p-adic dynamics of Hecke operators

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This is a report on a Research-in-Teams project.

Participants.

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4. Juan Rivera-Letelier, University of Rochester, New York, USA.

The main short-term goal of this research-in-teams project was to extend work done by two separate groups to a wide class of Shimura curves and level structures. The work of the first group, consisting of Herrero-Menares-Rivera-Letelier, studied the dynamics of Hecke operators T_n acting on the j-line, in the p-adic topology, in terms of limit of measures. As a sample result we provide the following: given a j invariant x of a curve with ordinary mod p reduction, denote by $\delta(n)$ the probability measure supported on the orbit $T_n(x)$ of x under the Hecke operator T_n; then

$$\lim_{n\to\infty} \delta(n) = \delta_\zeta,$$

where δ_ζ is the Dirac distribution associated to the Gauss point ζ on the Berkovich projective line. The convergence is in the sense of "weak convergence" of measures. An interesting feature of this result that it requires extending the usual affine line to the Berkovich line to recognize the limit.

The work of the second group, consisting of Goren-Kassaei, in the same setting concerns the dynamics of the iterations $T_\ell^n(x)$ as n goes to infinity, $(\ell, p) = 1$ and ℓ a prime. It contains a weaker statement than the above, but describe the set $T_\ell^n(x)$ nicely using isogeny volcanoes and, more significantly, explains the "error term" in the following sense. There is a minimal n_o such that a point in $T_\ell^{n_o}x$ belongs to the same residue disc as x, due to a cyclic endomorphism f of the mod p reduction of x, and then the dynamics of T_ℓ on this residue disc is given by

$$q \mapsto (1+q)^{\bar{f}/f},$$

where q is the Serre-Tate parameter and $\bar{f}/f$ is a p-adic unit that is explicitly defined.

The papers referred to above are the following. A more precise formulation of these results and, in fact, many more and much harder results can be found there.

- Eyal Z. Goren & Payman L. Kassaei: p-adic dynamics of Hecke operators on modular curves. 38 pp. ArXiv:1711.00269 *Submitted*
- Sebastian Herrero, Ricard Menares & Juan Rivera-Letelier: p-adic distribution of CM points and Hecke orbits I. Convergence towards the Gauss point. *Submitted*

Our goal in this Research-in-Teams project was to take the whole spectrum of results both teams had and to develop them for the case of Shimura curves. Even the case of Shimura curves offers many, perhaps too many, variations. And, the case of modular curves is just a test-ground for a more general study of studying the action of Hecke operators on special subvarieties of Shimura varieties, in the p-adic topology. Such general studies were conducted by Clozel, Ullmo, and others over the complex numbers. The analogous questions for p-adic actions seem to be wide-open, although the mod p action was studied by Chai, Oort, and others.

That said, the setting we considered during this week is that of Shimura curves associated to a quaternion algebra B over $\mathbb{Q}$ and Drinfeld level structure of type $\Gamma_0(n)$. By allowing p to divide the discriminant of B, or not, the level n, or not, we have several situations to consider. During the week we came up with conjectural descriptions of the limit measures
$\lim_{n\to\infty,(n,p)=1}\delta(n)$ when x is ordinary (in most situations), and have made very significant progress towards completing the conjectural description also when x is not ordinary.

The nature of the answer depends very much on whether $p|\operatorname{disc} B$, or $p|n$. For $p|\operatorname{disc} B$ we were able to use work of Drinfeld and Ribet. For $p|n$ and $B = M_2(\mathbb{Q})$, we were able to use work of Katz-Mazur. The case of $\operatorname{disc} B \neq 1$ and $p|n$ is conjecturally understood, but in reality we will have to use Drinfeld level structures for quaternion algebras. This was partially done by K. Buzzard but because he put severe restrictions on n we will have to develop this theory further ourselves. Similar "side projects" – very substantial on their own – came up during the week, but we hope to have solved the conceptual difficulties in carrying them on.

While a lot of work remains, it is safe to say that many of the main obstacles were overcome during this week.

I believe that I describe everyone's experience in saying that this week was one of the best experiences of joint research I have ever participated in. The conditions in BIRS were ideal. For one, the freedom of not needing to worry about meals – three meals provided in the dining hall – freed a lot of time. 2) The room allocated to us had large boards, convenient desks and a laptop projector (and access for wifi). It was excellent for collaboration. 3) The athletic facilities in Banff, and the trekking trails, offered excellent opportunities to break from marathon discussion sessions running from morning till late at night.

Our team members had complementary expertise and that kept discussions lively and interesting. We leave BIRS with a long "to-do" list, among its items are developing further Drinfeld level structures for Shimura curves, using Bruhat-Tits buildings to understand Hecke orbits, studying quadratic forms and equidistribution results for Eichler orders in quaternion algebras, studying metric properties of period maps for moduli of p-divisible groups with a Drinfeld level structure, and much more. We are very tempted to return to Banff to put the finishing touches on the project in the future. In particular, we believe that the atmosphere and working conditions in BIRS will be conducive for the completion of Phase II of our project which is devoted to arithmetic applications.

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