Analysis and geometry of several complex variables

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Let $\Omega \subseteq \mathbb{C}^d$ be an open connected set, and let $L^2(\Omega)$ denote the Hilbert space of equivalence classes of measurable functions f that are square-integrable with respect to the Lebesgue measure on Ω, i.e., for which

$$||f||^2_{L^2(\Omega)} = \int_\Omega |f(\mathbf{z})|^2 \, d\mathbf{z} < +\infty.$$

The *Bergman space* $A^2(\Omega)$ given by

$$A^2(\Omega) = \left\{ h \in L^2(\Omega) : h \text{ is holomorphic on } \Omega \right\}$$

is a closed subspace of $L^2(\Omega)$. The orthogonal projection $\mathcal{P} : L^2(\Omega) \to A^2(\Omega)$ is called the *Bergman projection*. This operator is given by integration:

$$\mathcal{P}f(\mathbf{z}) = \int_\Omega B(\mathbf{z}, \mathbf{w}) f(\mathbf{w}) \, d\mathbf{w}$$

where $B(\mathbf{z}, \mathbf{w}) : \Omega \times \Omega \to \mathbb{C}$ is the *Bergman kernel*. Basic properties of the Bergman kernel can be found in [4]. The study of the Bergman kernel and projection has a long and rich history, but many interesting questions remain. Our collaborative research has focused on estimation of the Bergman kernel, see [5], for several different types of domains. We made additional progress during the week-long residence at BIRS. We are currently writing up our results for publication. We briefly summarize them below.

1. Tubes over convex sets and log-convex Reinhardt domains.

A *tube over a convex set* $\Sigma \subset \mathbb{R}^n$ is the domain

$$T = \left\{ \mathbf{x} + i\mathbf{y} \in \mathbb{C}^n : \mathbf{y} \in \Sigma \right\}.$$

We say that $\mathcal{R} \subset \mathbb{C}^n$ is a *Reinhardt domain* if it is invariant under rotations about each coordinate axis[1]. A Reinhardt domain $\mathcal{R}$ is *axis-deleted* if $\prod_{j=1}^n z_j \neq 0$ for all $(z_1, \ldots, z_n) \in \mathcal{R}$, and is *log-convex* if

$$\log |\mathcal{R}| = \left\{ y_1, \ldots, y_n) \in \mathbb{R}^n : (e^{y_1}, \ldots, e^{y_n}) \in \mathcal{R} \right\}$$

is a convex set. If $\mathcal{R}$ is an axis-deleted log-convex Reinhardt domain then $\Phi(z_1, \ldots, z_n) = (e^{-iz_1}, \ldots, e^{-iz_n})$ is a holomorphic mapping of the tube domain over $\log |\mathcal{R}|$ onto $\mathcal{R}$ and is the universal covering map.

[1]This means that if $(z_1, \ldots, z_n) \in \mathcal{R}$ then $(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \in \mathcal{R}$ for all $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$.

We obtain sharp geometric estimates for the Bergman kernel $B(\mathbf{z}, \mathbf{w})$ and its derivatives on the diagonal $\mathbf{z} = \mathbf{w}$ for tubes T over convex sets Σ and for axis-deleted log-convex Reinhardt domains $\mathcal{R}$. These estimates are given in terms of the volume and other geometric properties of minimal caps of the convex sets Σ or $\log|\mathcal{R}|$. To describe these caps, let $E \subset \mathbb{R}^n$ be a proper[2] open convex set. If $\mathbf{p} \in E$ and $\mathbf{n}$ is a unit vector in $\mathbb{R}^n$, the *cap of E through $\mathbf{p}$ in direction $\mathbf{n}$* is the set $C_{\mathbf{p}}(\mathbf{n}) = \big\{\mathbf{y} \in E : \langle \mathbf{y} - \mathbf{p}, \mathbf{n}\rangle \geq 0\big\}$. There is a unit vector $\nu_{\mathbf{p},n} \in \mathbb{R}^n$ so that, if $|E|$ denotes the volume of a set E,

$$|C_{\mathbf{p}}(\nu_{\mathbf{p},n})| = \inf_{\mathbf{n} \in \mathbb{R}^n} |C_{\mathbf{p}}(\mathbf{n})|.$$

Thus $C_{\mathbf{p}}(\nu_{\mathbf{p},n})$ is a cap[3] of E of minimal volume through $\mathbf{p}$. Moreover, $\mathbf{p}$ is the centroid of the convex $(n-1)$-dimensional slice $E_{\mathbf{p}}(\nu_{\mathbf{p}}) = \big\{\mathbf{y} \in E : \langle \mathbf{y} - \mathbf{p}, \nu_{\mathbf{p}}\rangle = 0\big\}$. It follows that there are mutually orthogonal unit vectors $\nu_{\mathbf{p},1}, \ldots, \nu_{\mathbf{p},n-1}$, all orthogonal to $\nu_{\mathbf{p},n}$, and positive constants $\mu_1(\mathbf{p}), \ldots, \mu_{n-1}(\mathbf{p})$ so that

$$E_{\mathbf{p}}(\nu_{\mathbf{p}}) \approx \Big\{\mathbf{y} \in \mathbb{R}^n : \langle \mathbf{y} - \mathbf{p}, \nu_{\mathbf{p},n}\rangle = 0 \text{ and } |\langle \mathbf{y} - \mathbf{p}, \nu_{\mathbf{p},j}\rangle| < \mu_j(\mathbf{p}) \text{ for } 1 \leq j \leq n-1\Big\}.$$

The notation above implies that $E_{\mathbf{p}}(\nu_{\mathbf{p}})$ contains, and is contained in, constant dilates of the parallelepiped appearing on the right hand side, where the implicit constants in the approximation depend only on the dimension n. Let $\Delta(\mathbf{p}) = |C_{\mathbf{p}}(\nu_{\mathbf{p}})|$ be the minimal volume, and let $\mu_n(\mathbf{p}) = \sup_{\mathbf{y} \in E} \langle \mathbf{y} - \mathbf{p}, \nu_{\mathbf{p}}\rangle$.

We now state our diagonal estimates of the Bergman kernel $B_T(\mathbf{z}, \mathbf{w})$ for the tube domain T over a convex set $\Sigma \subset \mathbb{R}^n$. Let $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, $\mathbf{w} = \mathbf{x} + i\mathbf{v} \in T$ and let $\mathbf{p} = \frac{1}{2}(\mathbf{y} + \mathbf{v}) \in \Sigma$. Then $B_T(\mathbf{z}, \mathbf{w}) \approx \Delta(\mathbf{p})^{-2}$ where the implicit constants in the approximation depend only on the dimension n. To estimate derivatives, choose coordinates in $\mathbb{R}^n$ so that the j^{th} coordinate axis is in the direction of $\nu_{\mathbf{p},j}$. Then we show

$$\Big|\partial_{\mathbf{z}}^{\alpha} \partial_{\bar{\mathbf{z}}}^{\beta} B_T(\mathbf{z}, \mathbf{w})\Big| \lesssim \Delta(\mathbf{p})^{-2} \prod_{j=1}^{n} \mu_j(\mathbf{p})^{-\alpha_j - \beta_j}.$$

We also obtain estimates of the Bergman kernel $B_{\mathcal{R}}$ for an axis-deleted log-convex Reinhardt domain $\mathcal{R}$. This time the estimates use the minimal caps of the convex set $\log|\mathcal{R}|$. When the parameters $\mu_1(\mathbf{p}), \ldots, \mu_n(\mathbf{p})$ are all small, the estimates are identical with the tube case. However if some of these parameters are large, the estimates also depend on the number of lattice points in sets of the form

$$\big\{(y_1, \ldots, y_n) \in \mathbb{R}^n : |y_j| < \mu_j(\mathbf{p})^{-1},\ 1 \leq j \leq n\big\}.$$

2. Monomial-type model domains and monomial balls.

For any $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ the function $F_{\mathbf{m}}(\mathbf{z}) = z_1^{m_1} \cdots z_n^{m_n}$ is a monomial which is holomorphic on $\mathbb{C}^n \setminus \big\{\mathbf{z} \in \mathbb{C}^n : \prod_{j=1}^n z_j = 0\big\}$. We obtain diagonal estimates of the Bergman kernel for model domains in $\mathbb{C}^{n+1}$ of the form

$$\Omega = \Big\{(z_1, \ldots, z_n, w) \in \mathbb{C}^{n+1} : \sum_{j=1}^{d} |F_{m_j}(\mathbf{z})|^2 < \operatorname{Im} w\Big\}.$$

To study the Bergman kernel on the diagonal at a point $(0, \ldots, 0, i\delta)$ above the origin, the problem is easily reduced to obtaining diagonal estimates of the Bergman kernel for the domain

$$\Omega_\delta = \Big\{(z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{d} |F_{m_j}(\mathbf{z})|^2 < \delta\Big\},$$

and this is possible since Ω_δ is a Reinhardt domain. To obtain estimates above some other point $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$ we observe that Ω is biholomorphic to the domain

$$\Omega(\mathbf{a}) = \Big\{(z_1, \ldots, z_n, w) \in \mathbb{C}^{n+1} : \sum_{j=1}^{d} |F_{m_j}(\mathbf{z}) - F_{\mathbf{m}_j}(\mathbf{a})|^2 < \operatorname{Im} w\Big\}.$$

[2] This means that E does not contain any entire straight line.

[3] Note that the minimal cap need not be unique.

The domain $\Omega(\mathbf{a})$ is called a *monomial ball*, and estimating the Bergman kernel on the diagonal then involves the following steps.

1. Obtain a structure theorem for $\Omega(\mathbf{a})$: after a monomial mapping, monomial balls can be written as the Cartesian product of a polydisk and an axis-deleted log-convex Reinhardt domain.

2. Study the behavior of the Bergman kernel under monomial mappings.

3. Compute weighted diagonal estimates for the Bergman kernel on Reinhardt domains.

3. Off-diagonal estimates for tubes over model polynomial domains.

If $\Sigma \subset \mathbb{R}^{n+1}$ is a proper open convex set, the Bergman kernel for the tube T over Σ is given by

$$B_T(\mathbf{z},\mathbf{w}) = \int_{\mathbb{R}^{n+1}} e^{-2\pi i\langle \mathbf{x}-\mathbf{u},\mathbf{t}\rangle} e^{4\pi\langle \mathbf{p},\mathbf{t}\rangle}\left[\int_\Sigma e^{4\pi\langle \mathbf{s},\mathbf{t}\rangle}\,d\mathbf{s}\right]^{-1} d\mathbf{t},$$

where $\mathbf{z} = \mathbf{x}+i\mathbf{y}$, $\mathbf{w} = \mathbf{u}+i\mathbf{v} \in T$ and $\mathbf{p} = \frac{1}{2}(\mathbf{y}+\mathbf{v}) \in \Sigma$. Here the understanding is that $\left[\int_\Sigma e^{4\pi\langle \mathbf{s},\mathbf{t}\rangle}\,d\mathbf{s}\right]^{-1} = 0$ if $\int_\Sigma e^{4\pi\langle \mathbf{s},\mathbf{t}\rangle}\,d\mathbf{s} = \infty$. To obtain estimates which are better than the estitmates on the diagonal, one must take advantage of the oscillation $e^{-2\pi i\langle \mathbf{x}-\mathbf{u},\mathbf{t}\rangle}$ in the outer integral. We obtain sharp estimates for domains

$$\Sigma = \left\{(y_1,\ldots,y_n,y) = (\mathbf{y}',y) \in \mathbb{R}^{n+1} : y > \Psi(\mathbf{y}')\right\}$$

where $\Psi : \mathbb{R}^n \to \mathbb{R}$ is a convex polynomial of finite line type[4].

For $\mathbf{p}' \in \mathbb{R}^n$ let $\Psi_{\mathbf{p}'}(\mathbf{y}) = \Psi(\mathbf{p}'+\mathbf{y}') - \Psi(\mathbf{p}') - \langle \nabla\Psi(\mathbf{p}'),\mathbf{y}'\rangle$, and note that $\Psi_{\mathbf{p}'}$ is normalized at $\mathbf{p}'$ in the sense that $\Psi_{\mathbf{p}'}(\mathbf{0}') = 0$ and $\nabla\Psi_{\mathbf{p}'}(\mathbf{0}') = \mathbf{0}'$. Our estimates are given in terms of the geometry of the convex set $\left\{\mathbf{s}' \in \mathbb{R}^n : t\Psi_{\mathbf{p}}(\mathbf{s}) \le 1\right\}$. We show that there is a constant $\kappa \ge 1$ and for each $\mathbf{p} \in \mathbb{R}^n$ and each $t > 0$ there is a choice of coordinates $(s_1,\ldots,s_n)$ in $\mathbb{R}^n$ and positive constants $\mu_1(\mathbf{p},t),\ldots,\mu_n(\mathbf{p},t)$ so that

$$\Big\{\mathbf{s} \in \mathbb{R}^n : |s_j| \le \mu_j(\mathbf{p},t^{-1})\Big\} \subset \Big\{\mathbf{s} \in \mathbb{R}^n : t\Psi_{\mathbf{p}}(\mathbf{s}) \le 1\Big\} \subset \Big\{\mathbf{s} \in \mathbb{R}^n : |s_j| \le \kappa\,\mu_j(\mathbf{p},t^{-1})\Big\}.$$

Now let $\mathbf{z} = (\mathbf{z}',z) = (\mathbf{x}'+i\mathbf{y}',x+iy)$, $\mathbf{w} = (\mathbf{w}',w) = (\mathbf{u}'+i\mathbf{v}',u+iv) \in \mathbb{C}^{n+1}$, and let

$$\begin{aligned} p &= \frac{1}{2}(y+v) \in \mathbb{R}, & \mathbf{p}' &= \frac{1}{2}(\mathbf{y}'+\mathbf{v}') \in \mathbb{R}^n, \\ \lambda &= (x-u) - \langle \mathbf{x}'-\mathbf{u}',\nabla\Psi(\mathbf{p}')\rangle, & \mu &= 2\big(p-\Psi(\mathbf{p}')\big) \in \mathbb{R}. \end{aligned} \tag{1}$$

Note that $(\mathbf{p}',p) \in \Sigma$ and that $\mu \ge 0$ with $\mu > 0$ if $(\mathbf{p}',p) \in \mathrm{Int}(\Sigma)$. We show that

$$\Big|B_T\big((\mathbf{z},z),(\mathbf{w},w)\big)\Big| \approx |\lambda+i\mu|^{-2}\prod_{j=1}^n \Big(|x_j-u_j| + \mu_j\big(\mathbf{p},|\lambda+i\mu|\big)\Big)^{-2}$$

with corresponding estimates for derivatives of B_T.

4. Additional questions.

We are also working on several related problems.

(a) There has been considerable interest in the formula for Bergman kernels and the L^p boundedness of Bergman projections in tubes over symmetric cones (see for example [3], [2], [1]). We are interested in extending these investigations to tubes over certain classes of non-symmetric cones.

(b) We would like to obtain sharp off-diagonal estimates of the Bergmen kernel for tubes T over bounded convex sets Σ. We can show for example that $|B_T(\mathbf{x}+i\mathbf{y},\mathbf{u}+i\mathbf{v})| \lesssim e^{-\epsilon|\mathbf{x}-\mathbf{u}|}$.

Finally we would like to thank BIRS for the opportunity to work together for a week and we greatly appreciate their hospitality.

[4]This means that any straight line in $\mathbb{R}^{n+1}$ makes at most finite order of contact with the graph $y_0 = \Psi(y_1,\ldots,y_n)$.

References

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