spatial dimensions. He demonstrated that the solutions to these equations yield both Type I and Type II gapped fracton phases – which are distinct as as translationally-invariant quantum phases — along with their dual subsystem symmetry-protected topological phases (SSPT). These constraint conditions present a constructive starting point for finding new fracton orders and provide a complementary perspective to understanding fracton phases through "foliated" equivalence relations. He noted that their constraint conditions are derived after taking a translation symmetry group and a subsystem symmetry group as input; he commented on (i) possible generalizations of this prescription with more general objects, beyond subsystem symmetry groups, and (ii) how these constraint equations may be parametrized to extract "universal" data that may characterize fracton orders.

Daniel Bulmash's talk focused on Type-II fracton models, in which all nontrivial point-like excitations are immobile. These models have eluded most systematic descriptions of fractons. Bulmash and collaborators showed that Haah's B code, which is a type-II model, can be described by using networks of defects in topological quantum field theories. They explicitly showed how type-II excitations can be created in these models using fractal-like networks of TQFT operators. Taken together with their results (described in prior talks) that show that defect networks can also describe examples of both Abelian and non-Abelian type-I fracton models, they conjecture that all fracton models can be described by a suitable defect network.

Dominic Williamson presented an overview of the origin of fracton topological order, recent progress that has been made towards classifying and characterizing fracton phases, and preliminary results of ongoing work to find a unified framework for the construction and classification of all gapped fracton phases of matter.

3.3 Field theory description

Kevin Slagle talked about how fracton phases can be described by a topological quantum field theory (TQFT) with a network of defects. Slagle focused on the X-cube model as the primary example. This helped to understand the new and novel fracton physics by making use of the much more mature TQFT formalism.

Michael Hermele's talk on "Symmetry in fracton phases" began with a discussion of one motivation to study fracton phases, namely the relationship between quantum phases of matter and continuum quantum field theory (QFT). By challenging our usual assumptions, fraction phases make it clear that this relationship needs to be understood better. Symmetry plays an important role as a tool for understanding both phases and QFT that is independent of particular models or constructions. The talk then proceeded to discuss two pieces of work exploring the role of symmetry in fracton phase. In the first part, Hermele used symmetry and duality arguments to explain how to think about one of the simplest fracton models – the so-called rank-2 $U(1)$ scalar charge theory in 2+1 dimensions – as a conventional theory but with unusual symmetries. In the second part, Hermele discussed using higher form symmetries as a means to understand p-string condensation mechanisms, where one starts in a conventional phase and enters a fracton phase by condensing certain extended objects.

Han Ma discussed the quantum critical points described by an emergent tensor gauge theory featuring sub-dimensional excitations, in close relation to fracton theories. She also showed that the critical theory of the Lifshitz transition between two valence bond solid (VBS) phases can be mapped to a symmetric tensor gauge theory featuring one-dimensional particles. Also, the same tensor gauge theory describes a quantum critical point between a two-dimensional superfluid and a finite-momentum Bose condensate. Furthermore, she presented a new finite-temperature phase of bosons at this critical point, in which boson-hole pairs are condensed but individual bosons are not. Finally, the whole finite temperature phase diagram of this system was discussed.

3.4 Realization

Michael Pretko gave an overview of progress towards connecting the field of fractons with experiments. Pretko provided a brief introduction to fractons, describing some of the models and phenomenology encountered in the field. He then described some advances in proposed spin models realizing fracton behavior, such as a putative fracton spin ice. He also described some experimental diagnostics which are useful for detecting fractons. Finally, he discussed some new platforms for realizing fracton physics, such as hole-doped antiferromagnets and electric circuits.

Leo Radzihovski first gave a review of quantum crystal elasticity to fracton gauge theory duality, then discussed anisotropic quantum melting of the crystal into a smectic. Dualizing the latter, Radzihovski will discuss the resulting gauge theory whose charges capture the restricted mobility of disclinations in the quantum smectic. As a consistency check this smectic description can also be obtained by Higgs'ing the dual gauge theory of the quantum smectic.

3.5 Dynamics

Juan P. Garrahan discussed constrained dynamics, both in classical and quantum systems. Garrahan reviewed the the rich dynamics that emerges in simple models endowed with kinetic constraints. He considered the classification of stochastic kinetically constrained models (KCMs), and the range of behaviour that they can display. He emphasised the connection between KCMs and classical ”fractons” and how these ideas were in partly the origin of the current interest in quantum fractons. He discussed how these classical ideas can be adapted to the problem of slow thermalisation and (apparent) non-ergodicity in quantum systems in the absence of quenched disorder. In particular he focussed on the quantum East model as a paradigmatic quantum KCM displaying a range of interesting dynamical behaviour. He explained how as a consequence of constraints one can construct analytically a very large class of non-thermal excited states with low entanglement. Towards the end he provided a trailer on similar concepts in the context of discrete ”Floquet” circuit settings.

Frank Pollmann talked about ergodicity-breaking arising from Hilbert space fragmentation in dipole-conserving Hamiltonians. Pollmann and collaborators showed that the combination of charge and dipole conservation—characteristic of fracton systems—leads to an extensive fragmentation of the Hilbert space, which in turn can lead to a breakdown of thermalization. As a concrete example, they investigated the out-of-equilibrium dynamics of one-dimensional spin-1 models that conserve charge (total Sz) and its associated dipole moment. First, they considered a minimal model including only three-site terms and find that the infinite temperature auto-correlation saturates to a finite value. The absence of thermalization is identified as a consequence of the strong fragmentation of the Hilbert space into exponentially many invariant subspaces in the local Sz basis, arising from the interplay of dipole conservation and local interactions. Second, they extended the model by including four-site terms and found that this perturbation leads to a weak fragmentation: the system still has exponentially many invariant subspaces, but they are no longer sufficient to avoid thermalization for typical initial states. More generally, for any finite range of interactions, the system still exhibits non-thermal eigenstates appearing throughout the entire spectrum.

Igor Lesanovsky's talk focused on Rydberg quantum simulators, i.e. highly excited atoms held in optical tweezer arrays. The Rydberg quantum simulators belong to the currently most advanced platforms for the implementation and study of strongly interacting spin systems. An interesting dynamical regime is reached when one atom that is brought to a Rydberg states facilitates the excitation of another nearby one. The resulting dynamics can be similar to that of epidemic spreading and also may form an ingredient for observing non-equilibrium phase transitions. In this talk, Lesanovsky discussed recent results concerning the analysis of constrained spin dynamics on Rydberg quantum simulators.

4 Scientific Progress Made

Extensive discussion was carried about among the workshop participants on all kinds of questions. Here are some of the scientific progress made at the conference as reported by the participants.

Dominic Williamson had many useful discussions about topics of current interest, including the role of 1-form symmetries in fracton topological order, the topological defect network construction of fracton phases, and linear subsystem SSPTs that are dual to topologically ordered models.

Michael Hermele had numerous useful discussions while in Banff. Particular notable was a discussion with Nat Tantivasadakarn, Hao Song, Sheng-Jie Huang and Juven Wang. Tantivasadakarn had recently constructed a new model where it appeared that the fractons were behaving like fractons. Together they came up with new statistical processes by which two fractons can in some sense be exchanged with one another, and showed that these processes give rise to a robust statistical phase of -1 in the model. Many open questions remain, and Tantivasadakarn and Hermele and others are planning to explore this further. Hermele also discussed this progress with Wilbur Shirley, who had been working on similar ideas, and Wilbur immediately had a number of very interesting further observations and questions. Also notable were productive discussions with Wilbur Shirley, Kevin Slagle and Xie Chen, where it seems like some progress was made on a project they have been working on.

Juan P. Garrahan had discussions mostly on the connections between classical and quantum concepts relating to slow dynamics and non-ergodicity, the emergence of constraints, and how idealized models can be realized in atomic settings.

Michael Pretko had the opportunity to discuss with numerous experts from around the globe, both in the fracton field and from the broader condensed matter community. These discussions provided various ideas for future investigations into fractons, as well as other topics at the forefront of condensed matter research. For example, Fakher Assad and Pretko discussed several potential interesting features of fracton behavior in hole-doped antiferromagnets with unusual lattice structures. They suspected that the Bethe lattice may provide a platform for exact fracton behavior, which would provide an exciting new result. Pretko had a variety of other discussions about fractons, such as an exploration with Han Ma about extensions of the fracton gauge principle, as well as with Yongbaek Kim about new fracton spin models. Pretko also had discussions on other topics, such as an engaging conversation with Sagar Vijay about quantum chaos.

Leo Radzihovski discussed physics with a number of people: with Andrey Gromov he discussed his multipole gauge theory of fractons, as well as a way to couple vortex lattice in a superfluid to gauge fields; with Maissam Barkeshli he discussed critical phases and also the breakdown of homotopy classification of orientational defects in systems that break rotational and translational symmetry (e.g., smectics and crystals); with Sagar Vijay he discussed critical phases, quantum smectics and relation to the KPZ nonequilibrium dynamics; with Sheng-Jie Huang he discussed elasticity duality to gauging spatial symmetries; with Michael Pretko and Andrey Gromov he discussed my quantum GL theory and its relation to Michael's highly nonlinear theory of fractons.

Han Ma was mainly inspired by the studies about global symmetry of the fracton phase. It is a very unusual symmetry and may be important for the formation of the phases. There will be a lot to explore along this direction. She plans to study this type of global symmetry and also study its interplay with other ordinary global symmetries, and even generalized global symmetries.

Kevin Slagle and his collaborators explained the results of their forthcoming works and receive many useful comments. Michael Hermele, Xie Chen, Wilbur Shirley, and Slagle made progress on their ongoing project regarding how to coarse grain the gapless $U(1)$ fracton models.

Nat Tantivasadakarn had many helpful discussions about his upcoming work on Jordan-Wigner transformations for translation-invariant Hamiltonians in higher dimensions. By bosonizing a fermion model with subsystem fermion parity symmetry, he can obtain a ”twisted” X-cube model where fractons behave like fermions. In particular, he had an insightful discussion with Mike Hermele, Sheng-Jie Huang, Hao Song, and Juven Wang about a certain braiding process that can give a minus sign in this model, but not in the usual X-cube model. This opens the question of whether there are in general meaningful ways one can braid immobile objects such as fractons without pairing them up into mobile particles, an aspect they hope to explore in the future.

Hao Song learned about the latest progress in the study of fracton orders, especially about the idea of realizing fractons as defect networks of topological orders. It is very insightful and he would like to explore problems like whether other interesting gapped non-liquid states can also be constructed in this way. During the workshop, he also had good discussions with many people, especially Mike Hermele, Sheng-Jie Huang, Nathanan Tantivasadakarn, and Juven Wang. These discussions improved his understanding on some field theoretical descriptions of fracton phases. They also got stimulated and made some new progress on characterizing fractons by exchange and braiding, which they will explore further.

Arpit Dua talked to various scientists including Xie Chen, Jeongwan Haah and Zhenghan Wang about his recent work on the structure of three-dimensional stabilizer models.

Also, the visit led to a new project on the entanglement renormalization and ungauging of Chamon's model with Xie Chen's group. Dua also discussed with Dominic Williamson and others about an ongoing project of entanglement renormalization of subsystem symmetry protected topological phases and potential future ideas like structure theorems for models described by X-S stabilizer formalism.

Daniel Bulmash made progress on writing his defect network paper, and had discussions on issues such as uses of defect networks to classify fracton phases, the relation between local operator structure and excitation mobility, and dualities in generalized gauge theories.

Sagar Vijay find the workshop very helpful in furthering his understanding of fracton orders, in making progress in new directions and in understanding open questions within this field. In particular, he had several useful discussions with colleagues (Andrey Gromov, Maissam Barkeshli, Jeongwan Haah, Zhenghan Wang, Michael Pretko) on understanding $U(1)$ fracton orders that may lie beyond what we currently understand using tensor gauge theories. As an example, he discussed generalizations of an algebraic framework for understanding gapped fracton orders (originally proposed by Haah) and how this may be useful for understanding when two $U(1)$ fracton orders — specified by a generalized "Gauss' law" — are distinct and/or are stable quantum phases. he is now pursuing these ideas and other open questions related to $U(1)$ fracton orders as a consequence of this workshop.

Wilbur Shirley find this conference to be a great opportunity to discuss new ideas with colleagues. He had stimulating discussions about a variety of topics related to fractons, including: fracton exchange statistics, fracton models with emergent fermionic gauge theory, TQFT defect network constructions of fracton models, fermionic 'strong' SSPTs, 'panoptic' fracton orders and new types of SPT and SET phases, and potential structure theorems for fractonic stabilizer codes. Several of these discussions have evolved into ongoing fruitful collaborations with various participants, including Mike Hermele, Nathanan Tantivasadakarn, Hao Song, Juven Wang, Arpit Dua, and Dominic Williamson.

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