Unifying 4-Dimensional Knot Theory

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1 Overview of the Field

Our workshop focused on an exciting facet of smooth 4–manifold topology: the theory of knotted surfaces. Topology in dimension three has seen an explosion of activity over the last 30 years, reaching a peak with Perelman's proof of the Poincaré Conjecture in 2003 [40]. Other notable highlights include the ground-breaking discovery of the Jones Polynomial in knot theory [27] and the chain reaction Jones' work set off in quantum topology, algebra, and physics, including Witten's work connecting the Jones Polynomial to Chern-Simons theory [45]. In stark contrast, our understanding of 4–dimensional topology has many prominent gaps, including the intractable Smooth 4–dimensional Poincaré Conjecture, and breakthroughs have been more difficult to attain.

In 4–dimensional spaces, the interesting knotted objects are not loops but rather surfaces, and the first step in studying these objects is to find appropriate geometric representations, in the sense that these pictures should be easily manipulated and lend themselves to performing computations. Classical approaches to the theory of knotted surfaces include movies [14], broken surface diagrams [46] (defined below), and marked vertex diagrams [31, 47]. For example, whereas a classical knot diagram is the projection of a loop in $\mathbb{R}^3$ to an immersed curve in $\mathbb{R}^2$ equipped with crossing information, a broken surface diagram is the projection of a surface in $\mathbb{R}^4$ to generic surface in $\mathbb{R}^3$, which has also been equipped with "crossing information," suitably defined. Carter, Kamada, Roseman, Saito, and many others have made an industry out of developing this diagrammatic and combinatorial theory [5, 41], notably using *quandles*, an algebraic structure generalizing Fox's notion of n–colorability [6].

People study knotted surfaces from through a variety of lenses. Some attempt to import 3–dimensional ideas to dimension four. Others emphasize aspects of 4–dimensional knot theory that fit coherently within the theory of codimension two embeddings in dimension four and higher. While these two groups often consider surfaces in S^4, yet another active area of research draws from classical results in 4-manifold topology, considering the broader class of surfaces in arbitrary 4-manifolds. Some utilize the additional structure afforded by symplectic and complex manifolds. Researchers study geometric, algebraic, and combinatorial aspects of knotted surfaces. The purpose of our workshop was to encourage a dialogue among this wide variety of researchers, with a central goal of better understanding knotted surfaces. We wanted participants to share their most interesting tools and problems so that the entire field could benefit from novel, multi-faceted approaches to both new and old challenges. Overall, we were impressed by the engagement of everyone involved, and based on the feedback that reached us, the workshop was universally considered a success.

2 Recent Developments and Open Problems

One of the most important breakthroughs in the theory of knotted surfacers over the last couple of years was Gabai's surprising proof of the 4-dimensional Light Bulb Theorem, an incredible triumph of 3–dimensional techniques applied novelly and elegantly in a brand new setting [17]. The classical version of this theorem in dimension three states that any knot in $S^1 \times S^2$ that meets $\{pt\} \times S^2$ in a single point is isotopic to $S^1 \times \{pt\}$. In its full generality, Gabai's theorem asserts that if S and T are homotopic, smoothly embedded 2-spheres in a smooth 4-manifold X with a common dual (square-zero sphere meeting each of S and T in a single point), and in addition $\pi_1(X)$ has no 2-torsion, then S and T are smoothly ambient isotopic in X. An ongoing and popular trend in 4-dimensional research is to develop new invariants to prove that spaces are different; Gabai's theorem stands out as remarkable in that it shows instead that a vast collection of surfaces are the same. His work also elegantly generalizes an elementary result from one dimension lower.

Another recent development in knotted surface theory also connects ideas from dimensions three and four. In 2012, David Gay and Robion Kirby introduced *trisections* of smooth 4-manifolds, decompositions of 4-manifolds into three simple pieces, as an analogue of 3-dimensional Heegaard splittings in dimension four [18]. Following their work, Meier and Zupan adapted bridge splittings of classical knots to develop the theory of bridge trisections, decompositions of embedded surfaces in 4-manifolds into trivial disks and trivial tangles within the components of a 4-manifold trisection [38, 39]. Bridge trisections also yield a new structure called *tri-plane diagrams*. Tri-plane diagrams encode knotted surfaces in a novel way, and this area is ripe for the same types of advances obtained in the theory of broken surface diagrams.

A third area that has seen an explosion of activity over the last two decades is *knot concordance*. Two classical knots K_1 and K_2 in S^3 are (smoothly) concordant if there exists a (smoothly) embedded annulus $A \subset S^3 \times I$ connecting $K_1 \subset S^3 \times \{0\}$ to $K_2 \subset S^3 \times \{1\}$. As such, knot concordance occurs at the nexus of 3–manifold and 4–manifold topology, often employing tools from both areas. The theory of concordance also interacts with several specialized topics in geometric topology, including contact/symplectic topology, and the powerful Heegaard Floer and Khovanov homology theories [36]. Through these and other avenues, many researchers have made great gains in studying knot concordance by developing an arsenal of invariants to obstruct two knots from being concordant.

Our workshop connected researchers studying these related but somewhat disparate areas of knotted surface theory for the mutual benefit of all parties. By encouraging a dialogue, we brought together subject-matter experts in all of these sub-fields for the purpose of developing a more unified theory of knotted surfaces in dimension four.

In the view of the organizers, the most important open problems in knotted surfaces theory involve drawing out differences (if they exist) between the topological and smooth categories. The Smooth 4-dimensional Poincaré Conjecture (S4PC), which asserts that every smooth 4-manifold homeomorphic to S^4 is also diffeomorphic to S^4, is the last remaining case of the Generalized Poincaré Conjecture. The relative version of the S4PC is an important open problem in 2-knot theory, the Unknotting Conjecture, positing that if $K \subset S^4$ is a smoothly embedded 2-sphere such that $\pi_1(S^4 \setminus K) = \mathbb{Z}$, then K is isotopic to the unknotted 2-sphere. Also related to the S4PC is the classification of unit 2-knots in $\mathbb{CP}^2$: Suppose that K is a smoothly embedded 2-sphere in $\mathbb{CP}^2$ with square one. Conjecturally, K is isotopic to the standard unit 2-knot, $\mathbb{CP}^1 \subset \mathbb{CP}^2$. Proving this conjecture is equivalent to showing that all Gluck twists in S^4 produce standard smooth S^4, which would be an incredible step toward a proof of the S4PC.

3 Presentation Highlights

Before discussing the content of some of the presentations, we wish to discuss the manner in which they were selected and organized. One of our goals was to organize an inclusive and open workshop, and to do so, we decided that instead of inviting particular speakers to give talks, we would ask every one of our 45 registered participants to submit an abstract if they wished to speak. In addition, when we solicited abstracts, we emphasized the varied areas of expertise among participants, expressing a desire for survey talks or at least some survey material in each talk. As a result of this process, we ended up with 23 talks, which we scheduled in 45 minute time slots for our allotted five days. The totality of topics covered by these talks was both broader and more representative than we could have achieved by selecting the talks ourselves, including diverse representation in terms of gender, career stage, and geographic location. We believe that going forward, this talk selection model is a fantastic one. Despite knowing how talks were "chosen," several participants commented that we did a great job creating the conference program (for which we deserved little credit).

Once we had our finalized list of talks, we set out to develop the schedule. The talks were split into rough categories, with similar talks grouped in morning or afternoon blocks. For example, the majority of the Tuesday talks centered on surfaces in manifolds other than S^4, and the three Wednesday talks focused on understanding differences between the smooth and topological categories. The talks that more closely resembled survey talks were planned for earlier in the week, while more technical results came later. We also ensured that each day's list of speakers was balanced by gender. In the following paragraphs, we highlight a few of the presentations that were a part of the workshop, but this list is by no means complete. We received considerable feedback about the quality of the entire program, and in all honesty, we could include any one of the 23 talks that were given as a highlight of the workshop.

The workshop commenced with a terrific talk from Cameron Gordon surveying known, mostly classical, results about knot groups in dimension four and other dimensions. Gordon's talk provided substantial historical context and set an excellent tone for a convivial and productive meeting. Following Gordon, Masahico Saito delivered an excellent survey about applications of quandle co-cycle invariants, a familiar topic for one cadre of participants and esoteric for another, well serving the workshop goal of unifying participants around common themes. The third talk of the first day, delivered by Shin Satoh, presented a surprising and important result related to the triple point number of knotted surfaces. In spirit, the triple point number is meant to be an analogue of crossing number. However, despite the fact that for each n there are finitely many classical knots with crossing number at most n, there are infinitely many 2-knots with triple point number zero (namely, ribbon 2-knots). Satoh demonstrated that the three-dimensional finiteness may be able to be recovered by passing to ribbon concordance classes of 2-knots, in the sense that every 2-knot with triple point number at most n might be ribbon concordant to a finite family of 2-knots, with some fairly convincing evidence.

The second day of the workshop opened with a comprehensive survey talk by Laura Starkston about the extra structure afforded by examining the symplectic topology of certain surfaces in 4-manifolds, including a variety of open problems, which are incorporated into the problem list below. As mentioned elsewhere in this report, Gabai used his Tuesday afternoon talk to announce his surprising, fundamental theorem proved jointly with Budney, that spanning 3-balls for the unknotted 2-sphere in S^4 are not unique up to isotopy [4]. This development is discussed in greater detail below.

The third day opened with Arunima Ray's talk about a piece of Freedman and Quinn's disk embedding theorem [16]. Over the last ten years, a group has researchers has pushed for a more modern treatment of Freedman's pioneering work underlying his now nearly 40-year-old proof of the topological Poincaré conjecture in dimension four. Ray is a leader of a group of low-dimensional topologists writing a new exposition and expansion of Freedman's work, and her talk covered joint work giving some explicit constructions lacking in Freedman and Quinn's original work.

As mentioned above, the model of soliciting abstracts gave many graduate students speaking opportunities. Highlights from the graduate student talks included a pair of talks about concordance of knotted surfaces. On Tuesday afternoon, Maggie Miller presented of her work generalizing Gabai's Light Bulb Theorem to a broader result about in which concordance of 2-knots replaces isotopy. On Thursday afternoon, Jason Joseph discussed his dissertation research, in which he uses Alexander modules to obstruct concordance of 2-knots, obtaining an alternative answer to a recently solved Kirby problem, and giving the first proof certain elements of the concordance monoid do not have inverses, implying that it is indeed a monoid and not a group [28].

4 Scientific Progress Made

While the most important aspect of this workshop was the presentation of new results and the motivation of open problems, another secondary purpose was achieved. Namely, through the processes of socialization and shared values, the workshop was able to forge new relationships between several communities of 4-dimensional topologist that include an older and younger generation, people whose work focused on disparate areas (e.g., topological, smooth, and symplectic categories), people who are normally geographically isolated from each other (our participants represented four continents), and groups whose work embraces different styles of approach (e.g., traditional diagrammatic aspects vs. surgery-theoretic, say).

One noteworthy event related the conference was the inception of an important collaboration between Ryan Budney and David Gabai. Gabai had been working in earnest in the months leading up to the workshop on the question of the uniqueness up to isotopy of embeddings of B^3 in S^4 such that the boundary of the embedded ball is the standard S^2 in S^4. (Note, this is but one facet of the research, which touches upon multiple important aspects of 4-dimensional topology. See, for example, Problem 30 below.) What follows is a paraphrasing of Budney's description of the collaboration.

Budney and Gabai had been discussing the problem for about a month by email prior to the workshop. In the first two days of the workshop, they outlined the proof of a theorem that could be phrased in a variety of ways – a few of which were mentioned in Gabai's talk. From one perspective, the theorem shows that the exterior of the trivial knot in S^4 admits infinitely many distinct fiberings over S^1 (with fibre an B^3), up to isotopy.

The exterior is $S^1 \times B^3$, and all the fiberings developed by Budney and Gabai come from the orbit of $\pi_0(\text{Diff}(S^1 \times D^3))$ acting on the standard fibering. In other words, they show that the mapping class group of the unknot exterior (leaving the boundary fixed pointwise) is not finitely generated. Another way of stating the result is that the component of the unknot in the space of embeddings $\text{Emb}(S^2, S^3)$ has a non-finitely generated fundamental group. So this is a large contrast from Hatcher's result (using the Smale conjecture) that the component of the unknot in $\text{Emb}(S^1, S^3)$ has fundamental group $\mathbb{Z}/2\mathbb{Z}$. In general their argument extends to say that $\pi_{n-3}(\text{Diff}(S^1 \times D^n))$ is not finitely generated when $n \geq 3$.

Another notable development came from Bob Gompf's contributions. According to Gompf, who proved the existence of infinitely many exotic $\mathbb{R}^4$'s, as a direct result of the invitation to the workshop and the workshop's theme, he developed a relative theory of exotic planes in $\mathbb{R}^4$ [20]. In Gompf's words, this work is "the threshold of a whole new research area," motivated directly by his workshop participation, and it is the subject of a forthcoming paper he is in the process of writing. The theme of adapting results about manifold to the relative cases of knots (in dimension three) or surfaces (in dimension four) is pervasive, and Gompf's work is a terrific concrete example of this principle at work. Five contributed problems stemming from his work are included in the problem list below.

The collaboration of Budney and Gabai described above is the most prominent example of a number of new collaborations that were initiated or strengthened during the workshop. As another example, Meier, Thompson, and Zupan initiated a new collaboration as a direct result of discussions that took place during the coffee breaks at the workshop. They revisited a problem they had considered with little progress before: Every bridge trisection of a knotted surface induces a cell decomposition of the underlying surface whose 1-skeleton is a cubic graph. An open question is whether the reverse is true: Given an abstract cubic graph Γ with a 3-coloring of its edges, is there a bridge trisected surface $\mathcal{K}$ inducing Γ? Together, they proved that the answer is yes for bipartite graphs and have continued to collaborate about the non-orientable case following the workshop. They expect to write a joint paper in the near future.

Joseph, Meier, Miller, and Zupan also initiated a new collaboration resulting from the workshop. In shuttle to the Calgary airport, Miller and Zupan realized that altogether, the four of them had established sufficiently many new results relating bridge trisections to other aspects of 2-knot theory that these results should collected in a coherent manuscript to be shared with the community. In particular, they have developed a algorithm to produce Seifert solids, the punctured 3-manifolds bounded by 2-knots in S^4 using tri-plane diagrams, a procedure to convert a tri-plane diagram to a broken surface diagram, and an elegant computation of the normal Euler number of a surface knot. They expect to write a joint paper in the near future.

Of course, we are limited to describing the collaborations including ourselves and those that were communicated, but we also witnessed the continuation of existing collaborations, for instance between Bar-Natan and Dancsworkshopo and between Auckly and Ruberman.

5 Outcome of the Meeting

In our proposal, we listed a number of goals for the workshop. The manner in which the workshop and its participants addressed several of these is articulated.

1. **Raise awareness of the state-of-the-art within each sub-field.** While our original goal was to include expository talks primarily at the beginning of the conference, this was not always possible. Talks by

Dave Auckly, Hans Boden, Scott Carter, Celeste Damiani, Zsuzsanna Dancso, Bob Gompf, Cameron Gordon, Mark Hughes, Sashka Kjuchukova, Laura Starkston, Arunima Ray, and Masahico Saito included introductory and background material that informed the participants of the scope of the results that were presented.

2. **Adapt machinery across different representations of knotted surfaces.** Talks by Jason Joseph, Seungwong Kim, Vince Longo, Maggie Miller, and Danny Ruberman helped describe relations among the various sub-genres of surfaces in 4-spaces. In general, we feel that the workshop did an excellent job of exposing researchers to approaches to knotted surface theory that might represent a departure from their usual techniques and skill sets.

3. **Develop new knotted surface invariants.** Talks of Hans Boden, Dave Gabai, Mark Hughes, Byeorhi Kim, Masahico Saito, and Laura Starkston addressed the development of invariants. The development of invariants remains an important goal of the field, broadly construed. One approach to finding invariants is sufficiently develop diagrammatic theories over which the invariants might be defined. The workshop greatly enabled the development of diagrammatic theories, both classical and more recent.

4. **Understand various representations of symplectic surfaces.** Informal discussions among presenters and non-presenters occurred in these regards. Talks of Dave Auckly, Maggie Miller, Arunima Ray, and Laura Starkston are also relevant.

5. **Explore the implications of the 4–dimensional Light Bulb Theorem.** This was one of our greatest successes of the workshop. Ryan Budney and Dave Gabai used Gabai's techniques to establish that there are non-unique smooth spanning 3-balls that are bounded by an unknotted sphere. Maggie Miller and Dave Gabai both gave excellent talks related to recent work that has grown out of Gabai's theorem.

6. **Compile a list of important problems.** Immediately following the workshop, speakers and other participants were asked to contribute one or more problems related to the contents of the workshop program. The organizers curated that list, and the resulting product was a comprehensive and wide-ranging list of problems that can help shape the directions of the theory of embedded surfaces in S^4 and in other 4-manifold in the future. That problem list is included as the next section.

In short, all of our stated objectives were addressed. Many were fully achieved, and others will shape the directions of research that occurs in the near future and beyond.

6 Problem List

After the workshop, we requested that all presenters (and anyone else interested) submit a problem or two related the their talk or to some conversation they had during the workshop. This problem list was then curated, distributed to conference participants, and posted online so that a broader audience could access some of the workshop developments. The problem list follows below.

6.1 Problems about knotted surfaces in S^4

Problem 1 (Cameron Gordon)**.** *Is there an algorithm to decide whether or not a (PL locally flat) 2-knot is (PL or TOP) unknotted?*

Problem 2 (Gordon)**.** *Is there a 2-knot whose group has unsolvable word problem?*

The reader is encouraged to refer to [21, Theorem 7.1], which shows that a positive answer to Problem 2 implies a negative answer to Problem 1.

Problem 3 (Masahico Saito)**.** *Find algebraic and diagrammatic interpretations of the quandle 3-cocycle invariant for knotted surfaces.*

For classical knots, the quandle 2-cocycle invariant has an algebraic interpretation as obstructions of extending colorings to extensions of quandles, and a diagrammatic aspect in relation to the fundamental quandle class. Such interpretations for knotted surfaces will be desirable both for computations and applications.

Problem 4 (Vincent Longo)**.** *Is the connected sum of an even twist spun knot and an unknotted projective plane is diffeomorphic to the connected sum of the spin of the same knot and the unknotted projective plane?*

Longo proved that up to the parity of an integer n, all connected sums of an n-twist spun knot and an unknotted projective plane become diffeomorphic after a single trivial 1 handle addition, assuming either n is odd or K is a 2 bridge knot [37]. The remaining case of n an even integer and the knot having bridge number at least 3 is still open. The version of this problem for n odd is Kirby Problem 4.58 [33].

The triple point number of an S^2-knot is the minimal number of triple points for all possible diagrams of the 2-knot. It is known that a 2-knot has triple point number 0 if and only if it is a ribbon 2-knot, and that there is no 2-knot of triple point number 1, 2, nor 3. Recently Satoh proved that a 2-knot has triple point number 4 if and only if it is ribbon concordant to the 2-twist-spun trefoil knot. The following question is still open.

Problem 5 (Shin Satoh)**.** *For any integer $n > 4$, is there a minimal "finite" set S_n of 2-knots such that any 2-knot of triple point number n is ribbon concordant to some 2-knot belonging to S_n? In particular, does it hold $S_n = \emptyset$ if n is odd and $S_6 = \{$the 3-twist-spun trefoil knot$\}$?*

Two 2-knots K and J are n-concordant if there is a smooth concordance joining them such that the regular level sets consist of surfaces of genus at most n. All 2-knots are concordant [32], so any two are n-concordant for some n. Melvin proved that 0-concordant 2-knots have diffeomorphic Gluck twists and asked if every 2-knot is 0-slice, i.e. 0-concordant to the unknot. Recently Sunukjian [44], Dai-Miller [7], and Joseph [28] have shown the existence of infinitely many 0-concordance classes, but there are still many unanswered questions. As in the classical case, ribbon 2-knots are clearly 0-slice, but there are no known examples of nonribbon, 0-slice 2-knots. Joseph has shown that any 2-knot with nonprincipal Alexander ideal is not invertible in the 0-concordance monoid. There are no known examples of 2-knots which do have an inverse under 0-concordance, i.e. K and J which are not 0-slice but so that $K \# J$ is.

Problem 6 (Jason Joseph)**.** *Is every 0-slice 2-knot ribbon?*

Problem 7 (Joseph)**.** *Is any nontrivial 0-concordance class invertible?*

Problem 8 (Joseph)**.** *Are all 2-knots 1-concordant to the unknot?*

Problem 9 (Zsuzsanna Dancso)**.** *Is the conjectured set of Reidemeister moves for ribbon 2-knots (see for example [2] for a description) complete? Is the same true for ribbon 2-tangles and w-foams (see [3])?*

Problem 10 (Robin Gaudreau)**.** *Generalize the index of a crossing to the double curves of the projection of a knotted sphere in a 4-manifold.*

The index of a crossing in an oriented knot diagram on a surface is the signed intersection number of the curves obtained by smoothing the crossing with the orientation. Definitions for other knot theories, when applicable, follow from that one.

In the following, $M(K)$ is the closed orientable 4-manifold obtained by elementary surgery on a 2-knot K in S^4.

Problem 11 (Hillman). *Let π be a (high-dimensional) knot group with commutator subgroup free of rank r ($\pi' \cong F(r)$). Then π has a presentation of deficiency 1. Is such a group $\pi \cong F(r) \rtimes \mathbb{Z}$ the group of a (fibred) ribbon 2-knot?*

See [23, Chapter 17.6] for more on 2-knots with groups of cohomological dimension 2.

Problem 12 (Hillman). *Is there a 2-knot K such that $M(K)$ is an orbifold bundle over a flat 2-orbifold with general fibre an hyperbolic surface?*

See [24] for what little is known about constraints on such knots.

Problem 13 (Jonathan Hillman). *Let $L = \sqcup L_i$ be a surface link in S^4, with regular neighbourhood $N(L) \cong L \times D^2$ and exterior $X(L) = \overline{S^4 \setminus N(L)}$. If $X(L)$ fibres over S^1 then $\chi(X(L) = 0$, and so*

$$\Sigma\chi(L_i) = \chi(N(L)) = \chi(N(L)) + \chi(X(L)) = \chi(S^4) = 2.$$

It is easy to find examples with one component a 2-sphere and the rest tori. Are there any fibred surface links with at least one hyperbolic component?

Problem 14 (Hillman: An old question of Hosokawa and Kawauchi [26]). *Is there a surface knot of genus ≥ 1 with aspherical exterior? More specifically, is there a knotted torus with exterior a compact finite volume $\mathbb{H}^4$-manifold?*

(With recent constructions of cusped hyperbolic 4-manifolds of small volume, this might have a chance.). Finally, the issue of the topological unknotting theorem for surfaces in S^4 should be revisited.

Problem 15 (Hillman). *If $\pi_1(S^4 \setminus F) \cong \mathbb{Z}$, is F isotopic into the equator $S^3 \subset S^4$?*

6.2 Problems about knotted surfaces in other 4-manifolds

Problem 16 (Laura Starkston). *Is every smooth 2-sphere in $\mathbb{CP}^2$ in the homology class $[\mathbb{CP}^1]$ or $2[\mathbb{CP}^1]$ isotopic to a symplectic 2-sphere?*

These are the only two homology classes in $\mathbb{CP}^2$ that can be represented by symplectic spheres (higher multiples are represented by symplectic surfaces of higher genus). Gromov proved using pseudoholomorphic curves that there is a unique symplectic isotopy class of symplectic 2-spheres in each of these two classes [22]. Therefore if any such smooth 2-sphere was isotopic to a symplectic one, there would be a unique smooth isotopy class of spheres in these homology classes. By contrast, for surfaces of higher genus in homology class $d[\mathbb{CP}^1]$ for d greater than or equal to 3, examples of Finashin [10] and (independently) Hee Jung Kim [29] show that there are smooth surfaces in $\mathbb{CP}^2$ such that the fundamental group of the complement is the same as that of a standard algebraic curve in the same homology class, but which are not isotopic to a standard algebraic curve.

Problem 17 (Starkston). *Are there symplectic surfaces in $\mathbb{CP}^2$ which are not isotopic to complex curves? (This problem is open in both the smooth and singular cases.)*

The version for smooth surfaces is a long standing open question, and it has been proven that every symplectic surface homologous to $d[\mathbb{CP}^1]$ (degree d) for d less than or equal to 17 is isotopic to a complex curve. There are a few examples of singular symplectic surfaces which have a collection of singularity types which cannot be realized by a complex curve–(see Ruberman-Starkston [42] for line arrangement examples and Golla-Starkston [19] section 8 for an irreducible example), but there is no clear explanation for when exactly symplectic surfaces align with complex algebraic curves. A key direction which is currently lacking is constructive techniques–how can we build interestingly embedded examples of smooth or singular symplectic surfaces? Another direction on this problem is to change from surfaces in $\mathbb{CP}^2$ to more general 4-manifolds. There are examples of exotic isotopy classes of symplectic surfaces in the same homology class in some 4-manifolds, starting with examples of tori in elliptic surfaces by Fintushel and Stern [13]. However, looking at $S^2 \times S^2$ or $\mathbb{CP}^2$ blown-up once, one can find similar interesting open problems.

An outstanding problem for surface bundles over surfaces is the following:

Problem 18 (Inanc Baykur)**.** *Geography of surface bundles: For which pairs of integers (g, h) are there genus–g surface bundles over genus–h surfaces with positive signatures?*

A necessary condition is $g \geq 3$ and $h \geq 2$, and there are many examples going back to pioneering works of Kodaira, Atiyah, Hirzebruch, and Endo et al. Recent progress suggests there are only a few such pairs of (g, h) (at most two dozen) for which there may be no positive signature examples. This realization problem is harder for holomorphic surface bundles.

A much more advanced geography problem is:

Problem 19 (Baykur)**.** *Topography of surface bundles: How high can the signature of a genus–g surface bundle over a genus–h surface be?*

Increasingly sharper upper bounds have been provided by the works of Taubes, Kotschick and Hamenstädt. For $g, h > 1$, let $m = \frac{3\sigma}{\chi}$ be the *slope* of a surface bundle, where σ and χ are the signature and the Euler characteristic of the bundle. A more tractable variant of the above problem is:

Problem 20 (Baykur)**.** *Slopes of surface bundles: What is the highest slope $m = \frac{3\sigma}{\chi}$ for a surface bundle over a surface, with fiber and base genera greater than one?*

The highest known slope known to date is $2/3$, realized by holomorphic surface bundles constructed by Catanese-Solenske. Can one get to $m = 1$? (This is not possible for holomorphic surface-bundles.)

For two smooth surfaces R_1 and R_2 in a 4-manifold X, $fq(R_1, R_2)$ denote the Freedman-Quinn invariant of R_1 and R_2 [43].

Problem 21 (Maggie Miller)**.** *Construct a smooth 4-manifold X and positive-genus surfaces R_1 and R_2 smoothly embedded in X that are homotopic with $fq(R_1, R_2) = 0$ so that there exists a 2-sphere $G \subset X$ with trivial normal bundle which intersects R_1 transversely in one point but:*

(a) R_2 also intersects G transversely in one point yet R_1 and R_2 are not smoothly isotopic,

(b) R_1 and R_2 are not smoothly concordant.

Gabai's light bulb theorem [17] doesn't apply to most positive genus surfaces – one has to assume that the fundamental group of the surface includes trivially. But there is no known example where the theorem actually fails (although probably it does). Thus, this problem is asking for counterexamples to the light bulb theorem and concordance analogue when considering surfaces whose fundamental group includes non-trivially into the ambient manifold.

Problem 22 (Seungwon Kim)**.** *Adapt the approaches of Lee and others to find a new isotopy invariant of knotted surfaces in a closed orientable 4-manifold and in particular, an isotopy invariant of knotted surfaces in $\mathbb{CP}^2$.*

Lee and his collaborators developed ambient isotopy invariants of knotted surfaces in S^4 [30, 34, 35]; these invariants are analogues of the Kauffman bracket polynomial of knots in S^3. Hence, it is reasonable to ask whether similar invariants can be constructed for knotted surfaces in closed orientable 4-manifold.

6.3 Topological vs. Smooth problems

Problem 23 (Bob Gompf)**.** *Are there unsmoothable 2-knots in S^4? Distinct smooth 2-knots that are topologically isotopic? Is smooth isotopy of 2-knots different from pairwise diffeomorphism?*

Such phenomena are well-known in other 4-manifolds. There are exotically embedded nonorientable surfaces in S^4 [9, 11, 12].

In the proper setting, up to ambient isotopy, there are uncountably many *exotic planes* $\mathbb{R}^2 \hookrightarrow \mathbb{R}^4$ that are smoothly, but not topologically, isotopic to the standard plane. (There are also embeddings that are smooth except at one point and are topologically standard, but not smoothable by a compactly supported isotopy.

Smooth isotopy is the same as orientation-preserving pairwise diffeomorphism since $\mathbb{R}^n$ has no exotic self-diffeomorphisms.) There is an explicit movie description of an exotic plane (with time given by the radius function in $\mathbb{R}^4$), and uncountably many others obtained by suitable ramification. These all have the property that they can be modified, working outside an arbitrarily large ball in $\mathbb{R}^4$, to obtain a sphere (in fact, an unknotted sphere). There are uncountably many exotic planes for which such modification can only produce higher genus surfaces, but the known examples seem intractable to describe by movies, requiring infinitely many 2-handles. (Their construction involves taking the complement of an infinite intersection of reimbedded Casson towers.)

Problem 24 (Gompf)**.** *Give an explicit description (movie) of an exotic plane that cannot be modified to a sphere outside every preassigned ball.*

Problem 25 (Gompf)**.** *Are there combinatorial invariants that can distinguish exotic planes?*

Such invariants could not be determined by the underlying topology, e.g., the knot group. The known examples are distinguished by their double branched covers, which are potentially exotic smoothings of $\mathbb{R}^4$, or by the potentially exotic smoothings of $D^2 \times \mathbb{R}^2$ obtained by using the annulus at infinity to add boundary. Are double (or higher degree) branched covers of exotic planes always exotic?

The general theory of smooth proper 2-knots $\mathbb{R}^2 \hookrightarrow \mathbb{R}^4$ (or more general proper knotted surfaces) splits in two directions, namely studying exotic planes and working modulo exotic planes: Call two proper 2-knots equivalent if they become the same after pairwise end sum with suitable exotic planes. Every compact, orientable surface embedded in B^4 with nonempty boundary in ∂B^4 generates an infinite equivalence class (through passing to the interior and end summing with exotic planes). There is a forgetful map from equivalence classes to topological isotopy classes.

Problem 26 (Gompf)**.** *Is this map a bijection?*

Presumably not, but nothing concrete appears to be known.

Problem 27 (Gompf)**.** *Are any exotic planes in $\mathbb{C}^2$ holomorphic? Symplectic? Lagrangian?*

The movies mentioned above seem unlikely to yield symplectic surfaces, since they have many antiparallel sheets.

Problem 28 (Danny Ruberman)**.** *Does every simply connected 4-manifold with $H_2 = \mathbb{Z}$ have a topological spine?*

This problem could be interpreted as in Ruberman's workshop talk – i.e. the spine is a tamely (locally PL) 2-sphere whose inclusion map is a homotopy equivalence). Or it could have no restriction at all – i.e. the sphere could be wildly embedded with no restrictions on local behavior.

Problem 29 (Dave Auckly)**.** *All of the following problems concern the notion of* stabilization.

*(a) Does every pair of homeomorphic, smooth 4-manifolds become diffeomorphic after taking the connected sum with *one* copy of $S^2 \times S^2$?*

(b) Do any homotopy class in the diffeomorphism group Diff(Z, D) that is trivial in Homeo(Z, D) become trivial after one stabilization?

(c) It is known that there are surfaces that require several internal stabilizations, i.e., $\#(S^4, T^2)$ to become equivalent. Are there examples of topologically isotopic surfaces with this property so that the complement of the surface is simply-connected?

6.4 Spaces of diffeomorphisms and embeddings

Problem 30 (David Gabai)**.** *Is Diff$_0(S^3 \times S^1)/$Diff$_0(B^4, \partial)$ non trivial?*

The conjectured answer is "yes." Budney and Gabai proved (during the conference) that

$$\mathrm{Diff}_0(B^3 \times S^1, \partial)/\mathrm{Diff}_0(B^4, \partial)$$

is non-trivial, but by construction, all of their maps are isotopically trivial when extended to $S^3 \times S^1$ [4].

Given a knot in S^3, if you include the knot into S^4, it becomes trivial. But there are two ways to trivialize the knot, and this question concerns whether or not those two trivialization (processes) are distinct, in a homotopy-sense. Let K_n be the space of smooth embeddings of $\mathbb{R}$ into $\mathbb{R}^n$ which agree with the map $x \mapsto (x, 0)$ outside of the interval $[-1, 1]$.

The inclusion $K_3 \to K_4$ is null homotopic. Here are two null homotopies of that inclusion map. In the first one, once the knot is included into K_4, you perturb it slightly so that it increases rapidly (in the 4th dimension) near the "start" of the interval $[-1, 1]$, and have the 4th coordinate decrease slowly along the interval $[-1, 1]$. Now there is a straight line homotopy in the $\mathbb{R}^3$ coordinates. When you remove the bump in the 4th dimension, that gives one null homotopy of the inclusion $K_3 \to K_4$. Repeat the same process, but using a bump function that increases along the interval $[-1, 1]$.

Combining the two maps together yields a map: $K_3 \to \Omega K_4$

Problem 31 (Ryan Budney)**.** *Is the map $K_3 \to \Omega K_4$ null-homotopic?*

It is known to be trivial on rational homology, and rational homotopy groups. Budney believes this map is null-homotopic as well, and conjectures that there exists a non-trivial map $K_3 \to \Omega^2 K_4$ which is non-trivial on rational homology.

Problem 32 (Auckly)**.** *Forthcoming work of Auckly and Ruberman implies that for some manifolds*

$$\ker(\pi_n(\textit{Diff}(Z)) \to \pi_n(\textit{Homeo}(Z)))$$

has high rank summands, and for some manifolds $\pi_n(\textit{Diff}(Z)) \to \pi_n(\textit{Homeo}(Z))$ has a large cokernel.

(a) Are there manifolds for which the kernel has an infinite rank summand? (Conjecturally, the answer is yes.)

(b) Is it non-trivial for every manifold? (This is difficult with current technology.)

(c) Is there one manifold for which all of these groups are non-trivial? For which all of these groups have an infinite rank summand?

(d) Is there a manifold so that the kernel and cokernel are both non-trivial? Both large? Both very large (e.g., contain infinite rank summands)?

(e) Is (d) be true for all 4-manifolds?

(f) What about the analogous questions for spaces of embeddings? (Results about a space of diffeomorphsims seems to yield results about spaces of embeddings for free.)

6.5 Additional problems

Let G be a group, and S a set of generators for G. Given $\beta \in G$, an *S-band decomposition of β* is a product of the form

$$\beta = w_1 s_1^{\pm 1} w_1^{-1} \cdots w_p s_p^{\pm 1} w_p^{-1}$$

where each $s_j \in S$ and $w_j \in G$. The *S-rank* of β, denoted $\mathrm{rk}_S(\beta)$, is defined to be the smallest p for which such an S-band decomposition exists.

Problem 33 (Mark Hughes)**.** *Let B_n be the n-strand braid group, and $\Sigma = \{\sigma_1, \dots, \sigma_{n-1}\}$ the Artin generators. Given a braid $\beta \in B_n$, is there a finite group H and a homomorphism $\phi : B_n \to H$ such that* $\mathrm{rk}_{\phi(\Sigma)}(\phi(\beta)) = \mathrm{rk}_\Sigma(\beta)$*? Is there a procedure for producing such a group H given β?*

Problem 34 (Hughes)**.** *A braid β is called* quasipositive *if it can be written as*

$$\beta = w_1 \sigma_{i_1} w_1^{-1} \cdots w_p \sigma_{i_p} w_p^{-1}$$

for some $i_j \in \{1, 2, \dots, n-1\}$ and $w_j \in B_n$. Is there a algorithm to determine whether or not a braid is quasipositive?

Problem 35 (Hans Boden)**.** *Is the Tait flyping conjecture true for links in thickened surfaces and/or virtual links?*

Problem 36 (Boden)**.** *Do the Tait conjectures hold for welded links?*

The following is a sub-problem of the previous one:

Problem 37 (Boden)**.** *Does a reduced alternating diagram of a welded knot have minimal crossing number?*

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