Representation Theory Connections to (q, t)-Combinatorics
19w5131

François Bergeron (UQaM)
Jim Haglund (UPenn)
Mike Zabrocki (York University)

January 20–25, 2019

In 2007, a workshop was held at the Banff research station entitled *Applications of Macdonald polynomials*[1]. The focus of that meeting was related to a number of mathematical problems that arose in the roughly 20 years prior.

That workshop served to highlight some of the directions of research related to the following discoveries.

- In 1988, Macdonald [Mac88] introduced a family of symmetric functions $P_\lambda(X; q, t)$ that depended on parameters q and t and a family of non-symmetric polynomials $E_\alpha(X; q, t)$ which are a basis for the polynomial ring. An open problem remains to find a combinatorial interpretation for the transition coefficients $K_{\lambda\mu}(q, t)$ from the Macdonald symmetric functions to a Schur basis.

- In 1994, Garsia, Haiman [Hai94, GarHai96] stated a series of conjectures related to a symmetric group representation known as the diagonal harmonics which is the quotient ring of polynomials in two sets of variables of size n by the ideal generated by the ideal generated by the symmetric group invariants.

- One of the conjectures stated that this space was of dimension $(n + 1)^{n-1}$ (and this was resolved in 2001 by Haiman [Hai99, Hai01, Hai02, Hai02b]), but further investigation into the combinatorics of parking functions by Haglund, Haiman, Loehr, Remmel and Ulyanov [HHLRU05] lead them to 'the shuffle conjecture' in 2004, a q, t-combinatorial formula for the graded Frobenius character of the module.

In the last eleven and a half years since that conference, there have been a number of remarkable breakthroughs related to these and the other research problems highlighted at that conference. A number of exciting connections have arisen over the last few years between Macdonald polynomials and invariants of torus knots and torus links [Gor13, GN15, GORS14, Hag16].

These new connections lead to the development of extensions of the shuffle conjecture to labeled paths that lie above the diagonal of an $m \times n$ rectangle [ALW14, GORS14, GMV16, GM16, BGLX16, GLXW15].

In addition, many new questions in this area arose out of the operator realization of the elliptic Hall algebra of Burban, Schiffman and Vasserot [BS, SV13] that left many open questions and connections to the representation theory and knot invariants left to be explored [MS, Ber16].

In 2015, Carlsson and Mellit [CM15] proved the shuffle conjecture and shortly after Mellit [Mel16] proved the rational shuffle conjecture. This progress resolved a major open problem in the area and caused a shift of research in this area.

[1] `https://www.math.upenn.edu/~jhaglund/conf/finalreportbirs07.pdf`

In 2016, Haglund, Remmel and Wilson [HRW15] discovered a conjecture which proposed a combinatorial model for the symmetric function expression $\Delta_{e_k}(e_n)$ and called this 'the Delta conjecture.' In the case that $k = n$, the Delta conjecture reduces to the shuffle conjecture, but it currently remains unresolved.

We saw the workshop "Representation Theory Connections to (q, t)-Combinatorics" at BIRS as an opportunity to re-visit some of the remaining open problems and re-assess what are the most interesting open questions in this area. The focus of research in the field has shifted in the last decade given that the research leading up to the proof of the shuffle conjecture uncovered many new connections between representation theory, the elliptic Hall algebra and knot invariants.

Three of the objectives that were listed in the proposal of this workshop are summarized in the following bullet points:

1. to connect some of the combinatorial and algebraic methods recently developed with quasi-symmetric functions. [ELW10, HLMvW11a, BBSSZ14, As15, AsSer16, Rob14]

2. consider positivity of symmetric function expressions arising in this context from the perspective of representation theoretical constructions [HRW15, GORS14, HRS1, HRS2, Rho16, GHRY]

3. make connections between symmetric functions, combinatorial formulae, the elliptic Hall algebra and torus knot invariants. [Ber16, BS, Gor13, GN15, Hik14, LW08, MS, MMR14, Mel16, Neg13, Neg14, Sch12, SV13]

We chose to have a small number of presentations and give participants ample time to discuss ideas. The following is a list of the presentations that we will discuss in summary below.

1. Speaker: **François Bergeron**
   Title : *Multivariate modules for (m, n)-rectangular combinatorics I and II*

2. Speaker: **Adriano Garsia**
   Title: *Some Conjectures with Surprising Consequences*

3. Speaker: **Matthew Hogancamp**
   Title: *How to compute superpolynomials*

4. Speaker: **Lauren Williams**
   Title: *From multiline queues to Macdonald polynomials via the exclusion process*

5. Speaker: **Sami Assaf**
   Title: *Nonsymmetric Macdonald polynomials and Demazure characters*

6. Speaker: **Brendon Rhoades**
   Title: *Spanning configurations*

7. Speaker: **Gabriel Frieden**
   Title: *Kostka–Foulkes polynomials at $q = -1$*

8. Speaker: **Luc Lapointe**
   Title: *m-symmetric Macdonald polynomials*

9. Speaker: **Hugh Morton**
   Title: *A skein-theoretic model for the double affine Hecke algebras*

We mentioned that the Macdonald polynomials was one of the discoveries that motivated this area of research and was one of the topics highlighted in in the workshop in 2007. They reappeared in several of the talks in this workshop. The talk by Sami Assaf was on non-symmetric Macdonald polynomials and their expansion into Demazure characters using a crystal structure. The talk by Luc Lapointe showed a new approach to the problem of finding a combinatorial interpretation for the Macdonald q, t-Kostka coefficients using larger algebras to lift their structure. The talk by Lauren Williams showed how Macdonald polynomials arise in two combinatorial models, one called multi-line queues and another was the multispecies asymmetric simple exclusion process.

The talks by Matthew Hogencamp and Hugh Morton explained connections between knot invariants and q, t-combinatorics and connections with representation theory. In particular, the talk on "how to compute super-polynomials" discussed a combinatorial technique for Khovanov-Rozansky homology which has led to the computation of the rational q, t-Catalan.

A number of the other talks focused on the connections of representation theory and their connections to q, t-combinatorics. The extension of the shuffle conjecture known as the Delta conjecture appeared tangentially in a number of the talks. We mention that one of the basic open problems related to the Delta conjecture was to find a plausible candidate for a module of the symmetric group S_n whose bigraded Frobenius characteristic is given by the symmetric function side of the Delta conjecture. As luck would have it, organizer M. Zabrocki found such a candidate (see (1) and [Zab19]) the day before the conference, and shared it with the conference attendees, which led to a lot of interesting discussions and further conjectures. The presentation of Adriano Garsia discussed the phenomenon (outlined in [Ber16]) of replacing q with $q + 1$ in a Schur positive symmetric function arising from q, t-combinatorics and the expression is very often positive when expanded in the e-basis. He presented a growing list of expressions where this phenomenon can be proven. François Bergeron presented a number of results and broad conjectures about symmetric functions which encode the $Gl_k \times S_n$ character of multidiagonal harmonics. In particular he proposed explicit modules for which the aforementioned symmetric functions appear as characters,. This gives a entirely new representation theoretic foundation for the result established by Mellit and Carlsson which ties together the $m \times n$-rectangular path combinatorics and the elliptic Hall algebra based formulas appearing in the rectangular compositional shuffle formula (see [BGLX16, HMZ12, Hic14]). Not only do these new symmetric functions open the way to a broad program of research, but they tie together most of the previous areas of research in the domain, including the Delta conjecture, and suggest many natural generalization. In the final discussion, it was realized that they also merge nicely with the candidate proposed for the Delta conjecture by Mike Zabrocki, as well as suggesting generalizations to both the multivariate case and the $m \times n$-rectangular one.

In addition, on the Thursday of the workshop meeting, the afternoon session was devoted to asking participants to submit a summary of the major open problems in this area. The participants also submitted a written summary after the meeting which was published on the webpage. While a longer more detailed versions of these problems appears in the summary on the web page, we provide a summary below that gives a flavor of the mathematics discussed at this conference.

Presenter: **François Bergeron**
Outline: The probability that a monomial positive symmetric function is actually Schur positive is

$$\frac{1}{\prod_{\mu \vdash n} \sum_{\lambda \vdash n} K_{\lambda\mu}} .$$

"Consider the q-analogue of these formulae obtained by replacing the Kostka numbers by the q-Kostka polynomials (or even (q, t)), can we give a natural interpretation of this as a probability of some sort?"

Presenter: **Lauren Williams**
Outline: Given a matrix $M = (m_{ij})$ with entries are symmetric polynomials. M is totally Schur positive if each square sub-matrix has Schur positive determinant. An example of this are the Jacobi-Trudi matrices, but are there others?

Presenter: **Hugh Morton**
Outline: Is it possible to create a non-degenerate bilinear form on braids in the torus that can be used to determine linear independence? What is the HOMFLY skein in this case?

Presenter: **Brendan Pawlowski**
Define Stanley symmetric functions, affine Stanley symmetric functions, involutive Stanley symmetric functions and (finally) affine involutive Stanley symmetric functions. It can be shown that the affine involutive

Stanley symmetric functions have positive coefficients when expanded in the affine Stanley Schur functions, but what other properties do they have?

Presenter: **Peter Samuelson**
Outline: The elliptic Hall algebra $E_{q,t}$ is an algebra over $C(q,t)$ which was defined by Burban and Schiffmann as a "universal Hall algebra of elliptic curves over finite fields." There is an action of $E^+_{q,t=q}$ action on Sym. Question: Does the action of $E_{q,t=q}$ on $Sym \otimes Sym$ extend to generic t?. The "vertical subalgebra" (generated by the $u_{0,n}$) acts on Sym (essentially) by Macdonald operators, which are diagonalized by Macdonald polynomials. The module $Sym \otimes Sym$ has a basis $s_{\lambda,\mu}$ of "double Schur functions," and this diagonalizes the "vertical subalgebra" of the skein algebra. A positive answer to this question would (presumably) lead to "double" Macdonald polynomials, which would be indexed by pairs of Young diagrams, so a followup question is "how much of Macdonald theory carries through to double Macdonald polynomials"?

Presenter: **Marino Romero**
Outline: In the 1990's Bergeron and Garsia presented a sequence of conjectures that they labelled Science Fiction [BG99]. For α a partition, let M_α be the Garsia-Haiman module. One approach to proving the $n!$ conjecture is to understand why for partitions α and β each found by removing a single cell from a larger partition μ, then

$$\dim(M_\alpha \bigwedge M_\beta) = n!/2 \, .$$

More generally, the Frobenius image is given by

$$Frob_{qt}\left(M_\alpha \bigwedge M_\beta\right) = \frac{T_\beta \tilde{H}_\alpha - T_\alpha \tilde{H}_\beta}{T_\beta - T_\alpha} = \left(\frac{1}{1 - T_\alpha/T_\beta}\right)\tilde{H}_\alpha + \left(\frac{1}{1 - T_\beta/T_\alpha}\right)\tilde{H}_\beta$$

where $\tilde{H}_\beta$ is the Macdonald symmetric function where $T_\mu = \prod_{(i,j)\in\mu} q^i t^j$. These conjectures extend to larger collections of partitions.

Presenter: **Mikhail Mazin**
Outline: Take m and n relatively prime. There is a bijection between lattice paths which lie below the diagonal in an $n \times m$ rectangle and partitions which are simultaneously m and n cores. There are q,t-countings of both sides which agree. That is, on the left hand side you have a q,t counting of paths below the diagonal in an $m \times n$ rectangle and this is equal to a right hand side of a q,t counting of the partitions which are simultaneously m and n cores.

Consider again the m and n relatively prime and let $(M,N) = (dm, dn)$. The q,t-counting can be naturally generalized both for lattice paths that stay below the diagonal in an $M \times N$ rectangle and for the simultaneous M,N-core partitions. However, they are not equal anymore. In fact, the set of simultaneous M,N-core partitions is infinite in the non relatively prime case, and the resulting generating function is not a polynomial, but rather a power series (a rational function with denominator $(1-q)^{d-1}$). Both counts are related to some deep and interesting mathematics. The paths under the diagonal appear in the compositional Shuffle theorem by Erik Carlsson and Anton Mellit [CM15], and simultaneous cores correspond to invariants of torus links.

The open question is what is the precise relation between these two q,t-countings in the non relatively prime case.

Presenter: **Gabriel Frieden**
Outline: Haglund, Haiman, and Loehr gave a combinatorial formula for the monomial expansion [HHL05] One appealing approach to finding the Schur expansion would be to define a crystal structure on the set of fillings of μ that preserves the statistics maj and inv; the Schur expansion would then be given by the weights of the highest elements in the crystal structure. In the case where μ has two columns, Haglund-Haiman-Loehr defined a suitable crystal structure on the fillings [HHL05].

In the case where μ has three columns, Blasiak [BL] recently proved a conjectured rule of Haglund [Hag04] for the Schur expansion. We propose the problem of finding a maj- and inv-preserving crystal structure on fillings of partitions with three columns.

Presenter: **François Bergeron**
Outline: For any operator $F_{m,n}$, on symmetric functions coming from the elliptic Hall algebra. Define the operation $\hat{F}_{m,n} := F_{m,n}|_{t=1}$. There is an identity that is simple to state, but begs for a proof:

$$\hat{F}_{m,n} \cdot f = f \cdot (\hat{F}_{m,n} \cdot 1).$$

In other words, $\hat{F}_{m,n}$ is simply a multiplication operator. This is certainly not the case (in general) for $F_{m,n}$.

Presenter: **Mike Zabrocki**
Outline: Let

$$Q[X_n, Y_n; \Theta_n] := Q[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n; \theta_1, \theta_2, \ldots, \theta_n]$$

be the polynomial ring in three sets of variables, the first two are commuting and the third one is anti-commuting (and the variables of different flavors commute). The invariants of this polynomial ring (here denoted by Sym, with Sym^+ standing for those that are constant term free) are generated by analogues of the power sums

$$p_{r,s} = \sum_{i=1}^{n} x_i^r y_i^s \text{ and } \tilde{p}_{r',s'} = \sum_{i=1}^{n} \theta_i x_i^{r'} y_i^{s'}$$

for $0 < r + s \leq n$ and $0 \leq r' + s' < n$. Define the analogue of the diagonal harmonics as

$$\mathrm{SDCoinv}_n := Q[X_n, Y_n; \Theta_n] / \left\langle p_{r,s}, \tilde{p}_{r',s'} | 0 < r + s \leq n, 0 \leq r' + s' < n \right\rangle \ .$$

Based on computational evidence of the Frobenius series, $n \geq 1$,

$$\mathrm{Frob}_{qtz}(\mathrm{SDCoinv}_n) = \Delta'_{e_{n-1}[X-\epsilon z]}(e_n), \tag{1}$$

where $e_{n-1}[X - \epsilon z] = e_{n-1} + z e_{n-2} + z^2 e_{n-3} + \cdots + z^{n-1}$ and $\Delta'_f(\tilde{H}_\mu[X; q, t]) = f[B_\mu - 1]\tilde{H}_\mu[X; q, t]$.

We can make the following interesting further conjecture. Let $\mathcal{E}_n[Q; Z] = \sum_\mu a_\mu[Q] s_\mu[Z]$ be the symmetric functions described in François Bergeron's talk on Monday and Tuesday as a symmetric function expression for the multivariate analogue of the diagonal harmonics, then let

$$\mathrm{Coinv}_n^{k,k'} := Q[X_n^{(1)}, \ldots, X_n^{(k)}; \Theta_n^{(1)}, \ldots, \Theta_n^{(k')}] / \left\langle \mathrm{Sym}^+ \right\rangle$$

be the coinvariant space in k sets of commuting variables and k' sets of anti-commuting variables (such that the anticommuting variables also anticommute among themselves).

Let Q_k represent the alphabet $q_1, q_2, \ldots, q_k$ to keep track of the degrees in the $X_n^{(i)}$ variables and $T_{k'}$ represent the alphabet $t_1, t_2, \ldots, t_{k'}$ as variables which keep track of the degrees in the $\Theta_n^{(i)}$ variables, then we conjecture that

$$\mathrm{Frob}_{Q_k, T_{k'}}(\mathrm{Coinv}_n^{k,k'}) = \mathcal{E}_n[Q_k - \epsilon T_{k'}; Z]$$

for $k + k' > 0$.

References

[AGHRS12] D. Armstrong, A. Garsia, J. Haglund, B. Rhoades, and B. Sagan. *Combinatorics of Tesler matrices in the theory of parking functions and diagonal harmonics*. J. Comb. 3 (2012), 451–494.

[ALW14] D. Armstrong, N. Loehr, G. Warrington, *Rational parking functions and Catalan numbers* `arXiv:1403.1845`

[As15] S. Assaf, *Dual equivalence graphs I: A new paradigm for Schur positivity*, Forum of Mathematics, Sigma, Volume 3, 2015. (`arXiv:1506.03798`)

[AsSer16] S. Assaf, D. Searles, Kohnert tableaux and a lifting of quasi-Schur functions `arXiv:1609.03507`

[BBSSZ14] C. Berg, N. Bergeron, F. Saliola, L. Serrano, M. Zabrocki, *A Lift of the Schur and Hall-Littlewood Bases to Non-commutative Symmetric Functions*, Canad. J. Math. 66 (2014), 525–565

[Ber09] F. Bergeron, *Algebraic Combinatorics and Coinvariant Spaces*, CMS Treatise in Mathematics, CRC Press, 2009. 221 pages.

[Ber13] F. Bergeron, *Multivariate Diagonal Coinvariant Spaces for Complex Reflection Groups*, Advances in Mathematics, Volume 239 (2013) pp. 97–108. (arXiv:1105.4358)

[Ber16] F. Bergeron, *Open Questions for operators related to Rectangular Catalan Combinatorics*, arXiv:1603.04476.

[BDZ10] N. Bergeron, F. Descouens and M. Zabrocki, *A filtration of (q, t)-Catalan numbers*, Adv. in Applied Mathematics 44 (1), (2010) 16–36.

[BG99] F. Bergeron and A. M. Garsia, *Science fiction and Macdonald's polynomials*, Algebraic methods and q-special functions, (Montréal, QC, 1996), CRM Proc. Lecture Notes, vol. 22, Amer. Math. Soc., Providence, RI, 1999, pp. 1–52.

[BGHT99] F. Bergeron, A. M. Garsia, M. Haiman, and G. Tesler, *Identities and positivity conjectures for some remarkable operators in the theory of symmetric functions*, Methods Appl. Anal. **6** (1999), 363–420.

[BGLX14] F. Bergeron, A. Garsia, E. Leven, G. Xin, *Some remarkable new Plethystic Operators in the Theory of Macdonald Polynomials*, arXiv:1405.0316.

[BGLX16] F. Bergeron, A. Garsia, E. Leven, G. Xin, *Compositional (km, kn)-Shuffle Conjectures*, Inter. Math. Res. Not., Vol. 2016, 4229–4270, doi:10.1093/imrn/rnv272

[BPR12] F. Bergeron, L.F. Préville-Ratelle, *Higher Trivariate Diagonal Harmonics via generalized Tamari Posets*, Journal of Combinatorics, Volume 3 (2012) Number 3, pp. 317–341. (arXiv:1105.3738)

[BL] J. Blasiak. *Haglund's conjecture on 3-column Macdonald polynomials*. Math. Z., 283(1-2):601–628, 2016.

[BS] I. Burban, O. Schiffmann, *On the Hall algebra of an elliptic curve*, Duke Math. J., Volume 161, Number 7 (2012), 1171–1231.

[CL06] M. Can, N. Loehr, *A proof of the q, t-square conjecture*, J. of Comb. Th., Series A Volume 113, Issue 7, October 2006, Pages 1419–1434.

[CM15] E. Carlsson and A. Mellit. *A proof of the shuffle conjecture*, arXiv:1508.06239, August 2015.

[CL] C. Carré and B. Leclerc. Splitting the square of a Schur function into its symmetric and antisymmetric parts. J. Algebraic Combin., 4(3):201–231, 1995.

[DGZ13] A. Duane, A. M. Garsia, M. Zabrocki, *A new "dinv" arising from the two part case of the Shuffle Conjecture*, Journal of Alg. Comb., Volume 37, Issue 4 (2013), Page 683–715. (DOI: 10.1007/s10801-012-0382-0)

[EHKK03] E. Egge, J. Haglund, D. Kremer and K. Killpatrick, *A Schröder generalization of Haglund's statistic on Catalan paths*. Electron. J. of Combin., 10 (2003), Research Paper 16, 21 pages (electronic)

[ELW10] E. Egge, N. Loehr, G. Warrington, *From quasisymmetric expansions to Schur expansions via a modified inverse Kostka matrix*, European J. Combin. 31 (2010), no. 8, 2014–2027.

[G92] A. M. Garsia, *Orthogonality of Milne's polynomials and raising operators*, Discrete Math. **99**, Issue 1-3 (1992), 247–264.

[GarHag02] A. M. Garsia and J. Haglund, *A proof of the q, t-Catalan positivity conjecture*, Discrete Math. **256** (2002), 677–717.

[GarHag15] A. M. Garsia and J. Haglund, *A polynomial expression for the character of diagonal harmonics*, Ann. Combin., 19 (2015), 693–703.

[GarHai93] A. Garsia and M. Haiman, *A graded representation model for Macdonald's polynomials* Proc. Nat. Acad. Sci. U.S.A. 90 (1993), no. 8, 3607–3610.

[GarHai95] A. Garsia and M. Haiman, *A random q, t-hook walk and a sum of Pieri coefficients*, J. Combin. Theory, Ser. A 82 (1998), 74–111.

[GarHai96] A. Garsia and M. Haiman, *A remarkable q,t-Catalan sequence and q-Lagrange inversion*, J. Algebraic Combinatorics **5**, (1996), 191–244.

[GHT99] A. M. Garsia, M. Haiman, and G. Tesler, *Explicit plethystic formulas for Macdonald q,t-Kostka coefficients*, Sém. Lothar. Combin. 42 (1999), Art. B42m, 45 pp. (electronic), The Andrews Festschrift (Maratea, 1998).

[GHRY] A. Garsia, J. Haglund, J. Remmel, M. Yoo, *A proof of the Delta Conjecture when $q = 0$*, `arXiv:1710.07078`.

[GHX14] A.M. Garsia, J. Haglund, and G. Xin. *Constant Term methods in the theory of Tesler matrices and Macdonald polynomial operators*, Ann. Comb. 18 (2014) 83–109.

[GHXZ16] A. M. Garsia, J. Haglund, G. Xin, M. Zabrocki, *Some new applications of the Stanley-Macdonald Pieri Rules*, for Stanley@70 (MIT, June 23-27, 2014).

[GLXW15] A. Garsia, E. Sergel Leven, N. Wallach, G. Xin, *A new Plethystic Symmetric Function Operator and the Rational Shuffle Conjecture at $t = 1/q$*, `arXiv:1501.00631`

[GXZ10] A. Garsia, G. Xin, and M. Zabrocki, *Hall-Littlewood operators in the theory of parking functions and diagonal harmonics*, Int. Math. Res. Notices, 2012 (6): 1264–1299. doi: 10.1093/imrn/rnr060

[GXZ14a] A. Garsia, G. Xin, and M. Zabrocki, *A three shuffle case of the compositional parking function conjecture*, Journal of Combinatorial Theory, Series A 123/1 (2014), pp. 202–238.

[GXZ14b] A. Garsia, G. Xin, and M. Zabrocki, *Proof of the 2-part Compositional Shuffle Conjecture*, Progress in Mathematics (Book 257), Birkhäuser; 2014, 538 pages, pp. 227–257.

[Gor13] E. Gorsky, *q,t-Catalan numbers and knot homology*, Zeta Functions in Algebra and Geometry, 213–232. Contemp. Math. 566 , Amer. Math. Soc., Providence, RI, 2012.

[GM13] E. Gorsky, M. Mazin, *Compactified Jacobians and q,t-Catalan Numbers, I*, J. of Comb. Theory, Series A, 120 (2013) 49–63.

[GM14] E. Gorsky, M. Mazin, *Compactified Jacobians and q,t-Catalan numbers, II*, J. of Alg. Comb., 39 (2014), no. 1, 153–186.

[GM16] E. Gorsky, M. Mazin, *Rational Parking Functions and LLT Polynomials*, J. of Comb. Theory, Series A, 140 (2016), 123–140.

[GMV16] E. Gorsky, M. Mazin, M. Vazirani, *Affine permutations and rational slope parking functions*, Trans. of the AMS, 2016, `http://dx.doi.org/10.1090/tran/6584`

[GN15] E. Gorsky and A. Negut. *Refined knot invariants and Hilbert schemes*, J. Math. Pures Appl. 104 (2015), 403–435.

[GORS14] E. Gorsky and A. Oblomkov and J. Rasmussen and V. Shende. *Torus knots and the rational DAHA*, Duke Math. J. **163** (2014), 2709-2794.

[Hag03] J. Haglund, *Conjectured statistics for the q,t-Catalan numbers*, Adv. Math. **175** (2003), no. 2, 319–334.

[Hag] J. Haglund. A combinatorial model for the Macdonald polynomials. Proc. Natl. Acad. Sci. USA, 101(46):16127–16131, 2004.

[Hag04] J. Haglund, *A proof of the q,t-Schröder conjecture*, Internat. Math. Res. Notices **11** (2004), 525–560.

[Hag08] J. Haglund, *The q,t-Catalan numbers and the space of diagonal harmonics*, University Lecture Series, vol. 41, American Mathematical Society, Providence, RI, 2008, With an appendix on the combinatorics of Macdonald polynomials.

[Hag11] J. Haglund, *A polynomial expression for the Hilbert series of the quotient ring of diagonal coinvariants*, Adv. Math. 227 (2011), 2092–2106.

[Hag16] J. Haglund, *The combinatorics of knot invariants arising from the study of Macdonald polynomials*, Recent Trends in Combinatorics, The IMA Volumes in Math. and its Applications, Vol. 159, Beveridge, A., Griggs, J.R., Hogben, L., Musiker, G., Tetali, P. (Eds.), 2016, pp. 579–600.

[HHL05] J. Haglund, M. Haiman, N. Loehr, *A Combinatorial Formula for Macdonald Polynomials*, J. Amer. Math. Soc. 18 (2005), 735–761.

[HLMvW11a] J. Haglund, K. Luoto, S. Mason, S. Van Willigenburg, *Quasisymmetric Schur functions*, J. Combin. Theory Ser. A, 118 (2011), pp. 463–490. (arXiv:0810.2489)

[HLMvW11b] J. Haglund, K. Luoto, S. Mason, S. Van Willigenburg, *Refinements of the Littlewood-Richardson rule*, Trans. Amer. Math. Soc., 363 (2011), 1665–1686. (arXiv:0908.3540)

[HMZ12] J. Haglund, J. Morse, M. Zabrocki, *A compositional shuffle conjecture specifying touch points of the Dyck path*, (arXiv:1008.0828v2), Canad. J. of Math. 64(2012), no. 4, 822–844.

[HRW15] J. Haglund, J. Remmel, A. Wilson, *The Delta Conjecture*, arXiv:1509.07058v2 .

[HRS1] J. Haglund, B. Rhoades, M. Shimozono, *Ordered set partitions, generalized coinvariant algebras, and the Delta Conjecture*, Advances in Mathematics, 329, (2018), 851–915.

[HRS2] J. Haglund, B. Rhoades, M. Shimozono, *Hall-Littlewood expansions of Schur delta operators at $t=0$*, arXiv:1801.08017.

[Hai94] M. Haiman, *Conjectures On The Quotient Ring By Diagonal Invariants*, J. Algebraic Combin, Volume 3, Issue 1 (1994), 17–76.

[Hai99] M. Haiman, *Macdonald polynomials and geometry*, In New Perspectives in Geometric Combinatorics, MSRI Publications 37 (1999), 207–254.

[Hai01] M. Haiman, *Hilbert schemes, polygraphs, and the Macdonald positivity conjecture*, J. Amer. Math. Soc. 14 (2001), 941–1006.

[Hai02] M. Haiman, *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149** (2002), 371–407.

[Hai02b] M. Haiman, *Combinatorics, Symmetric Functions and Hilbert Schemes*, Current Developments in Mathematics, Volume 2002 (2002), 39–111.

[HHLRU05] J. Haglund, M. Haiman, N. Loehr, J. B. Remmel, and A. Ulyanov, *A combinatorial formula for the character of the diagonal coinvariants*, Duke J. Math. **126** (2005), 195–232.

[Hic10] A. Hicks, *Two parking function bijections refining the q,t-Catalan and Schröder recursions*, Int. Math. Res. Notices, Volume 2012, No 16, July 2011.

[Hic14] A. Hicks, *A parking function bijection supporting the Haglund-Morse-Zabrocki conjectures*, Int. Math. Res. Notices, Volume 2014, No 7.

[HL14] A. Hicks and E. Leven, *A refinement of the Shuffle Conjecture with cars of two sizes and $t = 1/q$*, J. of Comb. 5.1 (2014): 31–50.

[HL15] A. Hicks and E. Leven, *A simpler formula for the number of diagonal inversions of an (m, n)-Parking Function and a returning Fermionic formula*, Discrete Math. 383.3 (2015), pp. 48–65.

[Hik14] T. Hikita, *Affine Springer fibers of type A and combinatorics of diagonal coinvariants*, Advances in Mathematics Vo. 263, 2014, Pages 88–122. (arXiv:1203.5878)

[KM] R. Kaliszewski and J. Morse. Colorful combinatorics and Macdonald polynomials. arXiv:1710.00801, 2017.

[LW07] N. A. Loehr and G. Warrington, *Square q,t-Lattice Paths and $\nabla(p_n)$*, Trans. of the Amer. Math. Soc., Vol. 359, No. 2 (Feb., 2007), pp. 649–669.

[LW08] N. A. Loehr and G. Warrington, *Nested quantum Dyck paths and $\nabla(s_\lambda)$*, Inter. Math. Res. Not. 2008; Vol 2008: article ID rnm157

[LMvW13] K. Luoto, S. Mykytiuk, S. van Willigenburg, *An introduction to quasisymmetric Schur functions - Hopf algebras, quasisymmetric functions, and Young composition tableaux*, Springer (2013).

[Mac88] I. G. Macdonald, *A New Class Of Symmetric Functions*, Publ. I.R.M.A. Strasbourg, 372/S20, Actes 20 Séminaire Lotharingien, 1988, 131–171.

[Mac95] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Mathematical Monographs, second ed., Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1995.

[Mel16] A. Mellit, *Toric braids and (m, n)-parking functions*, arXiv:1604.07456.

[MMR14] K. Mészáros, A. H. Morales, B. Rhoades, *The polytope of Tesler matrices*, arXiv:1409.8566.

[MS] H. Morton, P. Samuelson, *The Homflypt skein algebra of the torus and the elliptic Hall algebra*

[Neg13] A. Negut, *Operators on Symmetric Polynomials*, http://arxiv.org/abs/1310.3515

[Neg14] A. Negut, *The Shuffle Algebra Revisited*, Inter. Math. Res. Not., Volume 2014, issue 22, pages 6242–6275. http://dx.doi.org/10.1093/imrn/rnt156

[PvW] R. Patrias, S. van Willigenburg, *The probability of positivity in symmetric and quasisymmetric functions*, arXiv:1810.11038.

[PR] B. Pawlowski, B. Rhoades, *A flag variety for the Delta Conjecture*, arXiv:1711.08301.

[Rho16] B. Rhoades, *Ordered set partition statistics and the Delta Conjecture*, Journal of Combinatorial Theory, Series A, 154 (2018), 172–217.

[Rob14] A. Roberts, *Dual Equivalence Graphs Revisited and the Explicit Schur Expansion of a Family of LLT Polynomials*, J. Algebr. Comb. (2014) 39, Issue 2, pp 389–428. doi:10.1007/s10801-013-0452-y

[Sch12] O. Schiffmann, *Drinfeld realization of the elliptic Hall algebra*, J. Algebraic Combin. 35 (2012), no. 2, 237–262.

[SV13] O. Schiffmann, E. Vasserot, *The elliptic Hall algebra and the equivariant K-theory of the Hilbert scheme of A^2*, Duke Math. J. 162 (2013), no. 2, 279–366.

[Sh] M. Shimozono. *Crystals for dummies*. https://www.aimath.org/WWN/kostka/crysdumb.pdf, 2005.

[St99] R. Stanley, *Positivity Problems and Conjectures in Algebraic Combinatorics*, Mathematics: Frontiers and Perspectives, AMS, 1999, 295–319.

[W15] A.T. Wilson. *A weighted sum over generalized Tesler matrices*, arxiv:1510.02684.

[Zab16] M. Zabrocki, *A proof of the* 4*-variable Catalan polynomial of the Delta conjecture*, arXiv:1609.03497

[Zab19] M. Zabrocki, *A module for the Delta conjecture*, arXiv:1902.08966