Moduli Spaces: Birational Geometry and Wall Crossings

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1 Overview of the Field

In ancient times moduli spaces were lonely creations, appearing one at a time. Things changed with the advent of Variations of Geometric Invariant Theory, introduced in the toric case by Brion–Procesi [10] and in general by Thaddeus [32] and Dolgachev–Hu [17]. This theory shows that some moduli spaces come in collections of birationally equivalent versions, with closely related geometry. These moduli spaces occur as Geometric Invariant Theory quotients of the same space, and correspond to points in a space of linearizations which exhibits a Wall and Chamber structure — the moduli spaces change as one passes from one chamber to another. Thaddeus [32] provided a proof of the Verlinde Formula, demonstrating that this added flexibility of moduli spaces can be used to prove an important result on classical moduli spaces of vector bundles.

Birational transitions are inherent in the minimal model program (MMP), and wall-and-chamber transitions — in the form of the directed MMP — are a tool in the groundbreaking work on the MMP by Birkar–Cascini–Hacon–M^{c}Kernan [7]. Putting this together with moduli spaces, the MMP for moduli spaces is the central object of the Hassett–Keel program for the moduli spaces of curves. The paper of Gibney–Keel–Morrison [19] introduced the minimal model program into the birational geometry of the moduli spaces of curves, and the papers of Hassett–Hyeon [23, 24] and Alper–Fedorchuk–Smyth [2] described the first few birational transitions these moduli spaces undergo in the directed MMP.

The minimal model program is also a tool in the construction of moduli spaces of higher dimensional stable varieties, first introduced by Kollár–Shepherd-Barron [26] and Alexeev [1]. It is now the subject of a book project by Kollár, aided by recent advances of Hacon–M^{c}Kernan [20], Hacon–M^{c}Kernan–Xu [21], Hacon–Xu [22], and current work of Fujino, Hacon, Kovács, M^{c}Kernan, Patakfalvi, Xu, and others. In particular, these advances allow us to look at moduli spaces of pairs (X, D) of a variety X with a divisor D, which, as one changes the coefficients of D exhibit wall-and-chamber transitions analogous to those of the moduli spaces of curves. Many new results are cropping up studying these moduli spaces and the transitions between them. At the same time there is a critical issue: in a large range of coefficients (specifically, any coefficients < 0.5), there is no universally accepted definition of the moduli space.

We now turn our attention to another source of wall-crossing phenomena in moduli spaces. Stability conditions in triangulated categories were introduced in a seminal paper of Bridgeland [9], which was motivated by work in theoretical physics by Douglas and informed by GIT stability for vector bundles. Stability conditions on a fixed triangulated category are parametrized by a manifold, and, by design, the class of stable objects, and the corresponding moduli space, undergo wall-crossing transitions.

Stability conditions in triangulated categories, and the resulting transitions between moduli spaces of objects, were related to birational geometry already in early work of Bridgeland [8], and the relation is present to this day, for instance in work of Arcara–Bertram–Coskun–Huizenga [3], Bayer–Bertram–Macrì–Toda [5], and Toda [33]. But such transitions have enormous implications in a subject of intense current effort — namely curve counting invariants, where all the above techniques seem to converge.

Numerical invariants associated to these moduli spaces of stable objects, the so-called Donaldson–Thomas invariants, change via the wall-crossing formulas which were beautifully formalized in the theory of Kontsevich–Soibelman [27]. Wall-crossing of this sort has been used intensively in the last several years by Bridgeland, Calabrese, Chang, Ciocan-Fontanine, Kiem, Kim, J. Li, A.-M. Li, M. Liu, Szendroi, Toda, and others to prove deep relationships between various curve counting theories. The result is a magnificent interwoven picture, part proven and part conjectured, between Donaldson–Thomas invariants, Gromov–Witten invariants, Pandharipande–Thomas invariants, and the Landau–Ginzburg invariants of Fan–Jarvis–Ruan [18], with the quasimaps invariants of Ciocan-Fontanine–Kim ([16]) thrown in between specifically to enable wall-crossing transitions to connect between these invariants.

2 Recent Developments and Open Problems

2.1 Foundations of moduli spaces

Moduli spaces of semistable objects in the derived categories have been an important tool in Donaldson-Thomas theory and a major source of wall-crossings phenomena, yet they were mostly constructed case-by-case. Curent work of Alper–Halpern-Leistner–Heinloth, based on work of Artin–Zhang [4], provides foundations for the existence of an Artin stack of semistable objects admitting a proper good moduli space in fairly general situations. This work was the subject of the opening lecture by Alper. The work still posits numerous assumptions (in the spirit of "Artin's criteria"). It is an open question in what generality these assumptions hold.

2.2 Construction of stability conditions

Bridgeland's stability conditions were defined years ago, but their existence is in general hard to verify. A major breakthrough this decade was provided by work of Bayer, Macrì, and Toda, who showed that certain Gieseker type inequalities on a tilted heart of the derived category of a variety can be used to guarantee the existence of a stability condition. A major breakthrough was obtained by Maciocia and Piyaratne, who showed that the Bayer–Macri–Toda approach applies to certain abelian varieties. Bayer reported on stability conditions in Kuznetsov's category (see below). He also pointed out that during our meeting a preprint by C. Li appeared on the archive [28] which constructs a stability condition on the derived category of the quintic threefold, the first known example on a projective strict Calabi–Yau threefold. This was the subject of much discussion between participants. Once again this is one example — but one expects further examples to follow, and it remains an open question in what generality stability conditions exist.

2.3 Moduli in the MMP

The groundbreaking work of Birkar–Cascini–Hacon–McKernan [7] on the minimal model program, the work of Hacon–McKernan–Xu [20, 21, 22] on boundedness and moduli, and Kollár's forthcoming book on higher dimensional moduli opened the possibility of constructing meaningful moduli spaces of "stable pairs" in the sense of the MMP and wall–crossings transitions between them (it was pointed out that "stable pairs" is a loaded term). Two recent PhDs reported on their work in this direction: Bejleri reported on work with Ascher and related work of Inchiostro on moduli of elliptic surface pairs, where wall crossings transitions are used in the construction of the moduli spaces. DeVleming reported on work extending Hacking's work on curves in the projective plane to surfaces in the projective space — here one works with pairs near the boundary wall of moduli, and shows properness of moduli for surfaces of odd degree. The case of even degree, and the analogous questions for higher dimension, remain open.

2.4 Wall-Crossings for Donaldson–Thomas invariants.

The general machinery for computing the change in Donaldson–Thomas partition functions under the change of stability conditions (wall-crossing formulas), developed by Joyce–Song and Kontsevich–Soibelman several years ago, can be difficult to wield in practice. While there were early successful implementations of the wall-crossing formula (for example by Toda and Bridgeland), recent technical breakthroughs by Bentjees–Calabrese–Rennemo [6] have allowed them to prove the Donaldson-Thomas crepant resolution conjecture. As reported by Rennemo at the workshop, they have used wall-crossing techniques to relate the partition functions of four Donaldson-Thomas like theories associated to a Calabi–Yau orbifold $\mathcal{X}$ and its crepant resolution Y, namely, $DT(\mathcal{X})$, $PT(\mathcal{X})$, $DT(Y)$, and $BS(Y)$ the Donaldson–Thomas, Pandharipande–Thomas, and Bryan–Steinberg partition functions of $\mathcal{X}$ and Y. They provide relationships between all of them, most strikingly, they show that $PT(\mathcal{X}) = BS(Y)$, but only *after re-expansion as rational functions.* This last assertion, while easy to state, is a highly non-trivial consequence of subtle wall-crossing analysis.

3 Presentation Highlights

The workshop had 16 research talks, among which 5 were given by female speakers and 4 were given by junior scholars. All of the talks were beautifully presented and reported most recent advances in the field. Below we mention a few presentation highlights.

In the talk "Complete intersections of three quadrics and rationality" based on joint work with Pirutka and Tschinkel [25], Hassett described how the moduli of complete intersections of three quadrics vary from the perspective of rationality, with a focus on geometric and arithmetic structures that may induce rational parameterizations.

In the talk "Moduli spaces for Kuznetsov categories of Fano threefolds and cubic fourfolds" based on joint work with Lahoz, Macri, Nuer, Stellari and Perry, Bayer explained the construction of stability conditions for the Kuznetsov category of cubic fourfolds, and the notion and construction of stability conditions in families of varieties.

In the talk "The Donaldson–Thomas crepant resolution conjecture" based on joint work with Beentjes and Calabrese [6], Rennemo explained a proof of the crepant resolution conjecture for Donaldson-Thomas invariants of hard Lefschetz CY3 orbifolds, formulated by Bryan–Cadman–Young, after reinterpreting it as an equality of rational functions.

In the talk "Cohomological field theories and derivations" based on joint work with Oberdieck [29, 30], Pixton described connections between cohomological field theories, Givental's R-matrix action, and holomorphic anomaly equations from the Gromov–Witten theory of elliptic fibrations.

In the talk "MSP, NMSP and their applications" based on joint work with Chang, Guo, J. Li and Liu [11, 12, 13, 14], W.-P. Li surveyed latest developments about the theory of Mixed-Spin-P-fields, its applications to Gromov–Witten invariants and FJRW invariants, and in particular, a very recent proof of BCOV's Feynman Rule for quintic 3-folds.

In the talk "Logarithmic GLSM moduli spaces" based on joint work with Q. Chen, Ruan and Sauvaget [15], Janda described a construction of logarithmic GLSM moduli spaces via Witten's r-spin class and its application to cycle class calculations of the strata of holomorphic differentials.

4 Scientific Progress Made and Outcome of the Meeting

There were numerous active and inspiring discussions between the workshop participants.

Bayer had a discussion with Haseett on cubic fourfolds that was very helpful for his research.

Bayer, Bolognese, and D. Chen discussed quadratic differentials and stability conditions, aiming at counting invariants in Teichmüller dynamics by wall-crossing formulas in enumerative geometry.

D. Chen and Ionel discussed the three proofs of Witten's conjecture by Kontsevich, Okounkov–Pandharipande, and Mirzakhani, aiming at analogous proofs for the volumes of the strata of holomorphic differentials via intersection theory.

D. Chen and Pixton discussed asymptotic properties of certain Hurwitz numbers of torus coverings, aiming at volume calculations of the strata of holomorphic differentials via intersection theory.

Q. Chen, Janda, and Ruan made progress on several key steps in their project about higher genus Gromov–Witten invariants calculations via logarithmic compactifications, such as cosection and virtual cycle constructions. They also discussed with W.-P. Li about a different approach towards higher genus Gromov–Witten invariants via master space localization.

Deopurkar and Patel made significant progress understanding a particularly beautiful example which concerns the behavior of ramification divisors of linear projections of projective varieties. They also did some calculations and gained clarity on what questions they should have been asking about stabilization phenomena in the Chow rings of Hurwitz spaces. Moreover, they had fruitful discussions with Han about the problem of determining an upper bound on the number of planes that a smooth degree d hypersurface in $\mathbb{P}^5$ can have, and they made some progress in understanding the asymptotics of the upper bound in terms of d.

Rennemo was benefited from various discussions during the workshop. In particular, the comments he received after his talk improved the paper [6] he talked about.

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